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Fuzzy Minimal and Maximal δ-Open Sets

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ABSTRACT. The aim of this article is to introduce fuzzy minimal δ-open and fuzzy maximal δ-open sets in fuzzy topological space. Further, we investigate some properties with these new sets.

1. INTRODUCTION

Zadeh [6] established fuzzy set in 1965 and Chang [2] introduced fuzzy topology in 1968. Ittanagi and Wali [3] instigated the notions of fuzzy maximal and minimal open sets. The notion of fuzzy δ-open set introduced by Supriti Saha [5]. In this paper , we introduce fuzzy minimal δ-open and fuzzy maximal δ-open sets. Further some of their related results investigated.

A proper nonempty fuzzy open set U of $X$is said to be a fuzzy minimal open [3] set if U and 0_X are only fuzzy open sets contained in U.

A proper nonempty fuzzy open set U of $X$is said to be a fuzzy maximal open [3] set if 1_X and U are only fuzzy open sets containing U.

A fuzzy subset K of a space X is called fuzzy regular open [1] (resp. fuzzy regular closed) if $K = \text{Int}(\text{Cl}(K))$ (resp.$K =\text{Cl}(\text{Int}(K))$).

The fuzzy δ-interior of a fuzzy subset K of X is the union of all fuzzy regular open sets contained in K. A fuzzy subset K is called fuzzy δ-open [5] if $K = \text{Int}_\delta(K)$. The complement of fuzzy δ-open set is called fuzzy δ-closed (i.e, $K = \text{Cl}_\delta(K)$).

2. FUZZY MINIMAL δ-OPEN SETS

Definition 2.1. A proper nonzero fuzzy δ-open set U in fts (X, τ) is said to be fuzzy minimal δ-open if fuzzy δ-open set contained in U is 0_X or U.

Lemma 2.1. *Let (X, τ) be a fts.*
(i) If U_1 is fuzzy minimal δ-open and U_2 is fuzzy δ-open in X, then $U_1 \cap U_2 = 0_X$ or $U_1 \subset U_2$.
(ii) If U_1 and U_2 are fuzzy minimal δ-open , then $U_1 \cap U_2 = 0_X$ or $U_1 = U_2$.

Proof. (i) Let us assume that U_2 is fuzzy δ-open in X such that $U_1 \cap U_2 \neq 0_X$. Since U_1 is fuzzy minimal δ-open , and $U_1 \cap U_2 \subset U_1$, then $U_1 \cap U_2 = U_1$ implies that $U_1 \subset U_2$.

(ii) Suppose that $U_1 \cap U_2 \neq 0_X$, then clearly from(ii), $U_1 \subset U_2$ and $U_2 \subset U_1$ as U_1 and U_2 are fuzzy minimal δ-open . Hence $U_1 = U_2$. $\square$

Theorem 2.1. *Let U and U_i are fuzzy minimal δ-open sets for any $i \in M$. If $U \subseteq \bigcup_{i \in M} U_i$, then $U = U_j$ for any $j \in M$.*

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Proof. Suppose $U \subseteq \bigcup_{i\in M} U_i$, then $U = U \cap \left(\bigcup_{i\in M} U_i\right) = \bigcup_{i\in M} (U \cap U_i)$. By deploying lemma 2.1(ii), $U \cap U_i = 0_X$ or $U = U_i$ as U and U_i are fuzzy minimal δ-open sets. If $U \cap U_i = 0_X$, then $U = 0_X$ which contradicts that U is a fuzzy minimal δ-open set. Hence if $U \cap U_i \neq 0_X$ then $U = U_j$ for any $j \in M$. □

Theorem 2.2. *If U and U_i are fuzzy minimal δ-open sets for any $i \in M$ and $U \neq U_i$, then $U \cap \left(\bigcup_{i\in M} U_i\right) = 0_X$ for any $i \in M$.*

Proof. Let $U \cap \left(\bigcup_{i\in M} U_i\right) \neq 0_X$, then $U \cap U_i \neq 0_X$ for any $i \in M$. By deploying lemma 2.1(ii), $U = U_i$ contradictory to $U \neq U_i$. Hence $U \cap \left(\bigcup_{i\in M} U_i\right) = 0_X$. □

Theorem 2.3. *If U_i is a fuzzy minimal δ-open for any $i \in M$ ($|M| \geq 2$) and $U_i \neq U_j$ for any distinct $i, j \in M$, then $\left(\bigcup_{i\in M\setminus\{j\}} U_i\right) \cap U_j = 0_X$ for any $j \in M$.*

