CREAT. MATH. INFORM.
Volume **31** (2022), No. 2,
Pages 229 - 240

Online version at https://semnul.com/creative-mathematics/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2022.02.08

Fourth Hankel determinant for a subclass of analytic functions defined by generalized Sãlãgean operator

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ABSTRACT. This paper is concerned with the estimation of fourth Hankel determinant for a subclass of analytic functions defined by generalized Sãlãgean operator in the open unit disc $E = \{z : |z| < 1\}$. The present study sets the stage for other researchers to investigate the fourth Hankel determinant for some other subclasses of analytic functions.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

in the unit disc $E = \{z : |z| < 1\}$ and are further normalized by $f(0) = f'(0) - 1 = 0$. The subclass of $\mathcal{A}$ consisting of the functions of the form (1.1) and which are univalent in E, is denoted by $\mathcal{S}$.
Let $\mathcal{P}$ denote the class of analytic functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

whose real parts are positive in E.
Sãlãgean [27] established an operator, with the help of which many subclasses of $\mathcal{A}$ were introduced. As a generalization, Al-Oboudi [1] introduced the following differential operator:
For $\delta \geq 1$ and $f \in \mathcal{A}$,

$$D_\delta^0 f(z) = f(z),$$

$$D_\delta^1 f(z) = (1-\delta) f(z) + \delta z f'(z),$$

and in general,

$$D_\delta^n f(z) = D(D_\delta^{n-1} f(z)) = (1-\delta) D_\delta^{n-1} f(z) + \delta z (D_\delta^{n-1} f(z))', n \in \mathbb{N}$$

which is equivalent to

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

Received: 21.09.2021. In revised form: 04.03.2022. Accepted: 11.04.2022
2010 *Mathematics Subject Classification.* 30C45, 30C50.
Key words and phrases. *Analytic functions, univalent functions, Hankel determinant, coefficient bounds,Sãlãgean operator.*
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with $D_\delta^n f(0) = 0$. For $\delta = 1$, the operator $D_\delta^n f(z)$ reduces to Sãlãgean operator. So, it is natural to call $D_\delta^n f(z)$ as the Generalized Sãlãgean operator.
By $\mathcal{R}(\delta; n; \alpha)$, let us denote the class of functions in $\mathcal{A}$ and which satisfy the condition

$$Re\left\{(1-\alpha)\frac{D_\delta^n f(z)}{z} + \alpha(D_\delta^n f(z))'\right\} > 0, 0 \leq \alpha \leq 1, z \in E.$$

The following consequences can be easily observed:
(i) $\mathcal{R}(1; 0; \alpha) \equiv \mathcal{R}(\alpha)$, the class investigated by Murugusundramurthi and Magesh [22].
(ii) $\mathcal{R}(1; 1; \alpha) \equiv \mathcal{R}'(\alpha)$, the class discussed in [26].
(iii) $\mathcal{R}(1; 0; 1) \equiv \mathcal{R}$, the class introduced and studied by MacGregor [19].
(iv) $\mathcal{R}(1; 0; 0) \equiv \mathcal{R}_1$, the subclass of close-to-star functions introduced by MacGregor [20].
In their pioneering work, Noonan and Thomas [24] introduced the q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For $q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2$, the Hankel determinant simplifies respectively to $H_2(1) = a_3 - a_2^2$ and $H_2(2) = a_2a_4 - a_3^2$.
Numerous work has been done on the estimation of second Hankel determinant by various authors including Noor [25], Ehrenborg [11], Layman [15], Singh [29], Mehrok and Singh [21] and Janteng et al. [12].

For $q = 3, n = 1$, the Hankel determinant reduces to

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix},$$

which is known as the third Hankel determinant.
For $f \in \mathcal{S}, a_1 = 1$,

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

and by using the triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1.2}$$

The estimation of third Hankel determinant is little bit complicated. Babalola [5] was the first researcher who successfully obtained the upper bound of third Hankel determinant for the classes of starlike functions, convex functions and the class of functions with bounded boundary rotation. Further a few researchers in [14, 23] and also including Shanmugam et al. [28], Bucur et al. [8], Altinkaya and Yalcin [2], Singh and Singh [30] have been actively engaged in the study of third Hankel determinant for various subclasses of analytic functions.

