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New integral inequalities for geometrically convex functions via conformable fractional integral operators

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ABSTRACT. We establish several basic inequalities versions of the Hermite-Hadamard type inequalities for $GA-$ and $GG-$convexity for conformable fractional integrals. Several special cases are also discussed, which can be deduced from our main result.

1. INTRODUCTION

We will start with some well-known definitions and concepts as followings.

Anderson *et. al.* mentioned mean function in [6] as following:

Definition 1.1. A function $M:(0,\infty)\times(0,\infty)\to(0,\infty)$ is called a Mean function if

(1) $M(x,y)=M(y,x)$,
(2) $M(x,x)=x$,
(3) $x<M(x,y)<y$, whenever $x<y$,
(4) $M(ax,ay)=aM(x,y)$ for all $a>0$.

Based on the definition of mean function, let us recall special means (See [6])

1. Arithmetic Mean: $M(x,y)=A(x,y)=\frac{x+y}{2}$.
2. Geometric Mean: $M(x,y)=G(x,y)=\sqrt{xy}$.
3. Harmonic Mean: $M(x,y)=H(x,y)=1/A\left(\frac{1}{x},\frac{1}{y}\right)$.
4. Logarithmic Mean: $M(x,y)=L(x,y)=(x-y)/(\log x-\log y)$ for $x\neq y$ nd $L(x,x)=x$.
5. Identric Mean: $M(x,y)=I(x,y)=(1/e)\left(x^x/y^y\right)^{1/(x-y)}$ for $x\neq y$ nd $I(x,x)=x$.

In [6], Anderson *et. al.* also gave a definition that include several different classes of convex functions as the following:

Definition 1.2. Let $f:I\to(0,\infty)$ be continuous, where I is subinterval of $(0,\infty)$. Let M and N be any two Mean functions. We say f is MN-convex (concave) if

$$f(M(x,y))\leq(\geq)N(f(x),f(y))$$

for all $x,y\in I$.

In [20], Niculescu mentioned the following considerable definitions:

The $AG-$convex functions (usually known as $\log-$convex functions) are those functions $f:I\to(0,\infty)$ for which

$$x,y\in I \text{ and } \lambda\in[0,1]\Longrightarrow f(\lambda x+(1-\lambda)y)\leq f(x)^{1-\lambda}f(y)^{\lambda}, \tag{1.1}$$

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i.e., for which $\log f$ is convex.

The $GG-$convex functions (called in what follows multiplicatively convex functions) are those functions $f : I \to J$ (acting on subintervals of $(0, \infty)$) such that

$$x, y \in I \text{ and } \lambda \in [0,1] \Longrightarrow f\left(x^{1-\lambda}y^{\lambda}\right) \leq f(x)^{1-\lambda} f(y)^{\lambda}. \tag{1.2}$$

The class of all $GA-$convex functions is constituted by all functions $f : I \to \mathbb{R}$ (defined on subintervals of $(0, \infty)$) for which

$$x, y \in I \text{ and } \lambda \in [0,1] \Longrightarrow f\left(x^{1-\lambda}y^{\lambda}\right) \leq f(x)^{1-\lambda} + f(y)^{\lambda}. \tag{1.3}$$

Besides, recall that the condition of $GA-$convexity is $x^2 f'' + x f' \geq 0$ which implies all twice differentiable nondecreasing convex functions are also $GA-$convex.

Let us recall the definition of Riemann-Liouville fractional integrals as:

Definition 1.3. (See [18]) Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$J_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x) dx, \qquad t > a$$

and

$$J_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (x-t)^{\alpha-1} f(x) dx, \qquad t < b$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$. Here $J_{a+}^{0} f(t) = J_{b-}^{0} f(t) = f(t)$

In the case of $\alpha = 1$, the fractional integral reduces to classical integral.

We will mention the Beta function (See [38]):

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1} (1-t)^{b-1} dt, \qquad a, b > 0,$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$ is Gamma function.

Incomplete Beta function is defined as:

$$B_x(a,b) = \int_0^x t^{a-1} (1-t)^{b-1} dt, \qquad a, b > 0.$$

In spite of its valuable contributions to mathematical analysis, the Riemann-Liouvile Fractional integrals have deficiencies. For example the solution of the differential equation that is given as;

$$y^{\left(\frac{1}{2}\right)} + y = x^{\left(\frac{1}{2}\right)} + \frac{2}{\Gamma(2.5)} x^{\left(\frac{3}{2}\right)}, \quad y(0) = 0$$

where $y^{\left(\frac{1}{2}\right)}$ is the fractional derivative of y of order $\frac{1}{2}$.

The solution of the above differential equation have caused to imagine on a new and simple represention of the definition of fractional derivative. In [17] , Khalil et al. gave a new definition that is called "conformable fractional derivative". They not only proved further properties of this definitons but also gave the differences with the other fractional derivatives. Besides, another considerable study have presented by Abdeljawad to discuss the basic concepts of fractional calculus. In [1], Abdeljawad gave the following definitions of Right-Left conformable fractional integrals:

Definition 1.4. Let $\alpha \in (n, n+1]$, $n = 0, 1, 2, ...$ and set $\beta = \alpha - n$. Then the left conformable fractional integral of any order $\alpha > 0$ is defined by

$$(I_{\alpha}^{a} f)(t) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} f(x) dx$$

Definition 1.5. Analogously, the right conformable fractional integral of any order $\alpha > 0$ is defined by

$$({}^{b}I_{\alpha}f)(t) = \frac{1}{n!}\int_{t}^{b}(x-t)^{n}(b-x)^{\beta-1}f(x)dx.$$

Notice that if $\alpha = n+1$ then $\beta = \alpha - n = n+1-n = 1$, hence $(I^{a}_{n+1}f)(t) = (J^{n+1}_{a+}f)(t)$ and $({}^{b}I_{n+1}f)(t) = \left(J^{n+1}_{b-}f\right)(t)$.

