

# Denumerably many positive solutions for fractional order boundary value problems

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ABSTRACT. The sufficient conditions are derived for the existence of denumerably many positive solutions for fractional order boundary value problem

$$\begin{aligned} \mathfrak{D}_{0+}^\varepsilon \left[ \phi \left[ \mathfrak{D}_{0+}^\eta \omega(t) \right] \right] + f(\omega(t)) &= 0, \quad 0 < t < 1, \\ \omega(0) = \omega'(0) = \mathfrak{D}_{0+}^\eta \omega(0) &= 0, \\ \mathfrak{D}_{0+}^\varepsilon \omega(1) = I_{0+}^\delta \omega(1) & \end{aligned}$$

where  $\mathfrak{D}_{0+}^\varepsilon, \mathfrak{D}_{0+}^\eta$  denote fractional derivatives of Riemann-Liouville type with  $0 < \varepsilon < 1, 2 < \eta \leq 3, I_{0+}^\delta$  ( $\delta \in \mathbb{R}$ ) denotes Riemann-Liouville fractional integral,  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is an increasing homeomorphism and positive homomorphism operator (IHPHO), by applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space.

## 1. INTRODUCTION

The fractional derivatives are used for a better description of considered material properties, mathematical modelling based on enhanced rheological models naturally leads to differential equations of fractional order and to the necessity of the formulation of initial conditions to such equations. Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions. It should be noted that most papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations in terms of special functions. Recently, the researchers are shown a great deal of interest in establishing existence and multiplicity of solutions (or positive solutions) of nonlinear initial fractional differential equations by the use of techniques of nonlinear analysis, see [7, 12, 36] and reference therein. In fact, boundary conditions have the same requirements. However, few papers have considered the boundary value problems of fractional differential equations, see [25, 15, 16].

In [30], Y. Tian and Z. Bai considered the  $p$ -Laplacian fractional order boundary value problem

$$\begin{aligned} \mathfrak{D}_{0+}^\beta \left( \phi_p \left( \mathfrak{D}_{0+}^\alpha \omega(t) \right) \right) &= f(t, \omega(t)), \quad t \in (0, 1) \\ \omega(0) = \omega'(0) = 0, \omega'(1) &= a\omega'(\xi) \\ \mathfrak{D}_{0+}^\alpha \omega(0) = 0, \mathfrak{D}_{0+}^\alpha \omega(1) &= b\mathfrak{D}_{0+}^\alpha \omega(\eta), \end{aligned}$$

and established existence of positive solutions by applying monotone iterative method and the fixed point index theory on cones. In [9], Ege and Topal considered the fractional

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boundary value problem with IHPHO,

$$\begin{aligned} & {}^c\mathcal{D}^q(\phi({}^c\mathcal{D}^r\omega(t))) + f(t, \omega(t)) = 0, \\ \alpha_1\omega(0) - \beta_1\omega'(0) &= -\gamma_1\omega(\xi_1), \alpha_2x(1) + \beta_2\omega'(1) = -\gamma_2\omega(\xi_2) \\ & {}^c\mathcal{D}^r\omega(0) = 0, 0 < q \leq 1 < r \leq 2, 0 < t < 1 \end{aligned}$$

and established existence of positive solutions by utilizing Krasnoselskii’s and Legget-Williams cone fixed point.

Inspired by the works mentioned above, in this paper we consider Riemann-Liouville fractional order boundary value problem with IHPHO,

$$\begin{cases} \mathcal{D}_{0+}^\varepsilon [\phi[\mathcal{D}_{0+}^\eta\omega(t)]] + f(\omega(t)) = 0, & 0 < t < 1, \\ \omega(0) = \omega'(0) = \mathcal{D}_{0+}^\eta\omega(0) = 0, \\ \mathcal{D}_{0+}^\varepsilon\omega(1) = I_{0+}^\delta\omega(1) \end{cases} \tag{1.1}$$

where  $\mathcal{D}_{0+}^\varepsilon, \mathcal{D}_{0+}^\eta$  denote fractional derivatives of Riemann-Liouville type with  $0 < \varepsilon < 1, 2 < \eta \leq 3, I_{0+}^\delta (\delta \in \mathbb{R}^+)$  denotes Riemann-Liouville fractional integral and establish the existence of denumerably many positive solutions.

We assume that the following conditions are satisfied throughout the paper:

- (H<sub>1</sub>)  $f : [0, +\infty) \rightarrow [0, +\infty)$  is continuous,
- (H<sub>2</sub>) For  $0 \leq \tau \leq 1, 0 < \psi(\tau) < \infty$ , where

$$\psi(\tau) = \phi^{-1}\left(\frac{\tau^\varepsilon}{\Gamma(\varepsilon + 1)}\right), \quad 0 < \varepsilon \leq 1.$$

## 2. KERNEL AND IT’S BOUNDS

In this section, the basic definitions and lemmas which are useful for our later discussions are stated and constructed kernel for the boundary value problem (1.1) later we estimated bounds for the kernel.