For any $f \in \mathcal{A}$, we can represent the fourth Hankel determinant as

$$H_4(1) = a_7H_3(1) - a_6D_1 + a_5D_2 - a_4D_3, \tag{1.3}$$

where D_1, D_2 and D_3 are determinants of order 3 given by

$$D_1 = (a_3a_6 - a_4a_5) - a_2(a_2a_6 - a_3a_5) + a_4(a_2a_4 - a_3^2), \tag{1.4}$$

$$D_2 = (a_4a_6 - a_5^2) - a_2(a_3a_6 - a_4a_5) + a_3(a_3a_5 - a_4^2), \tag{1.5}$$

$$D_3 = a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2). \tag{1.6}$$

Fourth Hankel determinant is a generalization of second and third Hankel determinants. It is an interesting topic of current research in geometric function theory and so recently, the researchers has started to study the fourth Hankel determinant for various subclasses of $\mathcal{A}$. The process of finding the bounds of the fourth Hankel determinant is very lengthy and difficult. The initiative of finding the fourth Hankel determinant for subclasses of analytic functions was taken by Arif et al. [3] in 2018. After that, only a few reseracher worked in this direction including [32, 4, 13, 33, 10, 31].

Inspired from the above works, we study here the fourth Hankel determinant $H_4(1)$ for the class $\mathcal{R}(\delta; n; \alpha)$. The results already proved by various authors follow as special cases of our study.

2. Main results

Lemma 2.1. ([9, 18]) *If $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$, then for $n, k \in \mathbb{N} = \{1, 2, 3, ...\}$, we have the following inequalities:*

$$|c_{n+k} - \lambda c_n c_k| \leq 2, 0 \leq \lambda \leq 1$$

and

$$|c_n| \leq 2.$$

Lemma 2.2. ([16, 17, 7]) *If $p \in \mathcal{P}$, then*

$$2c_2 = c_1^2 + (4 - c_1^2)x,$$

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z,$$

and

$$8c_4 = c_1^4 + (4 - c_1^2)x(c_1^2(x^2 - 3x + 3) + 4x) - 4(4 - c_1^2)(1 - |x|^2)(c_1(x - 1)\eta + \bar{x}\eta^2 - (1 - |\eta|^2)z),$$

for some x, z and η satisfying $|x| \leq 1, |z| \leq 1, |\eta| \leq 1$ and $c_1 \in [0, 2]$.

Lemma 2.3. ([6]) *If $p \in \mathcal{P}$, then*

$$\left| c_2 - \sigma \frac{c_1^2}{2} \right| \leq \begin{cases} 2(1 - \sigma) & \text{if } \sigma \leq 0, \\ 2 & \text{if } 0 \leq \sigma \leq 2, \\ 2(\sigma - 1) & \text{if } \sigma \geq 2. \end{cases}$$

Theorem 2.1. *If $f \in \mathcal{R}(\delta; n; \alpha)$, then*

$$|a_n| \leq \frac{2}{[1 + (n-1)\alpha][1 + (n-1)\delta]^n}, n \geq 2. \tag{2.1}$$

The bound is sharp.

Proof. As $f \in \mathcal{R}(\delta; n; \alpha)$, therefore by definition, there exists a function $p \in \mathcal{P}$ such that

$$(1 - \alpha)\frac{D_\delta^n f(z)}{z} + \alpha(D_\delta^n f(z))' = p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

On expanding and equating the coefficients in the above equation, it yields

$$a_n = \frac{c_{n-1}}{[1 + (n-1)\alpha][1 + (n-1)\delta]^n}. \tag{2.2}$$

Using Lemma 2.1 in (2.2), the result (2.1) is obvious.
The estimate is sharp for the function defined as

$$(1-\alpha)\frac{D_\delta^n f(z)}{z}+\alpha(D_\delta^n f(z))'=\left(\frac{1+\delta z^n}{1-\delta z^n}\right), |\delta|=1.$$

□

For $\delta=1, n=0$, Theorem 2.1 gives the following result due to Singh et al. [32]:

Corollary 2.1. *If $f\in\mathcal{R}(\alpha)$, then*

$$|a_n|\leq\frac{2}{[1+(n-1)\alpha]}, n\geq 2.$$

For $\delta=1, n=1$, Theorem 2.1 coincides with the following result due to Sahoo [26]:

Corollary 2.2. *If $f\in\mathcal{R}'(\alpha)$, then*

$$|a_n|\leq\frac{2}{n[1+(n-1)\alpha]}, n\geq 2.$$

Theorem 2.2. *If $f\in\mathcal{R}(\delta;n;\alpha)$, then*

$$|a_3-a_2^2|\leq\frac{2}{(1+2\alpha)(1+2\delta)^n}. \tag{2.3}$$

The estimate is sharp.