In [17] , Khalil et al. gave the following definition:

Definition 1.6. (Conformable fractional integral) Let $\alpha \in (0,1]$, $0 \leq \kappa_1 < \kappa_2$. A function $h : [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}$ is α-fractional integrable on $[\kappa_1, \kappa_2]$ if the integral

$$\int\limits_{\kappa_1}^{\kappa_2} h(x)d_{\alpha}x = \int\limits_{\kappa_1}^{\kappa_2} h(x)x^{\alpha-1}dx$$

exists and is finite.

All $\alpha-$fractional integrable functions on $[\kappa_1, \kappa_2]$ is indicated by $L_{\alpha}([\kappa_1, \kappa_2])$.

Remark 1.1. (See [17])

$$I^{\kappa_1}_{\alpha}(h_1)(s) = I^{\kappa_1}_{1}(s^{\alpha-1}h_1) = \int\limits_{\kappa_1}^{s}\frac{h_1(x)}{x^{1-\alpha}}dx$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

In [1] and [17], authors have pointed that the Riemann-Liouville derivatives are not valid for product of two functions. In this case, the inequalities that have been proved by Riemann-Liouville integrals are not valid. The results which are obtained by using the conformable fractional integrals have a wide range of validity. (Let us consider the function f defined as $f : \mathbb{R}^{+} \to \mathbb{R}$, $f = x^2e^x$ which is convex.) For related papers see [36]-[4]. Recent results, generalizations, refinements and improvements on different kinds of convexity and inequalities involve fractional integrals can be found in [2]-[6], [8]-[16], [19]-[39], [5], [23]-[35].

In this paper, a new integral identity has been established and based on this new identity some new integral inequalities have been proved by using conformable fractional integrals for functions whose derivatives of absolute values are GA- and GG-convex functions. Several special cases of our results have also been considered.

2. Main results

In this section, we derive some new Hermite Hadamard type inequalities for conformable fractional integrals GA- and GG-convex functions. We denote $I = [a,b]$, unless otherwise specified.

In order to establish our main results, we need a lemma which we present in this section.

Lemma 2.1. *Let $f : I \subset \mathbb{R}_{+} = (0,\infty) \to \mathbb{R}$ be an α-fractional differentiable function on the interior I° of I, where $a,b \in I$ with $a < b$ and $f' \in L_{\alpha}[a,b]$. Then if $D_{\alpha} \in L_{\alpha}[a,b]$, the following* *equality holds:*

$$b^{\alpha}f(b)-a^{\alpha}f(a)-\alpha\int_{a}^{b}f(x)d_{\alpha}x=\frac{\ln b-\ln a}{2} \tag{2.4}$$
$$\int_{0}^{1}\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)^{2\alpha}D_{\alpha}(f)\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)dt$$
$$+\int_{0}^{1}\left(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\right)^{2\alpha}D_{\alpha}(f)\left(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\right)dt.$$

Proof. Assume that

$$\begin{aligned}
I &= \int_{0}^{1}\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)^{2\alpha}D_{\alpha}(f)\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)dt \\
&\quad +\int_{0}^{1}\left(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\right)^{2\alpha}D_{\alpha}(f)\left(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\right)dt \\
&= \int_{0}^{1}\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)^{\alpha+1}f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)dt \\
&\quad +\int_{0}^{1}\left(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\right)^{\alpha+1}f'\left(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\right)dt \\
&= I_1+I_2.
\end{aligned}$$

Now consider

$$I_1=\int_{0}^{1}\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)^{\alpha+1}f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)dt.$$

By the change of the variable $x=b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}$ and integration by parts, we have

$$\begin{aligned}
I_1 &= \frac{2}{\ln b-\ln a}\int_{\sqrt{ab}}^{b}x^{\alpha}f'(x)dx \qquad (2.5)\\
&= \frac{2[b^{\alpha}f(b)-(ab)^{\frac{\alpha}{2}}f(\sqrt{ab})]}{\ln b-\ln a}-\frac{2\alpha}{\ln b-\ln a}\int_{\sqrt{ab}}^{b}x^{\alpha-1}f(x)dx \\
&= \frac{2[b^{\alpha}f(b)-(ab)^{\frac{\alpha}{2}}f(\sqrt{ab})]}{\ln b-\ln a}-\frac{2\alpha}{\ln b-\ln a}\int_{\sqrt{ab}}^{b}f(x)d_{\alpha}x.
\end{aligned}$$

Similarly, one can get

$$I_2 = \frac{2}{\ln b - \ln a}\int_a^{\sqrt{ab}} x^\alpha f'(x)dx \tag{2.6}$$
$$= \frac{2[(ab)^{\frac{\alpha}{2}} f(\sqrt{ab}) - a^\alpha f(a)]}{\ln b - \ln a} - \frac{2\alpha}{\ln b - \ln a}\int_{\sqrt{ab}}^{b} f(x)d_\alpha x.$$

By adding 2.5 and 2.6, we obtain

$$I = \frac{2}{\ln b - \ln a}\big[b^\alpha f(b) - a^\alpha f(a) - \alpha \int_a^b f(x)d_\alpha x\big].$$

Multiplying by $\frac{\ln b - \ln a}{2}$ the resulting equality, we get the desired inequality. □

Remark 2.2. If we set $\alpha = 1$, then Lemma 2.1 reduces to Lemma 2.1 of [19].

Our first result is given in the following theorem.