**Definition 2.1.** [14] The Riemann-Liouville(RL) fractional integral of order  $\delta$  of a function  $\mathfrak{T} : (0, \infty) \rightarrow \mathbb{R}$  is defined by

$$\mathcal{I}_{0+}^\delta\mathfrak{T}(s) = \int_0^s \frac{(s-x)^{\delta-1}}{\Gamma(\delta)}\mathfrak{T}(x)dx, \quad \delta > 0.$$

**Lemma 2.1.** [14, 26] The general solution to  $\mathcal{D}_{0+}^\gamma\mathfrak{T}(t) = 0$  with  $\gamma \in (m - 1, m]$  and  $m > 1$  is the function

$$\mathfrak{T}(t) = \sum_{\ell=1}^m d_\ell t^{\gamma-\ell},$$

where  $d_\ell$  is a real number.

**Lemma 2.2.** [14, 26] Let  $\gamma > 0$ . Then for any function  $\mathfrak{T} : (0, \infty) \rightarrow \mathbb{R}$ , we have

$$\mathcal{D}_{0+}^{-\gamma}\mathcal{D}_{0+}^\gamma\mathfrak{T}(t) = \mathfrak{T}(t) + \sum_{\ell=1}^m d_\ell t^{\gamma-\ell},$$

where  $d_\ell$  is a real number and  $m \in \mathbb{Z}$  is the smallest integer greater than or equal to  $\gamma$ .

**Lemma 2.3.** Suppose (H<sub>1</sub>) and (H<sub>2</sub>) hold. Let  $\eta \in (1, 2]$  and  $g \in C^2[0, 1]$ . The boundary value problem

$$\mathcal{D}_{0+}^\eta\omega(t) + g(t) = 0, \quad t \in (0, 1), \tag{2.2}$$

$$\omega(0) = \omega'(0) = 0, \quad \mathcal{D}_{0+}^\varepsilon\omega(1) = I_{0+}^\delta\omega(1), \tag{2.3}$$

has a unique solution,

$$\omega(t) = \int_0^1 \chi(t, \tau)g(\tau)d\tau + \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(\tau, s)g(s)dsd\tau,$$

where

$$\chi(t, \tau) = \frac{1}{\Gamma(\eta)} \begin{cases} t^{\eta-1}(1-\tau)^{\eta-\varepsilon-1} - (t-\tau)^{\eta-1}, & \tau \leq t, \\ t^{\eta-1}(1-\tau)^{\eta-\varepsilon-1}, & t \leq \tau, \end{cases}$$

$$\mathcal{H}(t, \tau) = \frac{\Gamma(\eta-\varepsilon)\Gamma(\eta+\delta)}{[\Gamma(\eta+\delta) - \Gamma(\eta-\varepsilon)]\Gamma(\eta)\Gamma(\delta)} t^{\eta-1}(1-\tau)^{\delta-1}.$$

*Proof.* The equivalent fractional integral equation to the equation (2.2) is given by

$$\omega(t) = c_1 t^{\eta-1} + c_2 t^{\eta-2} + c_3 t^{\eta-3} - \int_0^t \frac{(t-\tau)^{\eta-1}}{\Gamma(\eta)} g(\tau)d\tau,$$

where  $c_1, c_2$  and  $c_3$  are constants. Applying boundary conditions 2.3 into the above equation, one can obtain  $c_2 = c_3 = 0$  and

$$c_1 = \int_0^1 \frac{(1-\tau)^{\eta-\varepsilon-1}}{\Gamma(\eta)} g(\tau)d\tau + \frac{\Gamma(\eta-\varepsilon)}{\Gamma(\eta)} I_{0+}^\delta \omega(1).$$

Therefore,

$$\omega(t) = \int_0^1 \chi(t, \tau)g(\tau)d\tau + \frac{\Gamma(\eta-\varepsilon)}{\Gamma(\eta)} t^{\eta-1} I_{0+}^\delta \omega(1). \tag{2.4}$$

By simple calculations, we get

$$I_{0+}^\delta \omega(1) = \frac{\Gamma(\eta+\delta)}{\Gamma(\eta+\delta) - \Gamma(\eta-\varepsilon)} \left[ \int_0^1 \frac{(1-\tau)^{\delta-1}}{\Gamma(\delta)} \times \int_0^1 \chi(\tau, s)g(s)dsd\tau \right].$$

Hence,

$$\omega(t) = \int_0^1 \chi(t, \tau)g(\tau)d\tau + \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(\tau, s)g(s)dsd\tau.$$

□

**Lemma 2.4.** The kernel  $\chi(t, \tau)$  have the following properties:

- (i)  $\chi(t, \tau) \geq 0$  and continuous on  $[0, 1] \times [0, 1]$ ,
- (ii)  $\chi(t, \tau) \leq \chi(1, \tau)$  for  $t, \tau \in [0, 1]$ ,
- (iii) there exists  $\varrho \in (0, \frac{1}{2})$  such that  $\varrho^{\eta-1}\chi(1, \tau) \leq \chi(t, \tau)$  for  $t \in [\varrho, 1 - \varrho], \tau \in [0, 1]$ .