Proof. Using (2.2), we find that

$$|a_3-a_2^2|=\frac{1}{(1+2\alpha)(1+2\delta)^n}\left|c_2-\frac{2(1+2\alpha)(1+2\delta)^n}{(1+\alpha)^2(1+\delta)^{2n}}.\frac{c_1^2}{2}\right|.$$

Since $0\leq\sigma=\dfrac{2(1+2\alpha)(1+2\delta)^n}{(1+\alpha)^2(1+\delta)^{2n}}\leq 2$, so by Lemma 2.3, the result (2.3) is obvious.
The bound is sharp for the function

$$f(z)=z+\frac{c_1}{(1+\alpha)(1+\delta)^2}z^2+\frac{c_1^2-2}{(1+2\alpha)(1+2\delta)^3}z^3+...$$

□

For $\delta=1, n=0$, Theorem 2.2 agrees with the following result due to Singh et al. [32]:

Corollary 2.3. *If $f\in\mathcal{R}(\alpha)$, then*

$$|a_3-a_2^2|\leq\frac{2}{(1+2\alpha)}.$$

For $\delta=1, n=1$, Theorem 2.2 gives the following result due to Sahoo [26]:

Corollary 2.4. *If $f\in\mathcal{R}'(\alpha)$, then*

$$|a_3-a_2^2|\leq\frac{2}{3(1+2\alpha)}.$$

Theorem 2.3. *If $f \in \mathcal{R}(\delta; n; \alpha)$, then*

$$|a_2 a_4 - a_3^2| \leq \frac{4}{(1+2\alpha)^2(1+2\delta)^{2n}}. \tag{2.4}$$

The bound is sharp.

Proof. Using (2.2), we have

$$|a_2 a_4 - a_3^2| = \left| \frac{c_1 c_3}{(1+\delta)^n(1+3\delta)^n(1+\alpha)(1+3\alpha)} - \frac{c_2^2}{(1+2\delta)^{2n}(1+2\alpha)^2} \right|.$$

Using Lemma 2.2, rearranging the terms and applying the triangle inequality along with the inequality $|z| \leq 1$, it yields

$$\begin{aligned}|a_2 a_4 - a_3^2| \leq \tfrac{T}{4} \Big[&[(1+2\delta)^{2n}(1+2\alpha)^2 - (1+\delta)^n(1+3\delta)^n(1+\alpha)(1+3\alpha)]c_1^4 \\ &+2[(1+2\delta)^{2n}(1+2\alpha)^2 - (1+\delta)^n(1+3\delta)^n(1+\alpha)(1+3\alpha)]c_1^2 x(4-c_1^2) \\ &+[\{(1+2\delta)^{2n}(1+2\alpha)^2 - (1+\delta)^n(1+3\delta)^n(1+\alpha)(1+3\alpha)\}c_1^2 \\ &+4(1+\delta)^n(1+3\delta)^n(1+\alpha)(1+3\alpha)](4-c_1^2)x^2 \\ &+2(1+2\delta)^{2n}(1+2\alpha)^2(4-c_1^2)c_1(1-|x|^2)\Big],\end{aligned}$$

where $T = \dfrac{1}{(1+\delta)^n(1+2\delta)^{2n}(1+3\delta)^n(1+\alpha)(1+2\alpha)^2(1+3\alpha)}$.

For $c_1 = c \in [0,2]$ and $|x| = \mu$, we have

$$\begin{aligned}|a_2 a_4 - a_3^2| \leq \frac{T}{4} \Big[&[(1+2\delta)^{2n}(1+2\alpha)^2 - (1+\delta)^n(1+3\delta)^n(1+\alpha)(1+3\alpha)]c^4 \\ &+2(1+2\delta)^{2n}(1+2\alpha)^2(4-c^2)c + 2[(1+2\delta)^{2n}(1+2\alpha)^2 \\ &-(1+\delta)^n(1+3\delta)^n(1+\alpha)(1+3\alpha)]c^2(4-c^2)\mu + \{(1+2\delta)^{2n}(1+2\alpha)^2 \\ &-(1+\delta)^n(1+3\delta)^n(1+\alpha)(1+3\alpha)\}(4-c^2)(c-2)(c-\beta)\mu^2\Big] = F(c,\mu),\end{aligned}$$

where $\beta = \beta(\alpha) = \dfrac{2(1+\delta)^n(1+3\delta)^n(1+\alpha)(1+3\alpha)}{(1+2\delta)^{2n}(1+2\alpha)^2 - (1+\delta)^n(1+3\delta)^n(1+\alpha)(1+3\alpha)}$.