Proof. Let $\left(\bigcup_{i\in M\setminus\{j\}} U_i\right) \cap U_j \neq 0_X$. Then $\bigcup_{i\in M\setminus\{j\}} (U_i \cap U_j) \neq 0_X \Rightarrow (U_i \cap U_j) \neq 0_X$. By lemma 2.1(ii), $U_i = U_j$, a contradiction. Hence $\left(\bigcup_{i\in M\setminus\{j\}} U_i\right) \cap U_j = 0_X$ for any $j \in M$. □

Theorem 2.4. *If U_i is a fuzzy minimal δ-open for any $i \in M$, ($|M| \geq 2$) and $U_i \neq U_j$ for any distinct $i, j \in M$. If K is a proper fuzzy set of M, then $\left(\bigcup_{i\in M\setminus K} U_i\right) \bigcap \left(\bigcup_{s\in K} U_s\right) = 0_X$.*

Proof. Let $\left(\bigcup_{i\in M\setminus K} U_i\right) \bigcap \left(\bigcup_{s\in K} U_s\right) \neq 0_X$. It implies that $\bigcup (U_i \cap U_s) \neq 0_X$ for $i \in M\setminus K$ and $s \in K$ implies that $U_i \cap U_s \neq 0_X$ for some $i \in M$ and $s \in K$. By lemma2.1(ii), $U_i = U_s$, which is a contradiction. Hence $\left(\bigcup_{i\in M\setminus K} U_i\right) \bigcap \left(\bigcup_{s\in K} U_s\right) = 0_X$. □

Theorem 2.5. *If U_i is a fuzzy minimal δ-open for any $i \in M$ such that $U_i \neq U_j$ for any distinct $i, j \in M$. If S is a proper nonzero fuzzy set of M, then $\left[\bigcup_{i\in M\setminus k} U_i\right] \cap \left[\bigcup_{k\in S} U_k\right] = 0_X$.*

Proof. Assume that $\cup [U_i \cap U_k] \neq 0_X$ for $i \in M\setminus k$,$k \in S$. Clearly, for some $i \in M$,$k \in S$ we have $[U_i \cap U_k] \neq 0_X$. By deploying lemma 2.1(ii) $U_i = U_k$, a contradiction. □

Theorem 2.6. *If U_i and U_k are fuzzy minimal δ-open sets for any $i \in M$ and $k \in S$ and if $\exists$ an $n \in S$ such that $U_i \neq U_n$ for any $i \in M$, then $\left[\bigcup_{n\in K} U_n\right] \not\subset \left[\bigcup_{i\in M} U_i\right]$.*

Proof. Assume that $\exists$ an $n \in S$ such that $U_i \neq U_n$ for any $i \in M$, then $\left[\bigcup_{n\in K} U_n\right] \subset \left[\bigcup_{i\in M} U_i\right]$.

$\Rightarrow U_n \subset \left[\bigcup_{i\in M} U_i\right]$ for some $n \in K$.

$\Rightarrow U_i \neq U_n$ for any $i \in M$, by theorem 2.1, which is a contradiction. Hence $\left[\bigcup_{n\in K} U_n\right] \not\subset \left[\bigcup_{i\in M} U_i\right]$. □

Theorem 2.7. *If U_i is a fuzzy minimal δ-open for any $i \in M$ such that $U_i \neq U_j$ for any distinct $i, j \in M$, then $\left[\bigcup_{k\in K} U_k\right] \subsetneqq \left[\bigcup_{i\in M} U_i\right]$ for any proper nonzero subset K of M.*

Proof. Let $m \in M\backslash K$, then U_m is a fuzzy minimal δ-open set of the family $\{U_m | m \in M\backslash K\}$ of fuzzy minimal δ-open sets. Clearly $U_m \cap \left[\bigcup_{k\in K} U_k\right] = \bigcup_{k\in K} [U_m \cap U_k] = 0_X$. Also $U_m \cap \left[\bigcup_{i\in M} U_i\right] = \bigcup_{i\in M} [U_m \cap U_i] = U_m$.