*Proof.* By simple calculations, (i) and (ii) can be established easily. We prove (iii). For  $0 \leq \tau \leq t \leq 1$ , let

$$F(t, \tau) = \frac{t^{\eta-1}(1-\tau)^{\eta-\varepsilon-1}}{\Gamma(\eta)} - \frac{(t-\tau)^{\eta-1}}{\Gamma(\eta)}.$$

Then for  $\varrho \in (0, \frac{1}{2})$ , we have

$$\begin{aligned} F(t, \tau) &\geq t^{\eta-1} \left[ \frac{(1-\tau)^{\eta-\varepsilon-1}}{\Gamma(\eta)} - \frac{(1-\tau)^{\eta-1}}{\Gamma(\eta)} \right] \\ &\geq \varrho^{\eta-1} \left[ \frac{(1-\tau)^{\eta-\varepsilon-1}}{\Gamma(\eta)} - \frac{(1-\tau)^{\eta-1}}{\Gamma(\eta)} \right] \\ &\geq \varrho^{\eta-1} \chi(1, \tau). \end{aligned}$$

□

**Lemma 2.5.** Suppose  $\Gamma(\eta + \delta) > \Gamma(\eta - \varepsilon)$  and  $\varrho \in (0, \frac{1}{2}]$ . Then

$$\min_{t \in [\varrho, 1-\varrho]} \mathcal{H}(t, \tau) \geq \varrho^{\eta-1} \max_{t \in [0,1]} \mathcal{H}(t, \tau).$$

*Proof.* From Lemma 2.2,

$$\frac{\min_{t \in [\varrho, 1-\varrho]} \mathcal{H}(t, \tau)}{\max_{t \in [0,1]} \mathcal{H}(t, \tau)} = \frac{\varrho^{\eta-1}(1-\tau)^{\delta-1}}{(1-\tau)^{\delta-1}} = \varrho^{\eta-1}.$$

□

**Lemma 2.6.** Suppose  $(H_1)$  and  $(H_2)$  hold, then the unique solution of (1.1) is given by

$$\begin{aligned} \omega(t) &= \int_0^1 \chi(t, \tau) \Phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(\tau))) \right) d\tau \\ &+ \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(\tau, s) \Phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(s))) \right) ds d\tau. \end{aligned}$$

*Proof.* The problem (1.1) can be written as

$$\begin{aligned} \mathfrak{D}_{0+}^\varepsilon v(t) + f(\omega(t)) &= 0, \\ v(0) &= 0, \end{aligned}$$

by letting  $\mathfrak{D}_{0+}^\eta \omega(t) = u(t)$  and  $v = \phi(u)$  with the condition  $\mathfrak{D}_{0+}^\eta \omega(0) = 0$ . The solution of the problem is given by

$$v(t) = ct^{\varepsilon-1} - \mathcal{I}_{0+}^\varepsilon (f(\omega(t))).$$

By the condition  $v(0) = 0$ , we get

$$v(t) = -\mathcal{I}_{0+}^\varepsilon (f(\omega(t))).$$

Hence, from the Lemma 2.4, the problem

$$\begin{cases} \mathfrak{D}_{0+}^\eta \omega(t) = -\Phi^{-1}(\mathcal{I}_{0+}^\varepsilon (f(\omega(t)))), \\ \mathfrak{D}_{0+}^\varepsilon \omega(1) = I_{0+}^\alpha \omega(1), \\ \omega(0) = \omega'(0) = 0, \end{cases} \tag{2.5}$$

has a unique solution given by

$$\begin{aligned} \omega(t) &= \int_0^1 \chi(t, \tau) \Phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(\tau))) \right) d\tau \\ &+ \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(\tau, s) \Phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(s))) \right) ds d\tau. \end{aligned}$$

□

We denote the Banach space  $C^4([0, 1], \mathbb{R})$  by  $\mathcal{X}$  with the norm

$$\|\omega\| = \max_{t \in [0,1]} |\omega(t)|.$$

For  $\varrho \in (0, 1/2)$ , the cone  $\mathcal{P}_\varrho \subset \mathcal{X}$  is defined by

$$\mathcal{P}_\varrho = \left\{ \omega \in \mathcal{X} : \omega(t) \geq 0, \min_{t \in [\varrho, 1-\varrho]} \omega(t) \geq \varrho^{\eta-1} \|\omega\| \right\},$$

For any  $\omega \in \mathcal{P}_\varrho$ , define an operator  $\mathcal{A} : \mathcal{P}_\varrho \rightarrow \mathcal{X}$  by

$$\begin{aligned} (\mathcal{A}\omega)(t) &= \int_0^1 \chi(t, \tau) \Phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(\tau))) \right) d\tau \\ &+ \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(\tau, s) \Phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(s))) \right) ds d\tau. \end{aligned}$$

**Lemma 2.7.** *Assume that  $(H_1)$  and  $(H_2)$  hold. Then for each  $\varrho \in (0, 1/2)$ ,  $\mathcal{A}(\mathcal{P}_\varrho) \subset \mathcal{P}_\varrho$  and  $\mathcal{A} : \mathcal{P}_\varrho \rightarrow \mathcal{P}_\varrho$  is completely continuous.*