Theorem 2.1. *Let $f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be an α-fractional differentiable function on the interior I° of I, where $a, b \in I$ with $a < b$ and $f' \in L_\alpha[a,b]$. If $|f'|^q$ is GA-convex on I for $q \geq 1$, then we have following inequality for conformable fractional integrals;*

$$\left| b^\alpha f(b) - a^\alpha f(a) - \alpha\int_a^b f(x)d_\alpha x \right| \tag{2.7}$$
$$\leq \frac{(\ln b - \ln a)^{1-\frac{1}{q}}}{2(\alpha+1)^{\frac{1}{q}}}\big[L(a^{\frac{\alpha+1}{2}}, b^{\frac{\alpha+1}{2}})\big]^{1-\frac{1}{q}}\Bigg[b^{\frac{\alpha+1}{2}}\Bigg\{|f'(a)|^q\big[L(a^{\frac{\alpha+1}{2}}, b^{\frac{\alpha+1}{2}}) - a^{\frac{\alpha+1}{2}}\big]$$
$$+|f'(b)|^q\big[(2b^{\frac{\alpha+1}{2}} - a^{\frac{\alpha+1}{2}}) - L(a^{\frac{\alpha+1}{2}}, b^{\frac{\alpha+1}{2}})\big]\Bigg\}^{\frac{1}{q}}$$
$$+a^{\frac{\alpha+1}{2}}\Bigg\{|f'(a)|^q\big[(b^{\frac{\alpha+1}{2}} - 2a^{\frac{\alpha+1}{2}}) + L(a^{\frac{\alpha+1}{2}}, b^{\frac{\alpha+1}{2}})\big]$$
$$+|f'(a)|^q\big[b^{\frac{\alpha+1}{2}} - L(a^{\frac{\alpha+1}{2}}, b^{\frac{\alpha+1}{2}})\big]\Bigg\}^{\frac{1}{q}}\Bigg].$$

Proof. From Lemma 2.1, using the property of modulus and GA-convexity of $|f'|^q$, we can write

$$\left| b^\alpha f(b) - a^\alpha f(a) - \alpha\int_a^b f(x)d_\alpha x \right| \tag{2.8}$$
$$= \frac{\ln b - \ln a}{2}\Bigg[\int_0^1 \big(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\big)^{2\alpha} D_\alpha(f)\big(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\big)dt$$
$$+\int_0^1 \big(b^{\frac{1-t}{2}} a^{\frac{1+t}{2}}\big)^{2\alpha} D_\alpha(f)\big(b^{\frac{1-t}{2}} a^{\frac{1+t}{2}}\big)dt\Bigg]$$

$$= \frac{(ab)^{\frac{\alpha+1}{2}}(\ln b-\ln a)}{2}\Bigg[\Big(\int_0^1 (\frac{b}{a})^{\frac{(\alpha+1)t}{2}}dt\Big)^{1-\frac{1}{q}}$$
$$\Big(\int_0^1 (\frac{b}{a})^{\frac{(\alpha+1)t}{2}}\big|f'(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})\big|^q dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1 (\frac{a}{b})^{\frac{(\alpha+1)t}{2}}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 (\frac{a}{b})^{\frac{(\alpha+1)t}{2}}\big|f'(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}})\big|^q dt\Big)^{\frac{1}{q}}\Bigg].$$

By computing the above integrals, we get

$$\begin{aligned} I_1 &= \int_0^1 (\frac{b}{a})^{\frac{(\alpha+1)t}{2}}\big|f'(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})\big|^q dt \\ &\leq \int_0^1 (\frac{b}{a})^{\frac{(\alpha+1)t}{2}}\Big[|f'(a)|^q(\frac{1-t}{2})+|f'(b)|^q(\frac{1+t}{2})\Big]dt \\ &= |f'(a)|^q\frac{L(a^{\frac{(\alpha+1)}{2}},b^{\frac{(\alpha+1)}{2}})-a^{\frac{(\alpha+1)}{2}}}{a^{\frac{(\alpha+1)}{2}}(\alpha+1)(\ln b-\ln a)} \\ &\quad +|f'(b)|^q\frac{(2b^{\frac{(\alpha+1)}{2}}-a^{\frac{(\alpha+1)}{2}})-L(a^{\frac{(\alpha+1)}{2}},b^{\frac{(\alpha+1)}{2}})}{a^{\frac{(\alpha+1)}{2}}(\alpha+1)(\ln b-\ln a)}. \end{aligned} \tag{2.9}$$

Similarly, one can compute I_2

$$\begin{aligned} &\int_0^1 (\frac{a}{b})^{\frac{(\alpha+1)t}{2}}\big|f'(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}})\big|^q dt \\ \leq\ & |f'(a)|^q\frac{(b^{\frac{\alpha+1}{2}}-2a^{\frac{(\alpha+1)}{2}})+L(a^{\frac{(\alpha+1)}{2}},b^{\frac{(\alpha+1)}{2}})}{b^{\frac{(\alpha+1)}{2}}(\alpha+1)(\ln b-\ln a)} \\ & +|f'(b)|^q\frac{b^{\frac{\alpha+1}{2}}-(L(a^{\frac{(\alpha+1)}{2}},b^{\frac{(\alpha+1)}{2}})}{b^{\frac{(\alpha+1)}{2}}(\alpha+1)(\ln b-\ln a)}. \end{aligned} \tag{2.10}$$

Also, it is easy to see that

$$\Big(\int_0^1 (\frac{b}{a})^{\frac{(\alpha+1)t}{2}}dt\Big)^{1-\frac{1}{q}} = \Big(\frac{L(a^{\frac{\alpha+1}{2}},b^{\frac{\alpha+1}{2}})}{.}a^{\frac{\alpha+1}{2}}\Big)^{\frac{1}{q}}. \tag{2.11}$$

Combining 2.8, 2.9, 2.10 and 2.11, we obtain the required result. □

Remark 2.3. If we choose $\alpha = 1$, then Theorem 2.1 reduces to Theorem 2.2 of [19].