If $\left[\bigcup_{k\in K} U_k\right] = \left[\bigcup_{i\in M} U_i\right]$, then $U_m = 0_X$ which is a contradiction that U_m is a fuzzy minimal δ-open set. Hence $\left[\bigcup_{k\in K} U_k\right] \subsetneqq \left[\bigcup_{i\in M} U_i\right]$. □

Theorem 2.8. *If U_i is a fuzzy minimal δ-open set for any $i \in M$ such that $U_i \neq U_j$ for any distinct $i, j \in M$, then*

(i) $U_j \subset \left[\bigcup_{i\in M\backslash\{j\}} U_i\right]^c$ for some $j \in M$.

(ii) $\bigcup_{i\in M\backslash\{j\}} U_i \neq 1_X$ for any $j \in M$.

Proof. (i) By hypothesis, $U_i \neq U_j$ for any distinct $i, j \in M$.

By theorem 2.2, $\left[\bigcup_{i\in M} U_i\right] \cap U_j = 0_X$ which is true for any $j \in M$.

$\Rightarrow \bigcup_{i\in M} [U_i \cap U_j] = 0_X$

$\Rightarrow U_i \cap U_j = 0_X$ (By Lemma 2.1(ii))

$\Rightarrow U_i \subset {U_j}^c$

$\Rightarrow \bigcup_{i\in M\backslash\{j\}} U_i \subset {U_j}^c$. Hence proved.

(ii) Let $j \in M$ such that $\bigcup_{i\in M\backslash\{j\}} U_i = 1_X$

$\Rightarrow U_i = 0_X$

$\Rightarrow U_i$ is not a fuzzy minimal δ-open set, a contradiction. Hence $\bigcup_{i\in M\backslash\{j\}} U_i \neq 1_X$ for any $j \in M$. □

Corollary 2.1. *If U_i is a fuzzy minimal δ-open set for any $i \in M$ such that $U_i \neq U_j$ for any distinct $i, j \in M$, then $U_i \cup U_j \neq 1_X$ for any distinct $i, j \in M$.*

Proof. Similar to that of "Theorem 2.8 (ii)." □

Theorem 2.9. *If U_i is a fuzzy minimal δ-open sets for any $i \in M$ such that $U_i \neq U_j$ for any distinct $i, j \in M$, then $U_j = \left[\bigcup_{i\in M} U_i\right] \cap \left[\bigcup_{i\in M\backslash\{j\}} U_i\right]^c$ for any $j \in M$.*

Proof. For any $j \in M \Rightarrow \left[\bigcup_{i\in M} U_i\right] \bigcap \left[\bigcup_{i\in M\setminus\{j\}} U_i\right]^c = \left[\bigcup_{i\in M\setminus\{j\}} U_i \bigcup U_j\right] \cap \left[\bigcup_{i\in M\setminus\{j\}} U_i\right]^c$
$= \left[\left(\bigcup_{i\in M\setminus\{j\}} U_i\right) \cap \left(\bigcup_{i\in M\setminus\{j\}} U_i\right)^c\right] \bigcup \left[U_j \cap \left(\bigcup_{i\in M\setminus\{j\}} U_i\right)^c\right]$
$= 0_X \cup U_j$
$= U_j$ for any $j \in M$. □

Proposition 2.1. *Let G be a fuzzy minimal δ-open set. If $x_\alpha \in G$, then $G \subset G_1$ for any fuzzy open neighbourhood G_1 of x_α.*

Proof. Let G_1 be an fuzzy δ-open neighborhood of x_α such that $G \not\subset G_1$. Clearly $G \cap G_1$ is an fuzzy δ-open such that $G \cap G_1 \subsetneq G$ and $G \cap G_1 \neq 0_X$. This implies that G is a fuzzy minimal δ-open set which a contradiction. □

Proposition 2.2. *Let G be a fuzzy minimal δ-open set. Then*
$G = \bigcap \{G_1 : G_1$ fuzzy e-open neighbourhood of x_α for any $x_\alpha \in G\}$.

Proof. By deploying proposition 2.1 and as G is an fuzzy δ-open neighborhood of x_α, we have G $\subset \bigcap \{G_1 : G_1$ fuzzy e-open neighbourhood of $x_\alpha \subset G\}$. This completes the proof. □

Theorem 2.10. *Let G be a fuzzy minimal δ-open set. Then the following conditions are equivalent.*
(i) G is fuzzy minimal δ-open set.
(ii)$G \subset eCl(K)$ for any nonzero fuzzy subset K of G.
(iii)$eCl(G) = eCl(K)$ for any nonzero fuzzy subset K of G.