Now $\frac{\partial F}{\partial \mu} = \frac{\{(1+2\delta)^{2n}(1+2\alpha)^2 - (1+\delta)^n(1+3\delta)^n(1+\alpha)(1+3\alpha)\}(4-c^2)[c^2+(c-2)(c-\beta)\mu]}{2(1+\delta)^n(1+2\delta)^{2n}(1+3\delta)(1+\alpha)(1+2\alpha)^2(1+3\alpha)} > 0$.

So, max.$F(c,\mu) = F(c,1) = G(c)$.

Therefore

$$\begin{aligned}G'(c) = W(\alpha,\delta)\Big[&\{(1+2\delta)^{2n}(1+2\alpha)^2 - (1+\delta)^n(1+3\delta)^n(1+\alpha)(1+3\alpha)\}c^3 \\ &+[4(1+\delta)^n(1+3\delta)^n(1+\alpha)(1+3\alpha) - 3(1+2\delta)^{2n}(1+2\alpha)^2]\Big] < 0,\end{aligned}$$

where

$W(\alpha,\delta) = -\dfrac{2}{(1+\delta)^n(1+2\delta)^{2n}(1+3\delta)(1+\alpha)(1+2\alpha)^2(1+3\alpha)}$.

So, max.$G(c) = G(0)$. Hence the result (2.4).

The result is sharp for the function

$$f(z) = z - \frac{2}{(1+2\alpha)(1+2\delta)^3} z^3 + \ldots$$

□

For $\delta = 1, n = 0$, Theorem 2.3 gives the following result due to Murugusundramurthi and Magesh [22]:

Corollary 2.5. *If $f \in \mathcal{R}(\alpha)$, then*

$$|a_2a_4 - a_3^2| \leq \frac{4}{(1+2\alpha)^2}.$$

For $\delta = 1, n = 1$, Theorem 2.3 coincides with the following result due to Sahoo [26]:

Corollary 2.6. *If $f \in \mathcal{R}'(\alpha)(0 \leq \alpha \leq \frac{1}{2})$, then*

$$|a_2a_4 - a_3^2| \leq \frac{4}{9(1+2\alpha)^2}.$$

Theorem 2.4. *If $f \in \mathcal{R}(\delta; n; \alpha)$, then*

$$|a_2a_3 - a_4| \tag{2.5}$$

$$\leq \frac{2[3(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha)-2(1+3\delta)^n(1+3\alpha)]^{\frac{3}{2}}}{3(1+\delta)^n(1+2\delta)^n(1+3\delta)^n(1+\alpha)(1+2\alpha)(1+3\alpha)\sqrt{3[(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha)-(1+3\delta)^n(1+3\alpha)]}}.$$

The estimate is sharp.

Proof. From (2.2), we have

$$|a_2a_3 - a_4| = \left|\frac{c_1c_2}{(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha)} - \frac{c_3}{(1+3\delta)^n(1+3\alpha)}\right|.$$

Using Lemma 2.2 and rearranging the terms, it yields

$$|a_2a_3 - a_4| = \left|\frac{\{2(1+3\delta)^n(1+3\alpha) - (1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha)\}c_1^3}{4(1+\delta)^n(1+2\delta)^n(1+3\delta)^n(1+\alpha)(1+2\alpha)(1+3\alpha)}\right.$$
$$-\frac{2\{(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha) - (1+3\delta)^n(1+3\alpha)\}}{4(1+\delta)^n(1+2\delta)^n(1+3\delta)^n(1+\alpha)(1+2\alpha)(1+3\alpha)}c_1(4-c_1^2)x$$
$$\left.+\frac{c_1(4-c_1^2)x^2}{4(1+3\delta)^n(1+3\alpha)} - \frac{(4-c_1^2)(1-|x|^2)z}{2(1+3\delta)^n(1+3\alpha)}\right|.$$