Theorem 2.2. *Let $f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be an α-fractional differentiable function on the interior I° of I, where $a, b \in I$ with $a < b$ and $f' \in L_\alpha[a,b]$. If $|f'|$ is GA-convex on I for $q > 1$*

and $p^{-1}+q^{-1}=1$*, then the following inequality holds for conformable fractional integrals;*

$$\left|b^{\alpha}f(b)-a^{\alpha}f(a)-\alpha\int_a^b f(x)d_{\alpha}x\right| \tag{2.12}$$

$$\leq \frac{\ln b-\ln a}{2^{1+\frac{1}{q}}}\left[L(a^{\frac{q(\alpha+1)t}{2(q-1)}},b^{\frac{q(\alpha+1)t}{2(q-1)}})\right]^{1-\frac{1}{q}}\left[b^{\frac{\alpha+1}{2}}\left\{A\Big(\left|f'(a)\right|^q,3\left|f'(b)\right|^q\Big)\right\}^{\frac{1}{q}}\right.$$

$$\left.+a^{\frac{\alpha+1}{2}}\left\{A\Big(3\left|f'(a)\right|^q,\left|f'(b)\right|^q\Big)\right\}^{\frac{1}{q}}\right].$$

Proof. From Lemma 2.1, using the property of modulus and GA-convexity of $|f'|^q$, we can write

$$\left|b^{\alpha}f(b)-a^{\alpha}f(a)-\alpha\int_a^b f(x)d_{\alpha}x\right| \tag{2.13}$$

$$= \frac{\ln b-\ln a}{2}\left[\int_0^1\big(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\big)^{2\alpha}D_{\alpha}(f)\big(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\big)dt\right.$$

$$\left.+\int_0^1\big(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\big)^{2\alpha}D_{\alpha}(f)\big(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\big)dt\right]$$

$$= \frac{(ab)^{\frac{\alpha+1}{2}}(\ln b-\ln a)}{2}\left[\left(\int_0^1\left(\frac{b}{a}\right)^{\frac{q(\alpha+1)t}{2(q-1)}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f'(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left(\frac{a}{b}\right)^{\frac{q(\alpha+1)t}{2(q-1)}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f'(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}})\right|^q dt\right)^{\frac{1}{q}}\right].$$

Now consider,

$$\begin{aligned} I_1 &= \int_0^1\left|f'(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})\right|^q dt \\ &\leq \left|f'(b)\right|^q\int_0^1\frac{1+t}{2}dt+\left|f'(a)\right|^q\int_0^1\frac{1-t}{2}dt \\ &= \frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}. \end{aligned} \tag{2.14}$$

Similarly I_2 becomes

$$\int_0^1\left|f'(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})\right|^q dt\leq\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}. \tag{2.15}$$

By a simple computation, one can see that

$$\left[\int_0^1\left(\frac{b}{a}\right)^{\frac{q(\alpha+1)t}{2(q-1)}}dt\right]^{1-\frac{1}{q}}=\left[\frac{L(a^{\frac{q(\alpha+1)t}{2(q-1)}},b^{\frac{q(\alpha+1)t}{2(q-1)}})}{a^{\frac{q(\alpha+1)t}{2(q-1)}}}\right]^{1-\frac{1}{q}}. \tag{2.16}$$

Combining 2.13, 2.14, 2.15 and 2.16, we have the required result. □

Remark 2.4. If we choose $\alpha = 1$, then Theorem 2.2 reduces to Theorem 2.4 of [19].

Theorem 2.3. *Let $f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be an $\alpha-$fractional differentiable function on the interior I° of I, where $a, b \in I$ with $a < b$ and $f' \in L_\alpha[a, b]$. If $|f'|^q$ is GA-convex on I for $q \geq 1$, then the following inequality holds for conformable fractional integrals;*

$$\begin{aligned} &\left|b^\alpha f(b) - a^\alpha f(a) - \alpha \int_a^b f(x) d_\alpha x\right| \qquad (2.17)\\ \leq\ & \frac{(\ln b - \ln a)^{1-\frac{1}{q}}}{2(q(\alpha+1))^{\frac{1}{q}}}\Big\{ b^{\frac{\alpha+1}{2}}\Big[|f'(a)|^q\big[L(a^{\frac{q(\alpha+1)}{2}}), b^{\frac{q(\alpha+1)}{2}}) - a^{\frac{q(\alpha+1)}{2}})\big]\\ &+|f'(b)|^q\big[(2b^{\frac{q(\alpha+1)}{2}} - a^{\frac{q(\alpha+1)}{2}}) - L(a^{\frac{q(\alpha+1)}{2}}), b^{\frac{q(\alpha+1)}{2}}))\big]\Big]^{\frac{1}{q}}\\ &+a^{\frac{\alpha+1}{2}}\Big[|f'(a)|^q\big[(b^{\frac{q(\alpha+1)}{2}} - 2a^{\frac{q(\alpha+1)}{2}}) + L(a^{\frac{q(\alpha+1)}{2}}, b^{\frac{q(\alpha+1)}{2}})\big] +\\ &|f'(b)|^q\big[b^{\frac{q(\alpha+1)}{2}} - L(a^{\frac{q(\alpha+1)}{2}}, b^{\frac{q(\alpha+1)}{2}})\big]\Big]^{\frac{1}{q}}\Big\}. \end{aligned}$$