Proof. $(i) \Rightarrow (ii)$: By deploying "proposition 2.1"for any $x_\alpha \in G$ and fuzzy δ-open neighborhood M of x_α, we have $K = (G \cap K) \subset (M \cap K)$ for any proper nonzero fuzzy subset $K \subset G$. Therefore, we have $(M \cap K) \neq 0_X$ and $x_\alpha \in eCl(K)$. It follows that $G \subset eCl(K)$.

$(ii) \Rightarrow (iii)$: For any proper fuzzy subset K of G, $eCl(G) \subset eCl(K)$. Also by(ii) $eCl(G) \subset eCl\left(eCl(K)\right) = eCl(K)$. Hence proved.

$(iii) \Rightarrow (i)$: Let us assume that G is not fuzzy minimal δ-open . Then there exists a proper fuzzy δ-open D such that $D \subset G$.Then $\exists\ y_\alpha \in G$ such that $y_\alpha \notin D$. Then $eCl(\{y_\alpha\}) \in D^c$ implies that $eCl(\{y_\alpha\}) \neq eCl(G)$, a contradiction. This completes our proof. □

3. Fuzzy minimal δ-open sets and its properties

Definition 3.2. A proper nonzero fuzzy δ-open set V of a fts (X, τ) is said to fuzzy maximal δ-open if any fuzzy δ-open set which contains V is either V or 1_X.

Example 3.1. Let $X = \{a, b, c, d\}$. Then fuzzy sets $\mathcal{F}_1 = \frac{0.3}{a} + \frac{0.4}{b} + \frac{0.3}{c} + \frac{0.4}{d}$; $\mathcal{F}_2 = \frac{0.5}{a} + \frac{0.5}{b} + \frac{0.5}{c} + \frac{0.5}{d}$; $\mathcal{F}_3 = \frac{0.7}{a} + \frac{0.6}{b} + \frac{0.7}{c} + \frac{0.6}{d}$; are defined as follows: Consider the fuzzy topology $\tau = \{0_X, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, 1_X\}$. Here $\mathcal{F}_1$ is fuzzy minimal δ-open and $\mathcal{F}_3$ fuzzy maximal δ-open set.

Lemma 3.2. Let (X, τ) be a fts. Then
(i) If V_1 is a fuzzy maximal δ-open and V_2 is fuzzy δ-open in X, then $V_1 \cup V_2 = 1_X$ or $V_2 \subset V_1$.
(ii) If V_1 and V_3 are fuzzy maximal δ-open sets, then either $V_1 \cup V_3 = 1_X$ or $V_1 = V_3$.

Proof. (i) Assume that $V_2 \not\subset V_1$. Clearly, $V_1 \subset (V_1 \cup V_2)$ a contrary to V_1 is a fuzzy maximal δ-open set if $V_1 \cup V_2 \neq 1_X$. Hence, $V_1 \cup V_2 = 1_X$.
(ii) Let V_1 and V_3 are fuzzy maximal δ-open sets. Then from(i) $V_3 \subset V_1$ and $V_1 \subset V_3$ implies that $V_1 = V_3$. □

Theorem 3.11. If V_1, V_2 and V_3 are fuzzy maximal δ-open sets such that $V_1 \neq V_2$ and $(V_1 \cap V_2) \subset V_3$, then either $V_1 = V_3$ or $V_2 = V_3$.

Proof. Suppose that V_1, V_2 and V_3 are fuzzy maximal δ-open sets with $V_1 \neq V_2$, $(V_1 \cap V_2) \subset V_3$ and if $V_1 \neq V_3$, then
$(V_2 \cap V_3) = V_2 \cap (V_3 \cap 1_X)$
$= V_2 \cap [V_3 \cap (V_1 \cup V_2)]$, by lemma 3.2(ii)
$= V_2 \cap [(V_3 \cap V_1) \cup (V_3 \cap V_2)]$
$= [V_2 \cap V_3 \cap V_1] \cup [V_2 \cap V_3 \cap V_2]$
$= [V_2 \cap V_1] \cup [V_2 \cap V_3]$
$= V_2 \cap [V_1 \cup V_3]$
$= V_2 \cap 1_X$
$= V_2$

$(V_2 \cap V_3) = V_2 \Rightarrow V_2 \subset V_3$. As V_2 and V_3 are fuzzy maximal δ-open sets, $V_3 \subset V_2$. Hence $V_2 = V_3$. □

Theorem 3.12. *For any distinct fuzzy maximal δ-open sets V_1, V_2, V_3*
$[V_1 \cap V_2] \not\subset [V_1 \cap V_3]$.