On applying the triangle inequality and using $c_1 = c \in [0,2]$ and $|x| = \rho, |z| \leq 1$, we have

$$|a_2a_3 - a_4| \leq \frac{\{2(1+3\delta)^n(1+3\alpha) - (1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha)\}c^3}{4(1+\delta)^n(1+2\delta)^n(1+3\delta)^n(1+\alpha)(1+2\alpha)(1+3\alpha)}$$
$$+\frac{\{(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha) - (1+3\delta)^n(1+3\alpha)\}}{2(1+\delta)^n(1+2\delta)^n(1+3\delta)^n(1+\alpha)(1+2\alpha)(1+3\alpha)}c(4-c^2)\rho$$
$$+\frac{(4-c^2)}{2(1+3\delta)^n(1+3\alpha)} + \frac{(c-2)(4-c^2)\rho^2}{4(1+3\delta)^n(1+3\alpha)} = F(c,\rho).$$

$$\frac{\partial F}{\partial \rho} = \frac{\{(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha)-(1+3\delta)^n(1+3\alpha)\}c(4-c^2)}{2(1+\delta)^n(1+2\delta)^n(1+3\delta)^n(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(c-2)(4-c^2)\rho}{2(1+3\delta)^n(1+3\alpha)} > 0.$$

Now $F(\rho) \leq F(1)$ and

$$F(c,1) = \frac{\{4(1+3\delta)^n(1+3\alpha)-3(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha)\}c^3}{4(1+\delta)^n(1+2\delta)^n(1+3\delta)^n(1+\alpha)(1+2\alpha)(1+3\alpha)}$$
$$+\frac{8\{(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha)-(1+3\delta)^n(1+3\alpha)\}c}{4(1+\delta)^n(1+2\delta)^n(1+3\delta)^n(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(4c-c^3)}{4(1+3\delta)^n(1+3\alpha)} = G(c).$$

$$G'(c) = \frac{3\{4(1+3\delta)^n(1+3\alpha) - 3(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha)\}c^2}{4(1+\delta)^n(1+2\delta)^n(1+3\delta)^n(1+\alpha)(1+2\alpha)(1+3\alpha)}$$
$$+\frac{8\{(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha) - (1+3\delta)^n(1+3\alpha)\}}{4(1+\delta)^n(1+2\delta)^n(1+3\delta)^n(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(4-3c^2)}{4(1+3\delta)^n(1+3\alpha)}.$$

$G''(c) = 0$ gives,

$$c = \sqrt{\frac{3(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha) - 2(1+3\delta)^n(1+3\alpha)}{3\{(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha) - (1+3\delta)^n(1+3\alpha)\}}} = c_0.$$

Since $G''(c_0) < 0$, so max. $G(c) = G(c_0)$ and hence the result (2.5) is obvious.
The bound is sharp for the function

$$f(z) = z + \frac{c_0}{(1+\alpha)(1+\delta)^2}z^2 + \frac{c_0^2 - 2}{(1+2\alpha)(1+2\delta)^3}z^3 + \frac{c_0(c_0^2-3)}{(1+3\alpha)(1+3\delta)^4}z^4 + \ldots$$

□

For $\delta = 1, n = 0$, Theorem 2.4 agrees with the following result due to Singh et al. [32]:

Corollary 2.7. *If $f \in \mathcal{R}(\alpha)$, then*

$$|a_2a_3 - a_4| \leq \begin{cases} 2 & \text{if } \alpha = 0, \\ \dfrac{2(1+3\alpha+6\alpha^2)^{\frac{3}{2}}}{3\sqrt{6}\alpha(1+\alpha)(1+2\alpha)(1+3\alpha)} & \text{if } 0 < \alpha \leq 1. \end{cases}$$

For $\delta = 1, n = 1$, Theorem 2.4 gives the following result due to Singh et al. [32]:

Corollary 2.8. *If $f \in \mathcal{R}'(\alpha)(0 \leq \alpha \leq \frac{1}{2})$, then*

$$|a_2a_3 - a_4| \leq \frac{(5+15\alpha+18\alpha^2)^{\frac{3}{2}}}{18(1+\alpha)(1+2\alpha)(1+3\alpha)\sqrt{3(1+3\alpha+6\alpha^2)}}.$$

For $\delta = 1, n = 0, \alpha = 1$, Theorem 2.4 coincides with the following result due to Babalola [5]:

Corollary 2.9. *If $f \in \mathcal{R}$, then*

$$|a_2a_3 - a_4| \leq \frac{5\sqrt{5}}{18\sqrt{3}}.$$

Theorem 2.5. *If $f \in \mathcal{R}(\delta; n; \alpha)$, then*

$$|H_3(1)| \leq \frac{4}{(1+2\delta)^n(1+2\alpha)}\left[\frac{2}{(1+2\delta)^{2n}(1+2\alpha)^2} + \frac{1}{(1+4\delta)^n(1+4\alpha)}\right.$$
$$\left.+\frac{[3(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha)-2(1+3\delta)^n(1+3\alpha)]^{\frac{3}{2}}}{3(1+\delta)^n(1+3\delta)^{2n}(1+\alpha)(1+3\alpha)^2\sqrt{3[(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha)-(1+3\delta)^n(1+3\alpha)]}}\right].$$

The result is sharp.

Proof. Using Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4 in (1.2), the result is obvious. The bound is sharp for the function

$$f(z) = z + \frac{c_0}{(1+\alpha)(1+\delta)^2}z^2 + \frac{c_0^2-2}{(1+2\alpha)(1+2\delta)^3}z^3 + \frac{c_0(c_0^2-3)}{(1+3\alpha)(1+3\delta)^4}z^4 + \frac{c_0^4-4c_0^2+2}{(1+4\alpha)(1+4\delta)^5}z^5 + \ldots$$

□

For $\delta = 1, n = 0$, Theorem 2.5 gives the following result due to Singh et al. [32]:

Corollary 2.10. *If $f \in \mathcal{R}(\alpha)$, then*

$$|H_3(1)| \leq \begin{cases} 16 & \text{if } \alpha = 0, \\ \dfrac{4}{1+2\alpha}\left[\dfrac{2}{(1+2\alpha)^2} + \dfrac{1}{1+4\alpha} + \dfrac{(1+3\alpha+6\alpha^2)^{\frac{3}{2}}}{3\sqrt{6}\alpha(1+\alpha)(1+3\alpha)^2}\right] & \text{if } 0 < \alpha \leq 1. \end{cases}$$

For $\delta = 1, n = 1$, Theorem 2.5 coincides with the following result due to Singh et al. [32]:

Corollary 2.11. *If $f \in \mathcal{R}'(\alpha)(0 \leq \alpha \leq \frac{1}{2})$, then*

$$|H_3(1)| \leq \frac{1}{3(1+2\alpha)}\left[\frac{8}{9(1+2\alpha)^2}+\frac{4}{5(1+4\alpha)}+\frac{(5+15\alpha+18\alpha^2)^{\frac{3}{2}}}{12(1+\alpha)(1+3\alpha)^2\sqrt{3(1+3\alpha+6\alpha^2)}}\right].$$

For $\delta = 1, n = 0, \alpha = 1$, Theorem 2.5 agrees with the following result due to Babalola [5]:

Corollary 2.12. *If $f \in \mathcal{R}$, then*

$$|H_3(1)| \leq 0.7423.$$

Theorem 2.6. *If $f \in \mathcal{R}(\delta; n; \alpha)$, then*

$$|H_4(1)| \leq \frac{8}{(1+2\delta)^n(1+6\delta)^n(1+2\alpha)(1+6\alpha)}\left[\frac{2}{(1+2\delta)^{2n}(1+2\alpha)^2}+\frac{1}{(1+4\delta)^{2n}(1+4\alpha)}\right. \tag{2.6}$$
$$\left.+\frac{[3(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha)-2(1+3\delta)^n(1+3\alpha)]^{\frac{3}{2}}}{3(1+\delta)^n(1+3\delta)^{2n}(1+\alpha)(1+3\alpha)^2\sqrt{3[(1+\delta)^n(1+2\delta)^n(1+\alpha)(1+2\alpha)-(1+3\delta)^n(1+3\alpha)]}}\right]$$
$$+\frac{2}{(1+5\delta)^n(1+5\alpha)}p(\delta,\alpha)+\frac{2}{(1+4\delta)^n(1+4\alpha)}q(\delta,\alpha)+\frac{2}{(1+3\delta)^n(1+3\alpha)}r(\delta,\alpha),$$