Proof. From Lemma 2.1, using the property of modulus and GA-convexity of $|f'|^q$, we can write

$$\begin{aligned} &\left|b^\alpha f(b) - a^\alpha f(a) - \alpha \int_a^b f(x) d_\alpha x\right| \qquad (2.18)\\ =\ & \frac{\ln b - \ln a}{2}\Big[\int_0^1 \big(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\big)^{2\alpha} D_\alpha(f)\big(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\big)dt\\ &+\int_0^1 \big(b^{\frac{1-t}{2}} a^{\frac{1+t}{2}}\big)^{2\alpha} D_\alpha(f)\big(b^{\frac{1-t}{2}} a^{\frac{1+t}{2}}\big)dt\Big]\\ =\ & \frac{(ab)^{\frac{\alpha+1}{2}}(\ln b - \ln a)}{2}\Big[\Big(\int_0^1 1dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 \big(\frac{b}{a}\big)^{\frac{q(\alpha+1)t}{2}}\big|f'(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}})\big|^q dt\Big)^{\frac{1}{q}}\\ &+\Big(\int_0^1 1dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 \big(\frac{a}{b}\big)^{\frac{q(\alpha+1)t}{2}}\big|f'(a^{\frac{1+t}{2}} b^{\frac{1-t}{2}})\big|^q dt\Big)^{\frac{1}{q}}\Big]. \qquad (2.19) \end{aligned}$$

By computing the above integrals, we have

$$\begin{aligned} I_1 &= \int_0^1 \big(\frac{b}{a}\big)^{\frac{q(\alpha+1)t}{2}}\big|f'(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}})\big|^q dt \qquad (2.20)\\ &\leq \int_0^1 \big(\frac{b}{a}\big)^{\frac{q(\alpha+1)t}{2}}\big[|f'(a)|^q\frac{1-t}{2} + |f'(b)|^q\frac{1+t}{2}\big]dt \end{aligned}$$

$$= |f'(a)|^q\Big[\frac{L(a^{\frac{q(\alpha+1)}{2}},b^{\frac{q(\alpha+1)}{2}})-a^{\frac{q(\alpha+1)}{2}}}{q(\alpha+1)(\ln b-\ln a)a^{\frac{q(\alpha+1)}{2}}}\Big]$$
$$+|f'(b)|^q\Big[\frac{(2b^{\frac{q(\alpha+1)}{2}}-a^{\frac{q(\alpha+1)}{2}})-L(a^{\frac{q(\alpha+1)}{2}},b^{\frac{q(\alpha+1)}{2}})}{q(\alpha+1)(\ln b-\ln a)a^{\frac{q(\alpha+1)}{2}}}\Big].$$

Similarly, one can see

$$\int_0^1 \left(\frac{a}{b}\right)^{\frac{q(\alpha+1)t}{2}}\left|f'(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}})\right|^q dt \tag{2.21}$$
$$\leq |f'(a)|^q\left[\frac{b^{\frac{q(\alpha+1)}{2}}-L(a^{\frac{q(\alpha+1)}{2}},b^{\frac{q(\alpha+1)}{2}})}{q(\alpha+1)(\ln b-\ln a)b^{\frac{q(\alpha+1)}{2}}}\right]$$
$$+|f'(b)|^q\left[\frac{(b^{\frac{q(\alpha+1)}{2}}-2a^{\frac{q(\alpha+1)}{2}})+L(a^{\frac{q(\alpha+1)}{2}},b^{\frac{q(\alpha+1)}{2}})}{q(\alpha+1)(\ln b-\ln a)b^{\frac{q(\alpha+1)}{2}}}\right].$$

Combining 2.18, 2.20 and 2.21, we obtain the required result. □

Remark 2.5. If we choose $\alpha = 1$, then Theorem 2.3 reduces to the Theorem 2.5 of [19].

Theorem 2.4. *Let $f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be an α-fractional differentiable function on the interior I° of I, where $a, b \in I$ with $a < b$ and $f' \in L_\alpha[a,b]$. If $|f'|^q$ is GA-convex on I for $q \geq 1$, then the following inequality holds for conformable fractional integrals;*

$$\left|b^\alpha f(b) - a^\alpha f(a) - \alpha\int_a^b f(x)d_\alpha x\right| \tag{2.22}$$
$$\leq \frac{(\ln b-\ln a)^{1-\frac{1}{q}}}{2\big(p(\alpha+1)\big)^{\frac{1}{q}}}\left[L\left(a^{\frac{(q-p)(\alpha+1)}{2(q-1)}},b^{\frac{(q-p)(\alpha+1)}{2(q-1)}}\right)\right]^{1-\frac{1}{q}}$$
$$\Bigg\{\Bigg[b^{\frac{\alpha+1}{2}}\Big\{\left|f'(a)\right|^q\big[L(a^{\frac{p(\alpha+1)}{2}},b^{\frac{p(\alpha+1)}{2}})-a^{\frac{p(\alpha+1)}{2}}\big]$$
$$+\left|f'(b)\right|^q\big[(2b^{\frac{p(\alpha+1)}{2}}-a^{\frac{p(\alpha+1)}{2}})-L(a^{\frac{p(\alpha+1)}{2}},b^{\frac{p(\alpha+1)}{2}})\big]\Big\}\Bigg]^{\frac{1}{q}}$$
$$+\Bigg[a^{\frac{\alpha+1}{2}}\Big\{\left|f'(a)\right|^q\big[(b^{\frac{p(\alpha+1)}{2}}-2a^{\frac{p(\alpha+1)}{2}})+L(a^{\frac{p(\alpha+1)}{2}},b^{\frac{p(\alpha+1)}{2}})\big]$$
$$+\left|f'(b)\right|^q\big[b^{\frac{p(\alpha+1)}{2}})-L(a^{\frac{p(\alpha+1)}{2}},b^{\frac{p(\alpha+1)}{2}})\big]\Big\}\Bigg]^{\frac{1}{q}}\Bigg\}.$$