*Proof.* Let  $\varrho \in (0, 1/2)$ . Since  $f(\omega(\tau))$  is nonnegative for  $\tau \in [0, 1]$ ,  $\omega \in \mathcal{P}_\varrho$ . Since  $\chi(t, \tau)$  is nonnegative for all  $t, \tau \in [0, 1]$ , it follows that  $\mathcal{A}(\omega(t)) \geq 0$  for all  $t \in [0, 1]$ ,  $\omega \in \mathcal{P}_\varrho$ . Now, by Lemma 2.5, we have

$$\begin{aligned} \min_{t \in [\varrho, 1-\varrho]} (\mathcal{A}\omega)(t) &= \min_{t \in [\varrho, 1-\varrho]} \left\{ \int_0^1 \chi(t, \tau) \Phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(\tau))) \right) d\tau \right. \\ &\quad \left. + \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(\tau, x) \Phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(x))) \right) dx d\tau \right\} \\ &\geq \int_0^1 \chi(1, \tau) \Phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(\tau))) \right) d\tau \\ &\quad + \max_{t \in [0, 1]} \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(\tau, x) \Phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(x))) \right) dx d\tau \\ &\geq \varrho^{\eta-1} \max_{t \in [0, 1]} \left\{ \int_0^1 \chi(t, \tau) \Phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(\tau))) \right) d\tau \right. \\ &\quad \left. + \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(\tau, x) \Phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(x))) \right) dx d\tau \right\} \\ &\geq \varrho^{\eta-1} \max_{t \in [0, 1]} |\mathcal{A}\omega(t)|. \end{aligned}$$

Thus  $\mathcal{A}(\mathcal{P}_\varrho) \subset \mathcal{P}_\varrho$ . Therefore, the operator  $\mathcal{A}$  is completely continuous by standard methods and by the Arzela-Ascoli theorem. □

### 3. DENUMERABLY INFINITELY MANY POSITIVE SOLUTIONS

In this section, we establish the existence of denumerably many positive solutions for bvp (1.1) by applying Krasnoselskii’s fixed point theorem and Hölder’s inequality. Firstly, we state fixed point theorem

**Theorem 3.1** (Krasnoselskii’s). [13] *Let  $\mathcal{E}$  be a cone in a Banach space  $\mathcal{X}$  and  $\Omega_1, \Omega_2$  are open sets with  $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$ . Let  $\mathcal{A} : \mathcal{E} \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow \mathcal{E}$  be a completely continuous operator such that*

- (a)  $\|\mathcal{A}z\| \leq \|z\|$ ,  $z \in \mathcal{E} \cap \partial\Omega_1$ , and  $\|\mathcal{A}z\| \geq \|z\|$ ,  $z \in \mathcal{E} \cap \partial\Omega_2$ , or
- (b)  $\|\mathcal{A}z\| \geq \|z\|$ ,  $z \in \mathcal{E} \cap \partial\Omega_1$ , and  $\|\mathcal{A}z\| \leq \|z\|$ ,  $z \in \mathcal{E} \cap \partial\Omega_2$ .

*Then  $\mathcal{A}$  has a fixed point in  $\mathcal{E} \cap (\bar{\Omega}_2 \setminus \Omega_1)$ .*

**Theorem 3.2.** (Hölder’s) *Let  $h \in L^p[0, 1]$  and  $g \in L^q[0, 1]$ , where  $p > 1, q > 1$ , with  $\frac{1}{p} + \frac{1}{q} = 1$ . Then  $hg \in L^1[0, 1]$  and  $\|hg\|_1 \leq \|h\|_p \|g\|_q$ . Further, if  $h \in L^1[0, 1]$  and  $g \in L^\infty[0, 1]$ . Then  $hg \in L^1[0, 1]$  and  $\|hg\|_1 \leq \|h\|_1 \|g\|_\infty$ .*

We establish the denumerably many positive solutions for the following three possible case for  $\omega \in L^p[0, 1] : 0 < p < 1, p = 1, p = \infty$ .

**Theorem 3.3.** *Suppose  $(H_1)$  and  $(H_2)$  hold, let  $\{\varrho_k\}_{k=1}^\infty$  be a sequence with  $t_{k+1} < \varrho_k < t_k$ . Let  $\{E_k\}_{k=1}^\infty$  and  $\{O_k\}_{k=1}^\infty$  be such that*

$$E_{k+1} < \varrho_k^{\eta-1} O_k < \beta O_k < E_k, \quad k \in \mathbb{N},$$

where

$$\beta = \max \left\{ \frac{1}{\varrho_1^{\eta-1} \sigma(\varrho_1) \int_{\varrho_1}^{1-\varrho_1} \chi(1, s) \psi(s) ds}, 1 \right\}, \quad \sigma(\varrho_1) = \max_{t \in [0,1]} \left\{ 1 + \int_{\varrho_1}^{1-\varrho_1} \mathcal{H}(t, \tau) d\tau \right\}.$$

Assume that

$$\lambda = \max_{t \in [0,1]} \left\{ \int_0^1 \mathcal{H}(t, \tau) d\tau \right\} < \infty$$

and  $f$  satisfies

(A1)  $f(\omega(t)) \leq \phi(K_1 E_k)$  for all  $t \in [0, 1]$ ,  $0 \leq \omega \leq E_k$ ,

where  $K_1 < \frac{1}{(1 + \lambda) \|\chi(1, s)\|_q \|\Psi\|_p}$ ,

(A2)  $f(\omega(t)) \geq \phi(\beta O_k)$  for all  $t \in [\varrho_k, 1 - \varrho_k]$ ,  $\varrho_k^{\eta-1} O_k \leq \omega \leq O_k$ .