where

$$p(\delta,\alpha) = 4\left[\frac{1}{(1+\delta)^{2n}(1+5\delta)^n(1+\alpha)^2(1+5\alpha)}+\frac{1}{(1+2\delta)^{2n}(1+3\delta)^n(1+3\alpha)(1+2\alpha)^2}\right. \tag{2.7}$$
$$\left.+\frac{1}{(1+\delta)^n(1+3\delta)^{2n}(1+\alpha)(1+3\alpha)^2}\right]+\frac{29}{4(1+\delta)^n(1+2\delta)^n(1+4\delta)^n(1+\alpha)(1+2\alpha)(1+4\alpha)},$$

$$q(\delta,\alpha) = 4\left[\frac{63}{50(1+\delta)^n(1+2\delta)^n(1+5\delta)^n(1+\alpha)(1+2\alpha)(1+5\alpha)}\right. \tag{2.8}$$
$$\left.+\frac{9}{5(1+2\delta)^{2n}(1+4\delta)^n(1+4\alpha)(1+2\alpha)^2}+\frac{76}{75(1+2\delta)^n(1+3\delta)^{2n}(1+2\alpha)(1+3\alpha)^2}\right]$$

and

$$r(\delta,\alpha) = 4\left[\frac{1}{(1+2\delta)^{2n}(1+5\delta)^n(1+2\alpha)^2(1+5\alpha)}\right. \tag{2.9}$$
$$+\frac{1}{(1+\delta)^n(1+3\delta)^n(1+5\delta)^n(1+\alpha)(1+3\alpha)(1+5\alpha)}+\frac{2}{(1+3\delta)^{3n}(1+3\alpha)^3}+\frac{1}{(1+\delta)^n(1+4\delta)^{2n}(1+\alpha)(1+4\alpha)^2}$$
$$\left.+\frac{17}{16(1+2\delta)^n(1+3\delta)^n(1+4\delta)^n(1+2\alpha)(1+3\alpha)(1+4\alpha)}\right]$$
$$+\frac{1}{(1+\delta)^n(1+2\delta)^{2n}(1+3\delta)^n(1+4\delta)^{2n}(1+5\delta)^n(1+\alpha)(1+2\alpha)^2(1+3\alpha)(1+4\alpha)^2(1+5\alpha)}.$$

Proof. Using (2.2) in (1.4), (1.5) and (1.6), it gives

$$D_1 = \frac{c_2c_5}{(1+2\delta)^n(1+5\delta)^n(1+2\alpha)(1+5\alpha)}-\frac{c_3c_4}{(1+3\delta)^n(1+4\delta)^n(1+3\alpha)(1+4\alpha)} \tag{2.10}$$
$$-\frac{c_1^2c_5}{(1+\delta)^{2n}(1+5\delta)^n(1+\alpha)^2(1+5\alpha)}+\frac{c_1c_2c_4}{(1+\delta)^n(1+2\delta)^n(1+4\delta)^n(1+\alpha)(1+2\alpha)(1+4\alpha)}$$
$$+\frac{c_1c_3^2}{(1+\delta)^n(1+3\delta)^n(1+\alpha)(1+3\alpha)^2}-\frac{c_3c_2^2}{(1+2\delta)^{2n}(1+3\delta)^n(1+3\alpha)(1+2\alpha)^2},$$

$$D_2 = \frac{c_3c_5}{(1+3\delta)^n(1+5\delta)^n(1+3\alpha)(1+5\alpha)}-\frac{c_4^2}{(1+4\delta)^{2n}(1+4\alpha)^2} \tag{2.11}$$

$$-\frac{c_1c_2c_5}{(1+\delta)^n(1+2\delta)^n(1+5\delta)^n(1+\alpha)(1+2\alpha)(1+5\alpha)}+\frac{c_1c_3c_4}{(1+\delta)^n(1+3\delta)^n(1+4\delta)^n(1+\alpha)(1+3\alpha)(1+4\alpha)}$$
$$+\frac{c_4c_2^2}{(1+2\delta)^{2n}(1+4\delta)^n(1+2\alpha)^2(1+4\alpha)}-\frac{c_2c_3^2}{(1+2\delta)^n(1+3\delta)^{2n}(1+2\alpha)(1+3\alpha)^2}$$

and

$$D_3=\frac{c_1c_3c_5}{(1+\delta)^n(1+3\delta)^n(1+5\delta)^n(1+\alpha)(1+3\alpha)(1+5\alpha)} \tag{2.12}$$
$$-\frac{c_1c_4^2}{(1+\delta)^n(1+4\delta)^{2n}(1+\alpha)(1+4\alpha)^2}-\frac{c_2^2c_5}{(1+2\delta)^{2n}(1+5\delta)^n(1+2\alpha)^2(1+5\alpha)}$$
$$+\frac{2c_2c_3c_4}{(1+2\delta)^n(1+3\delta)^n(1+4\delta)^n(1+2\alpha)(1+3\alpha)(1+4\alpha)}-\frac{c_3^3}{(1+3\delta)^{3n}(1+3\alpha)^3}.$$