Proof. From Lemma 2.1, using the property of modulus and GA-convexity of $|f'|$, we can write

$$\left|b^\alpha f(b) - a^\alpha f(a) - \alpha\int_a^b f(x)d_\alpha x\right| \tag{2.23}$$
$$= \frac{\ln b-\ln a}{2}\Big[\int_0^1 \big(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\big)^{2\alpha}D_\alpha(f)\big(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\big)dt$$

$$+\int_0^1 \left(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\right)^{2\alpha} D_\alpha(f)\left(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\right)dt\Bigg]$$

$$= \frac{(ab)^{\frac{\alpha+1}{2}}(\ln b-\ln a)}{2}\Bigg[\bigg(\int_0^1 \Big(\frac{b}{a}\Big)^{\frac{(\alpha+1)(q-p)t}{2(q-1)}}dt\bigg)^{1-\frac{1}{q}}\bigg(\int_0^1 \Big(\frac{b}{a}\Big)^{\frac{p(\alpha+1)t}{2}}\left|f'(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})\right|^q dt\bigg)^{\frac{1}{q}}$$

$$+\bigg(\int_0^1 \Big(\frac{a}{b}\Big)^{\frac{(\alpha+1)(q-p)t}{2(q-1)}}dt\bigg)^{1-\frac{1}{q}}\bigg(\int_0^1 \Big(\frac{a}{b}\Big)^{\frac{p(\alpha+1)t}{2}}\left|f'(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}})\right|^q dt\bigg)^{\frac{1}{q}}\Bigg].$$

By making use of necessary computations, we have

$$\begin{aligned} I_1 &= \int_0^1 \Big(\frac{b}{a}\Big)^{\frac{p(\alpha+1)t}{2}}\left|f'(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})\right|^q dt \\ &\le \int_0^1 \Big(\frac{b}{a}\Big)^{\frac{p(\alpha+1)t}{2}}\left[\frac{1-t}{2}|f'(a)|^q+\frac{1+t}{2}|f'(b)|^q\right]dt \\ &= |f'(a)|^q\left[\frac{L(a^{\frac{p(\alpha+1)}{2}},b^{\frac{p(\alpha+1)}{2}})-a^{\frac{p(\alpha+1)}{2}}}{a^{\frac{p(\alpha+1)}{2}}p(\alpha+1)(\ln b-\ln a)}\right] \\ &\quad +|f'(b)|^q\left[\frac{(2b^{\frac{p(\alpha+1)}{2}}-a^{\frac{p(\alpha+1)}{2}}-L(a^{\frac{p(\alpha+1)}{2}},b^{\frac{p(\alpha+1)}{2}}))}{a^{\frac{p(\alpha+1)}{2}}p(\alpha+1)(\ln b-\ln a)}\right]. \end{aligned} \tag{2.24}$$

Similarly I_2 becomes

$$\int_0^1 \Big(\frac{a}{b}\Big)^{\frac{p(\alpha+1)t}{2}}\left|f'(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}})\right|^q dt \le |f'(a)|^q\left[\frac{b^{\frac{p(\alpha+1)}{2}}-L((a^{\frac{p(\alpha+1)}{2}},b^{\frac{p(\alpha+1)}{2}})}{b^{\frac{p(\alpha+1)}{2}}p(\alpha+1)(\ln b-\ln a)}\right] \tag{2.25}$$

$$+|f'(b)|^q\left[\frac{b^{\frac{p(\alpha+1)}{2}}-2a^{\frac{p(\alpha+1)}{2}}+L(a^{\frac{p(\alpha+1)}{2}},b^{\frac{p(\alpha+1)}{2}})}{b^{\frac{p(\alpha+1)}{2}}p(\alpha+1)(\ln b-\ln a)}\right].$$

Also, it is easy to check that

$$\left[\int_0^1 \Big(\frac{b}{a}\Big)^{\frac{(q-p)(\alpha+1)t}{2(q-1)}}dt\right]^{1-\frac{1}{q}} = \left[\frac{L(a^{\frac{(q-p)(\alpha+1)}{2(q-1)}},b^{\frac{(q-p)(\alpha+1)}{2(q-1)}})}{a^{\frac{(q-p)(\alpha+1)}{2(q-1)}}}\right]^{1-\frac{1}{q}}. \tag{2.26}$$

Combining 2.23, 2.24, 2.25 and 2.26, we obtain the desired result. □

In this section,we will proceed a similar argument for $GG-$convex functions to obtain some new inequalities.

Theorem 2.5. *Let $f: I\subset\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be an α-fractional differentiable function on the interior I° of I, where $a,b\in I$ with $a<b$ and $f'\in L_\alpha[a,b]$. If $|f'|$ is GG-convex on I, then the following inequality holds for conformable fractional integrals;*

$$\left|b^\alpha f(b)-a^\alpha f(a)-\alpha\int_a^b f(x)d_\alpha x\right| \tag{2.27}$$

$$\le \frac{\ln b-\ln a}{2}\left[L\bigg(\left(a^{\alpha+1}|f'(a)|\right)^{\frac{1}{2}},\left(b^{\alpha+1}|f'(b)|\right)^{\frac{1}{2}}\bigg)\right]\left[\left(a^{\alpha+1}|f'(a)|\right)^{\frac{1}{2}}+\left(b^{\alpha+1}|f'(b)|\right)^{\frac{1}{2}}\right].$$