Then the bvp (1.1) has denumerably many positive solutions  $\{\omega_k\}_{k=1}^\infty$  such that  $O_k \leq \|\omega_k\| \leq E_k$  for  $k = 1, 2, 3 \dots$ .

*Proof.* Let  $\Omega_{1,k} = \{\omega \in \mathcal{X} : \|\omega\| < E_k\}$ ,  $\Omega_{2,k} = \{\omega \in \mathcal{X} : \|\omega\| < O_k\}$  be open subsets of  $\mathcal{X}$ . Let  $\{\varrho_k\}_{k=1}^\infty$  be given in the hypothesis and we note that  $t^* < t_{k+1} < \varrho_k < t_k < \frac{1}{2}$ , for all  $k \in \mathbb{N}$ .

For each  $k \in \mathbb{N}$ , we define the cone  $\mathcal{P}_{\varrho_k}$  by

$$\mathcal{P}_{\varrho_k} = \left\{ \omega \in \mathcal{X} : \omega(t) \geq 0, \min_{t \in [\varrho_k, 1-\varrho_k]} \omega(t) \geq \varrho_k^{\eta-1} \|\omega(t)\| \right\}.$$

Let  $\omega \in \mathcal{P}_{\varrho_k} \cap \partial\Omega_{1,k}$ . Then,  $\omega(s) \leq E_k = \|\omega\|$  for all  $s \in [0, 1]$ . By (A1),

$$\begin{aligned} \|\mathcal{A}\omega\| &= \max_{t \in [0,1]} \left\{ \int_0^1 \chi(t, \tau) \phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(\tau))) \right) d\tau \right. \\ &\quad \left. + \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(\tau, s) \phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(s))) \right) ds d\tau \right\} \\ &\leq \max_{t \in [0,1]} \left\{ K_1 E_k \int_0^1 \chi(1, \tau) \psi(\tau) d\tau \right. \\ &\quad \left. + K_1 E_k \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(1, s) \psi(s) ds d\tau \right\} \\ &\leq K_1 E_k \max_{t \in [0,1]} \left\{ 1 + \int_0^1 \mathcal{H}(t, \tau) d\tau \right\} \int_0^1 \chi(1, \tau) \psi(\tau) d\tau \\ &\leq (1 + \lambda) K_1 E_k \int_0^1 \chi(1, \tau) \psi(\tau) d\tau \\ &\leq (1 + \lambda) K_1 E_k \|\chi(1, \tau)\|_q \|\Psi\|_p \\ &\leq E_k. \end{aligned}$$

Since  $E_k = \|\omega\|$  for  $\omega \in \mathcal{P}_{\varrho_k} \cap \partial\Omega_{1,k}$ , we get

$$\|\mathcal{A}\omega\| \leq \|\omega\|. \tag{3.6}$$

Let  $t \in [\varrho_k, 1 - \varrho_k]$ . Then,

$$O_k = \|\omega\| \geq \omega(t) \geq \min_{t \in [\varrho_k, 1-\varrho_k]} \omega(t) \geq \varrho_k^{\eta-1} \|\omega\| \geq \varrho_k^{\eta-1} O_k.$$

By (A2),

$$\begin{aligned} \|\mathcal{A}\omega\| &= \max_{t \in [0,1]} \left\{ \int_0^1 \chi(t, \tau) \phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(\tau))) \right) d\tau \right. \\ &\quad \left. + \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(\tau, x) \phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(x))) \right) dx d\tau \right\} \\ &\geq \max_{t \in [0,1]} \left\{ \int_{\varrho_k}^{1-\varrho_k} \chi(t, \tau) \phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(\tau))) \right) d\tau \right. \\ &\quad \left. + \int_{\varrho_k}^{1-\varrho_k} \mathcal{H}(t, \tau) \int_{\varrho_k}^{1-\varrho_k} \chi(\tau, x) \phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(x))) \right) dx d\tau \right\} \end{aligned}$$

That is,

$$\begin{aligned} \|\mathcal{A}z\| &\geq \max_{t \in [0,1]} \left\{ \int_{\varrho_1}^{1-\varrho_1} \chi(t, \tau) \phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(\tau))) \right) d\tau \right. \\ &\quad \left. + \int_{\varrho_1}^{1-\varrho_1} \mathcal{H}(t, \tau) \int_{\varrho_1}^{1-\varrho_1} \chi(\tau, x) \phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(x))) \right) dx d\tau \right\} \\ &\geq \max_{t \in [0,1]} \left\{ \beta O_k \varrho_1^{\eta-1} \int_{\varrho_1}^{1-\varrho_1} \chi(1, \tau) \psi(\tau) \right. \\ &\quad \left. + \beta O_k \varrho_1^{\eta-1} \int_{\varrho_1}^{1-\varrho_1} \mathcal{H}(t, \tau) \int_{\varrho_1}^{1-\varrho_1} \chi(1, x) \psi(x) dx d\tau \right\} \\ &\geq \beta O_k \varrho_1^{\eta-1} \max_{t \in [0,1]} \left\{ 1 + \int_{\varrho_1}^{1-\varrho_1} \mathcal{H}(t, \tau) d\tau \right\} \int_{\varrho_1}^{1-\varrho_1} \chi(1, x) \psi(x) dx \\ &\geq \beta O_k \varrho_1^{\eta-1} \sigma(\varrho_1) \int_{\varrho_1}^{1-\varrho_1} \chi(1, x) \psi(x) dx \\ &\geq O_k = \|\omega\|. \end{aligned}$$