Proof. Consider $[V_1 \cap V_2] \subset [V_1 \cap V_3]$ for any distinct fuzzy maximal δ-open sets V_1, V_2 and V_3. Then
$[V_1 \cap V_2] \cup [V_2 \cap V_3] \subset [V_1 \cap V_3] \cup [V_2 \cap V_3]$
$= [V_1 \cup V_3] \cap V_2 \subset [V_1 \cup V_2] \cap V_3$
$= 1_X \cap V_2 \subset 1_X \cap V_3$
$= V_2$ is contained in V_3
a contradiction to V_1, V_2 and V_3 are distinct . Hence $[V_1 \cap V_2] \not\subset [V_1 \cap V_3]$. □

Remark 3.1. Proofs of "Theorem 3.13, Corollary 3.2, Theorem 3.14 and Theorem 3.15" are similar to proofs of "Theorem 2.8, Corollary 2.1, Theorem 2.9 and Theorem 2.7" respectively. Hence the proofs are omitted.

Theorem 3.13. *If V_i is a fuzzy maximal δ-open sets for any $i \in M$, M is a finite set and $V_i \neq V_j$ for any distinct $i, j \in M$, then*

(i) $\left[\bigcap_{i \in M \setminus \{j\}} V_i\right]^c \subset V_j$ *for any* $j \in M$

(ii) $\bigcap_{i \in M \setminus \{j\}} V_i \neq 0_X$ *for any* $j \in M$.

Corollary 3.2. *If V_i is a fuzzy maximal δ-open sets for any $i \in M$, M is a finite set and $V_i \neq V_j$ for any distinct $i, j \in M$ then,$V_i \cap V_j \neq 0_X$ for any distinct $i, j \in M$.*

Theorem 3.14. *If V_i is a fuzzy maximal δ-open sets for any $i \in M$, M is a finite set and $V_i \neq V_j$ for any distinct $i, j \in M$, then* $V_j = \left[\bigcap_{i \in M} V_i\right] \cup \left[\bigcap_{i \in M \setminus \{j\}} V_i\right]^c$ *for any* $j \in M$.

Theorem 3.15. *If V_i is a fuzzy maximal δ-open sets for any $i \in M$, M is a finite set and $V_i \neq V_j$ for any distinct $i, j \in M$ and if K is a proper nonzero fuzzy subset of M, then* $\bigcap_{i \in M} V_i \subsetneqq \bigcap_{k \in K} V_k$.

Theorem 3.16. *If V_i is a fuzzy maximal δ-open sets for any $i \in M$, M is a finite set and $V_i \neq V_j$ for any distinct $i, j \in M$ and if* $\bigcap_{i \in M} V_i$ *is a fuzzy subset, then V_j is a fuzzy subset for any $j \in M$.*

Proof. By "Theorem 3.14", we have $V_j = \left[\bigcap_{i\in M} V_i\right] \cup \left[\bigcap_{i\in M\setminus\{j\}} V_i\right]^c$ for any $j \in M$.

$V_j = \left[\bigcap_{i\in M} V_i\right] \cup \left[\bigcup_{i\in M\setminus\{j\}} V_i^c\right]$.

Since M is finite, $\bigcup_{i\in M\setminus\{j\}} V_i^c$ is fuzzy δ-closed. Hence V_j is fuzzy δ-closed for any $j \in M$. □

Theorem 3.17. *If V_i is a fuzzy maximal δ-open set for any $i \in M, M$ is a finite set and $V_i \neq V_j$ for any distinct $i, j \in M$. If $\bigcap_{i\in M} V_i = 0_X$,then $\{V_i/i \in M\}$ is the set of all fuzzy maximal δ-open sets of fts X.*

Proof. Suppose that $\exists$ another fuzzy maximal δ-open V_k of a fts X such that $V_k \neq V_i, \forall i \in M$. Clearly, $0_X = \bigcap_{i\in M} V_i = \bigcap_{i\in(M\cup k)\setminus\{k\}} V_i \neq 0_X$, by Theorem 3.13(ii), a contradiction.

Hence $\{V_i/i \in M\}$ is the family of all fuzzy maximal δ-open sets of fts X. □

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