Proof. From Lemma 2.1, using the property of the modulus and GG-convexity of $|f'|$, we have

$$\left|b^{\alpha}f(b)-a^{\alpha}f(a)-\alpha\int_a^b f(x)d_{\alpha}x\right| \tag{2.28}$$

$$\begin{aligned}
&= \frac{\ln b-\ln a}{2}\Bigg[\int_0^1 \big(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\big)^{2\alpha}D_{\alpha}(f)\big(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\big)dt \\
&\quad+\int_0^1 \big(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\big)^{2\alpha}D_{\alpha}(f)\big(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\big)dt\Bigg] \\
&\le \frac{\ln b-\ln a}{2}\Bigg[\int_0^1 \big(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\big)^{\alpha+1}\big|f'\big(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\big)\big|dt \\
&\quad+\int_0^1 \big(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\big)^{\alpha+1}\big|f'\big(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\big)\big|dt\Bigg] \\
&\le \frac{\ln b-\ln a}{2}\Bigg[\int_0^1 \big(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\big)^{\alpha+1}\big|f'(b)\big|^{\frac{1+t}{2}}\big|f'(a)\big|^{\frac{1-t}{2}}\big)\big|dt \\
&\quad+\int_0^1 \big(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\big)^{\alpha+1}\big|f'(b)\big|^{\frac{1-t}{2}}\big|f'(a)\big|^{\frac{1+t}{2}}\big)\big|dt\Bigg] \\
&\le \frac{(ab)^{\frac{\alpha+1}{2}}(\ln b-\ln a)}{2}\Bigg[\int_0^1 |f'(a)f'(b)|^{\frac{1}{2}}\left(\frac{b^{\alpha+1}|f'(b)|}{a^{\alpha+1}|f'(a)|}\right)^{\frac{t}{2}}dt \\
&\quad+\int_0^1 |f'(a)f'(b)|^{\frac{1}{2}}\left(\frac{a^{\alpha+1}|f'(a)|}{b^{\alpha+1}|f'(b)|}\right)^{\frac{t}{2}}dt\Bigg].
\end{aligned}$$

If we evaluate the above integrals, we get the desired results. □

Theorem 2.6. *Let $f: I\subset\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be an α-fractional differentiable function on the interior I° of I, where $a,b\in I$ with $a<b$ and $f'\in L_{\alpha}[a,b]$. If $|f'|^q$ is GG-convex on I for $q>1$ and $q>p>0$, then the following inequality holds for conformable fractional integrals;*

$$\left|b^{\alpha}f(b)-a^{\alpha}f(a)-\alpha\int_a^b f(x)d_{\alpha}x\right| \le \frac{\ln b-\ln a}{2}\left[L\big(a^{\frac{p(\alpha+1)}{2}},b^{\frac{p(\alpha+1)}{2}}\big)\right]^{\frac{1}{p}} \tag{2.29}$$

$$\left[L\big(|f'(a)|^{\frac{1}{2}},|f'(b)|^{\frac{1}{2}}\big)\right]^{\frac{1}{q}}\left[\big(a^{\frac{\alpha+1}{2}}|f'(a)|^{\frac{1}{2}}\big)+\big(b^{\frac{\alpha+1}{2}}|f'(b)|^{\frac{1}{2}}\big)\right].$$

Proof. From Lemma 2.1, using the property of the modulus and GG-convexity of $|f'|^q$ and Hölder integral inequality, we can write

$$\left|b^\alpha f(b) - a^\alpha f(a) - \alpha \int_a^b f(x) d_\alpha x\right| \tag{2.30}$$

$$= \frac{\ln b - \ln a}{2}\Bigg[\int_0^1 \big(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\big)^{2\alpha} D_\alpha(f)\big(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\big)dt$$

$$+ \int_0^1 \big(b^{\frac{1-t}{2}} a^{\frac{1+t}{2}}\big)^{2\alpha} D_\alpha(f)\big(b^{\frac{1-t}{2}} a^{\frac{1+t}{2}}\big)dt\Bigg]$$

$$\leq \frac{(ab)^{\frac{\alpha+1}{2}}(\ln b - \ln a)}{2}\Bigg[\bigg(\int_0^1 \big(\frac{b}{a}\big)^{\frac{t(\alpha+1)p}{2}} dt\bigg)^{\frac{1}{p}} \bigg(\int_0^1 \big|f'\big(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\big)\big| dt\bigg)^{\frac{1}{q}}$$

$$+\bigg(\int_0^1 \big(\frac{a}{b}\big)^{\frac{t(\alpha+1)p}{2}} dt\bigg)^{\frac{1}{p}} \bigg(\int_0^1 \big|f'\big(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\big)\big| dt\bigg)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{(ab)^{\frac{\alpha+1}{2}}(\ln b - \ln a)}{2}\Bigg[\bigg(\int_0^1 \big(\frac{b}{a}\big)^{\frac{t(\alpha+1)p}{2}} dt\bigg)^{\frac{1}{p}} \bigg(\int_0^1 |f'(b)|^{\frac{1+t}{2}} |f'(a)|^{\frac{1-t}{2}} dt\bigg)^{\frac{1}{q}}$$

$$+\bigg(\int_0^1 \big(\frac{a}{b}\big)^{\frac{t(\alpha+1)p}{2}} dt\bigg)^{\frac{1}{p}} \bigg(\int_0^1 |f'(b)|^{\frac{1-t}{2}} \big||f'(a)|^{\frac{1+t}{2}}\big)\big| dt\bigg)^{\frac{1}{q}}\Bigg].$$