Thus, if  $\omega \in \mathcal{P}_\varrho \cap \partial\Omega_{2,k}$ , then

$$\|\mathcal{A}\omega\| \geq \|\omega\|. \tag{3.7}$$

It is evident that  $0 \in \Omega_{2,k} \subset \bar{\Omega}_{2,k} \subset \Omega_{1,k}$ . From (3.6),(3.7), it follows from Theorem 3.1 that the operator  $\mathcal{F}$  has a fixed point  $\omega_k \in \mathcal{P}_{\varrho_k} \cap (\bar{\Omega}_{1,k} \setminus \Omega_{2,k}) \ni O_k \leq \|\omega_k\| \leq E_k$ . The proof is completed.  $\square$

For  $p = 1$ , we have the following theorem.

**Theorem 3.4.** *Suppose  $(H_1)$  and  $(H_2)$  hold, let  $\{\varrho_k\}_{k=1}^\infty$  be a sequence with  $t_{k+1} < \varrho_k < t_k$ . Let  $\{E_k\}_{k=1}^\infty$  and  $\{O_k\}_{k=1}^\infty$  be such that*

$$E_{k+1} < \varrho_k^{\eta-1} O_k < \alpha O_k < E_k, \quad k \in \mathbb{N},$$

Assume that  $f$  satisfies

(B1)  $f(\omega(t)) \leq \phi(K_2 E_k)$  for all  $t \in [0, 1]$ ,  $0 \leq \omega \leq E_k$ , where

$$K_2 < \min \left\{ \frac{1}{(1 + \lambda) \|\chi(1, \tau)\|_\infty \|\psi\|_1}, \beta \right\}$$

and (A2). Then the bvp (1.1) has denumerably many positive solutions  $\{\omega_k\}_{k=1}^\infty$ . Furthermore,  $O_k \leq \|\omega_k\| \leq E_k$  for each  $k \in \mathbb{N}$ .

*Proof.* For a fixed  $k$ , let  $\Omega_{1,k}$  be as in the proof of Theorem 3.3 and let  $\omega \in P_{\varrho_k} \cap \partial\Omega_{2,k}$ . Again

$$\omega(s) \leq E_k = \|\omega\|,$$

for all  $s \in [0, 1]$ . By (B1) and Theorem 3.3,

$$\begin{aligned} \|\mathcal{A}\omega\| &= \max_{t \in [0,1]} \left\{ \int_0^1 \chi(t, \tau) \Phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(\tau))) \right) d\tau \right. \\ &\quad \left. + \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(\tau, x) \Phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(x))) \right) dx d\tau \right\} \\ &\leq \max_{t \in [0,1]} \left\{ K_2 E_k \int_0^1 \chi(1, \tau) \psi(\tau) d\tau \right. \\ &\quad \left. + K_2 E_k \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(1, x) \psi(x) dx d\tau \right\} \\ &\leq K_2 E_k \max_{t \in [0,1]} \left\{ 1 + \int_0^1 \chi(t, \tau) d\tau \right\} \int_0^1 \chi(1, \tau) \psi(\tau) d\tau \\ &\leq (1 + \lambda) K_2 E_k \int_0^1 \chi(1, \tau) \psi(\tau) d\tau \\ &\leq (1 + \lambda) K_2 E_k \|\chi(1, \tau)\|_\infty \|\psi\|_1 \\ &\leq E_k. \end{aligned}$$

Thus,

$$\|\mathcal{A}\omega\| \leq \|\omega\|,$$

for  $\omega \in P_{\varrho_k} \cap \partial\Omega_{1,k}$ . Now define  $\Omega_{2,k} = \{\omega \in \mathcal{X} : \|\omega\| < O_k\}$ . Let  $\omega \in P_{\varrho_k} \cap \partial\Omega_{2,k}$  and let  $s \in [\varrho_k, 1 - \varrho_k]$ . Then, the argument leading to (3.7) can be done to the present case. Hence, the theorem.  $\square$

Lastly, the case  $p = \infty$ .

**Theorem 3.5.** Assume that  $(H_1)$  and  $(H_2)$  hold. Let  $\{E_k\}_{k=1}^\infty$  and  $\{O_k\}_{k=1}^\infty$  be such that

$$E_{k+1} < \varrho_k^{\eta-1} O_k < \beta O_k < E_k, \quad k \in \mathbb{N},$$

Assume that  $f$  satisfies

(E1)  $f(\omega(t)) \leq \phi(K_3 E_k)$  for all  $t \in [0, 1]$ ,  $0 \leq \omega \leq E_k$ , where

$$M_3 < \min \left\{ \frac{1}{(1 + \lambda) \|\chi(1, \tau)\|_1 \|\psi\|_\infty}, \beta \right\}$$

and (A2). Then the bvp (1.1) has denumerably many positive solutions  $\{\omega_k\}_{k=1}^\infty$  such that  $O_k \leq \|\omega_k\| \leq E_k$  for  $k = 1, 2, 3, \dots$ .



*Proof.* By (E1),

$$\begin{aligned} \|\mathcal{A}\omega\| &= \max_{t \in [0,1]} \left\{ \int_0^1 \chi(t, \tau) \phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(\tau))) \right) d\tau \right. \\ &\quad \left. + \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(\tau, x) \phi^{-1} \left( \mathcal{I}_{0+}^\varepsilon (f(\omega(x))) \right) dx d\tau \right\} \\ &\leq \max_{t \in [0,1]} \left\{ K_2 E_k \int_0^1 \chi(1, \tau) \psi(\tau) d\tau \right. \\ &\quad \left. + K_2 E_k \int_0^1 \mathcal{H}(t, \tau) \int_0^1 \chi(1, x) \psi(x) dx d\tau \right\} \\ &\leq K_2 E_k \max_{t \in [0,1]} \left\{ 1 + \int_0^1 \chi(t, \tau) d\tau \right\} \int_0^1 \chi(1, \tau) \psi(\tau) d\tau \\ &\leq (1 + \lambda) K_2 E_k \int_0^1 \chi(1, \tau) \psi(\tau) d\tau \\ &\leq (1 + \lambda) K_2 E_k \|\chi(1, \tau)\|_1 \|\psi\|_\infty \\ &\leq E_k. \end{aligned}$$

This shows that if  $\omega \in \mathcal{P}_{\varrho_k} \cap \partial\Omega_{1,k}$ , where  $\Omega_{1,k} = \{\omega \in \mathcal{X} : \|\omega\| < E_k\}$ , Then,

$$\|\mathcal{A}\omega\| \leq \|\omega\|.$$

Define  $\Omega_{2,k} = \{\omega \in \mathcal{X} : \|\omega\| < O_k\}$  and let  $z \in \mathcal{P}_{\varrho_k} \cap \partial\Omega_{2,k}$ . Then, the argument worked in the proof of Theorem 3.3 can be applied directly to get

$$\|\mathcal{A}\omega\| \geq \|\omega\|.$$

This completes proof of the present theorem. □

#### 4. EXAMPLES

In this section, we present an example to check validity of our main results.

**Example 4.1.** Consider the following fractional order boundary value problem,

$$\left. \begin{aligned} &\mathfrak{D}_{0+}^{3/4} (\phi(\mathfrak{D}_{0+}^{5/2} \omega(t))) + f(\omega(t)) = 0, \quad t \in (0, 1), \\ &\omega(0) = \mathfrak{D}_{0+}^{5/2} \omega(0) = 0, \quad \omega'(0) = 0, \quad \mathfrak{D}_{0+}^{3/4} \omega(1) = I_{0+}^2 \omega(1), \end{aligned} \right\} \tag{4.8}$$

where

$$\phi(\omega) = \begin{cases} \frac{\omega^3}{1 + \omega^2}, & \omega \leq 0, \\ \omega^2, & \omega > 0, \end{cases}$$

$$f(\omega) = \begin{cases} K_1^2 E_1^2, & \omega > E_1, \\ \beta^2 O_k^2 + \frac{K_1^2 E_k^2 - \beta^2 O_k^2}{E_k - O_k} (\omega - O_k), & O_k \leq \omega \leq E_k, k \in \mathbb{N}, \\ \beta^2 O_k^2, & \tau_k^{\alpha-1} O_k < \omega < E_k, k \in \mathbb{N}, \\ K_1^2 E_{k+1}^2 + \frac{\beta^2 O_k^2 - K_1^2 E_{k+1}^2}{\tau_k^{\alpha-1} O_k - E_{k+1}} (\omega - E_{k+1}), & E_{k+1} < \omega \leq \tau_k^{\alpha-1} O_k, k \in \mathbb{N}, \\ 0, & \omega = 0, \end{cases}$$

$$t_k = \frac{31}{64} - \sum_{j=1}^k \frac{1}{4(j+1)^4}, \quad k = 1, 2, 3, \dots$$

Let

$$p = q = 2, \quad t_k = \frac{31}{64} - \sum_{j=1}^k \frac{1}{4(j+1)^4}, \quad \text{for } k = 1, 2, 3, \dots, \quad \varrho_k = \frac{1}{2}(t_k + t_{k+1}),$$

then

$$\varrho_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32}, \quad \varrho_k > \frac{1}{5}$$

and

$$t_{k+1} < \varrho_k < t_k.$$

Therefore,

$$\varrho_k^{\eta-1} > \frac{1}{5^{3/2}}, \quad k = 1, 2, 3, \dots$$

It is easy to see

$$t_1 = \frac{15}{32} < \frac{1}{2}, \quad \text{for } k = 1, 2, 3, \dots, \quad t_k - t_{k+1} = \frac{1}{4(k+2)^4}.$$