If we calculate the above integrals, we get the desired result. □

Theorem 2.7. *Let $f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be an α-fractional differentiable function on the interior I° of I, where $a, b \in I$ with $a < b$ and $f' \in L_\alpha[a,b]$. If $|f'|^q$ is GG-convex on I for $q > 1$ and $q > p > 0$, then the following inequality holds for conformable fractional integrals;*

$$\left|b^\alpha f(b) - a^\alpha f(a) - \alpha \int_a^b f(x) d_\alpha x\right| \tag{2.31}$$

$$\leq \frac{\ln b - \ln a}{2}\left[L\big(a^{\frac{q(\alpha+1)}{2}}, b^{\frac{q(\alpha+1)}{2}}\big)\right]^{\frac{1}{q}}\left[\big(a^{\frac{\alpha+1}{2}}|f'(a)|^{\frac{1}{2}}\big) + \big(b^{\frac{\alpha+1}{2}}|f'(b)|^{\frac{1}{2}}\big)\right].$$

Proof. From Lemma 2.1, using the property of the modulus and GG-convexity of $|f'|^q$ and Hölder integral inequality, we can write

$$\left|b^\alpha f(b) - a^\alpha f(a) - \alpha \int_a^b f(x) d_\alpha x\right| \tag{2.32}$$

$$= \frac{\ln b - \ln a}{2}\Bigg[\int_0^1 \big(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\big)^{2\alpha} D_\alpha(f)\big(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\big)dt$$

$$+ \int_0^1 \big(b^{\frac{1-t}{2}} a^{\frac{1+t}{2}}\big)^{2\alpha} D_\alpha(f)\big(b^{\frac{1-t}{2}} a^{\frac{1+t}{2}}\big)dt\Bigg]$$

$$\leq \frac{(\ln b-\ln a)}{2}\Big[\Big(\int_0^1 1dt\Big)^{\frac{1}{p}}\Big(\int_0^1 (b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})^{q(\alpha+1)}\big|f'(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})\big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 1dt\Big)^{\frac{1}{p}}\Big(\int_0^1 (a^{\frac{1+t}{2}}b^{\frac{1-t}{2}})^{q(\alpha+1)}\big|f'(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})\big|^q dt\Big)^{\frac{1}{q}}\Big]$$

$$\leq \frac{(ab)^{\frac{\alpha+1}{2}}(\ln b-\ln a)}{2}\Big[\Big(\int_0^1 (\frac{b}{a})^{\frac{t(\alpha+1)q}{2}}\big|f'(b)\big|^{\frac{1+t}{2}}\big|f'(a)\big|^{\frac{1-t}{2}}dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 (\frac{a}{b})^{\frac{t(\alpha+1)q}{2}}\big|f'(b)\big|^{\frac{1-t}{2}}\big|f'(a)\big|^{\frac{1+t}{2}})\big|dt\Big)^{\frac{1}{q}}\Big].$$

By computing the above integrals, we get the desired result. □

Theorem 2.8. *Let $f: I\subset\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be an α-fractional differentiable function on the interior I° of I, where $a,b\in I$ with $a<b$ and $f'\in L_\alpha[a,b]$. If $|f'|^q$ is GG-convex on I for $q>1$ and $q>p>0$, then the following inequality holds for conformable fractional integrals;*

$$\Big|b^\alpha f(b)-a^\alpha f(a)-\alpha\int_a^b f(x)d_\alpha x\Big| \tag{2.33}$$

$$\leq \frac{\ln b-\ln a}{2}\Big[L\big(a^{\frac{(\alpha+1)}{2}},b^{\frac{(\alpha+1)}{2}}\big)\Big]^{1-\frac{1}{q}}\Big[L\Big(\big(a^{\alpha+1}|f'(a)|^q\big)^{\frac{1}{2}},\big(b^{\alpha+1}|f'(b)|^q\big)^{\frac{1}{2}}\Big)\Big]^{\frac{1}{q}}$$

$$\Big[\big(a^{\frac{\alpha+1}{2}}|f'(a)|^{\frac{1}{2}}\big)+\big(b^{\frac{\alpha+1}{2}}|f'(b)|^{\frac{1}{2}}\big)\Big].$$

Proof. From Lemma 2.1, using the property of the modulus and GG-convexity of $|f'|^q$ and Hölder integral inequality, we can write

$$\Big|b^\alpha f(b)-a^\alpha f(a)-\alpha\int_a^b f(x)d_\alpha x\Big| \tag{2.34}$$

$$= \frac{\ln b-\ln a}{2}\Big[\int_0^1 (b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})^{2\alpha}D_\alpha(f)(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})dt$$

$$+\int_0^1 (b^{\frac{1-t}{2}}a^{\frac{1+t}{2}})^{2\alpha}D_\alpha(f)(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}})dt\Big]$$

$$\leq \frac{(\ln b-\ln a)}{2}\Big[\Big(\int_0^1 (b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})^{\alpha+1}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 (b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})^{(\alpha+1)}\big|f'(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})\big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 (b^{\frac{1-t}{2}}a^{\frac{1+t}{2}})^{\alpha+1}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 (a^{\frac{1+t}{2}}b^{\frac{1-t}{2}})^{(\alpha+1)}\big|f'(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}})\big|^q dt\Big)^{\frac{1}{q}}\Big]$$

$$\leq \frac{(ab)^{\frac{\alpha+1}{2}}(\ln b-\ln a)}{2}\Big[\Big(\int_0^1 (\frac{b}{a})^{\frac{t(\alpha+1)}{2}}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 (\frac{b}{a})^{\frac{t(\alpha+1)}{2}}\big|f'(b)\big|^{\frac{q(1+t)}{2}}\big|f'(a)\big|^{\frac{q(1-t)}{2}}dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 (\frac{a}{b})^{\frac{t(\alpha+1)}{2}}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 (\frac{a}{b})^{\frac{t(\alpha+1)}{2}}\big|f'(b)\big|^{\frac{q(1-t)}{2}}\big|f'(a)\big|^{\frac{q(1+t)}{2}})\big|dt\Big)^{\frac{1}{q}}\Big].$$

If we calculate the integrals above, we get the desired result. □

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