Since  $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$  and  $\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$ , it follows that

$$\begin{aligned} t^* &= \lim_{k \rightarrow \infty} t_k = \frac{31}{64} - \sum_{j=1}^{\infty} \frac{1}{4(j+1)^4} \\ &= \frac{47}{64} - \frac{\pi^4}{360} > \frac{1}{5}, \end{aligned}$$

and  $\|\psi\|_2 = \left[ \int_0^1 \left[ \phi^{-1} \left( \frac{s^\varepsilon}{\Gamma(\varepsilon+1)} \right) \right]^2 ds \right]^{1/2} = \left[ \int_0^1 \left[ \frac{s^\varepsilon}{\Gamma(\varepsilon+1)} \right]^4 ds \right]^{1/2} \approx 0.5919429969.$

$$\begin{aligned} \lambda &= \max_{t \in [0,1]} \{ \mathcal{H}(t, s) \} \\ &= \frac{\Gamma(\eta - \varepsilon)\Gamma(\eta + \delta)}{[\Gamma(\eta + \delta) - \Gamma(\eta - \varepsilon)]\Gamma(\eta)\Gamma(\delta)} \max_{t \in [0,1]} \left\{ \int_0^1 t^{\eta-1} (1 - \tau)^{\delta-1} d\tau \right\} \\ &\approx 0.7506812224 \cdot \max_{t \in [0,1]} \left\{ \frac{1}{2} t^{\eta-1} \right\} \\ &\approx 0.3753406112, \end{aligned}$$

$$\begin{aligned} \sigma(\varrho_1) &= 1 + \max_{t \in [0,1]} \left\{ \int_{\varrho}^{1-\varrho_1} \mathcal{H}(t, \tau) d\tau \right\} \\ &= 1 + \frac{\Gamma(\eta - \varepsilon)\Gamma(\eta + \delta)}{[\Gamma(\eta + \delta) - \Gamma(\eta - \varepsilon)]\Gamma(\eta)\Gamma(\delta)} \max_{t \in [0,1]} \left\{ \int_{\varrho}^{1-\varrho_1} t^{\eta-1}(1-\tau)^{\delta-1} d\tau \right\} \\ &= 1.024617247 \end{aligned}$$

$$\int_0^1 [\chi(1, s)]^2 ds = \frac{4}{65\pi}, \text{ it follows that } \|\chi(1, s)\|_2 = \sqrt{\frac{4}{65\pi}} \approx 0.1399582105$$

$$\begin{aligned} \varrho_1^{\eta-1} \sigma(\varrho_1) \int_{\varrho_1}^{1-\varrho_1} \chi(1, s)\psi(s) ds &= 0.3193476457 \times 1.024617247 \\ &\times \int_{\varrho_1}^{1-\varrho_1} \frac{(1-s)^{\eta-\varepsilon-1} - (1-s)^{\eta-1}}{\Gamma(\eta)} \frac{s^\varepsilon}{\Gamma(\varepsilon+1)} ds \\ &\leq 0.3193476457 \times 1.024617247 \times 0.08733438065 \\ &= 0.02857660458. \end{aligned}$$

$$\beta = \max \left\{ \frac{1}{\varrho_1^{\eta-1} \sigma(\varrho_1) \int_{\varrho_1}^{1-\varrho_1} \chi(1, s)\psi(s) ds}, 1 \right\} > 34.99366054.$$

So, taking  $\beta = 35$ . Thus,

$$K_1 < \frac{1}{(1 + \lambda)\|\chi(1, s)\|_q \|\Psi\|_p} = 8.776300099.$$

So, taking,  $K_1 = 8$  and Next, taking

$$O_k = 10^{-2(2k+1)} \text{ and } E_k = 10^{-4k},$$

then

$$\begin{aligned} E_{k+1} = 10^{-(4k+4)} &< \frac{1}{5^{3/2}} \times 10^{-(4k+2)} < \varrho_k^{\eta-1} O_k \\ &< O_k = 10^{-(4k+2)} < E_k = 10^{-4k}, \end{aligned}$$

$$\beta O_k = 35 \times 10^{-2(2k+1)} < 8 \times 10^{-4k} = K_1 E_k.$$

Also,  $f$  satisfies conditions:

$$\begin{aligned} f(\omega) &\leq \phi(K_1 E_k) = K_1^2 E_k^2 = 64 \times 10^{-8k}, \quad \omega \in \left[ 0, 10^{-4k} \right] \\ f(\omega) &\geq \phi(\beta O_k) = \beta^2 O_k^2 \\ &= 1225 \times 10^{-(8k+4)}, \quad \omega \in \left[ \frac{1}{5^{3/2}} \times 10^{-2(2k+1)}, 10^{-2(2k+1)} \right]. \end{aligned}$$

Hence, by Theorem 3.2, the bvp (4.8) has denumerably infinitely many positive solutions  $\{\omega^{[k]}\}_{k=1}^\infty$  with  $10^{-6} \leq \|\omega^{[1]}\| \leq 10^{-4}$ ,  $10^{-10} \leq \|\omega^{[2]}\| \leq 10^{-8}$ ,  $10^{-14} \leq \|\omega^{[3]}\| \leq 10^{-12}$  and so on.

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