On rearranging the terms in (2.10), (2.11) and (2.12), it yields

$$D_1=\frac{c_5(c_2-c_1^2)}{(1+\delta)^{2n}(1+5\delta)^n(1+\alpha)^2(1+5\alpha)}+\frac{c_3(c_4-c_2^2)}{(1+2\delta)^{2n}(1+3\delta)^n(1+3\alpha)(1+2\alpha)^2} \tag{2.13}$$
$$-\frac{c_3(c_4-c_1c_3)}{(1+\delta)^n(1+3\delta)^{2n}(1+\alpha)(1+3\alpha)^2}-\frac{67c_4(c_3-c_1c_2)}{48(1+\delta)^n(1+2\delta)^n(1+4\delta)^n(1+\alpha)(1+2\alpha)(1+4\alpha)}$$
$$+\frac{19c_2(c_5-c_1c_4)}{48(1+\delta)^{2n}(1+2\delta)^n(1+4\delta)^n(1+\alpha)(1+2\alpha)(1+4\alpha)}$$
$$+\frac{c_2c_5}{48(1+\delta)^n(1+2\delta)^n(1+4\delta)^n(1+\alpha)(1+2\alpha)(1+4\alpha)},$$

$$D_2=\frac{c_5(c_3-c_1c_2)}{(1+\delta)^n(1+2\delta)^n(1+5\delta)^n(1+\alpha)(1+2\alpha)(1+5\alpha)} \tag{2.14}$$
$$-\frac{c_4(c_4-c_2^2)}{(1+2\delta)^n(1+4\delta)^n(1+4\alpha)(1+2\alpha)^2}$$
$$+\frac{c_3(c_5-c_2c_3)}{(1+2\delta)^n(1+3\delta)^{2n}(1+2\alpha)(1+3\alpha)^2}-\frac{4c_4(c_4-c_1c_3)}{5(1+2\delta)^n(1+3\delta)^{2n}(1+4\alpha)(1+2\alpha)^2}$$
$$-\frac{13c_3(c_5-c_1c_4)}{50(1+\delta)^n(1+2\delta)^n(1+5\delta)^n(1+\alpha)(1+2\alpha)(1+5\alpha)}+\frac{c_3c_5}{75(1+2\delta)^n(1+3\delta)^{2n}(1+2\alpha)(1+3\alpha)^2}$$

and

$$D_3=\frac{c_5(c_4-c_2^2)}{(1+2\delta)^{2n}(1+5\delta)^n(1+2\alpha)^2(1+5\alpha)} \tag{2.15}$$
$$-\frac{c_5(c_4-c_1c_3)}{(1+\delta)^n(1+3\delta)^n(1+5\delta)^n(1+\alpha)(1+3\alpha)(1+5\alpha)}$$
$$+\frac{c_3(c_6-c_3^2)}{(1+3\delta)^{3n}(1+3\alpha)^3}-\frac{c_3(c_6-c_2c_4)}{(1+3\delta)^{3n}(1+3\alpha)^3}+\frac{c_4(c_5-c_1c_4)}{(1+\delta)^n(1+4\delta)^{2n}(1+\alpha)(1+4\alpha)^2}$$
$$-\frac{17c_4(c_5-c_2c_3)}{16(1+2\delta)^n(1+3\delta)^n(1+4\delta)^{3n}(1+2\alpha)(1+3\alpha)(1+4\alpha)}$$
$$+\frac{c_4c_5}{4(1+\delta)^n(1+2\delta)^{2n}(1+3\delta)^n(1+4\delta)^n(1+5\delta)^n(1+\alpha)(1+2\alpha)^2(1+3\alpha)(1+4\alpha)^2(1+5\alpha)}.$$

Using Lemma 2.1 and applying triangle inequality in (2.13), (2.14) and (2.15), we obtain

$$|D_1|\leq p(\delta,\alpha), \tag{2.16}$$

$$|D_2|\leq q(\delta,\alpha), \tag{2.17}$$

and

$$|D_3|\leq r(\delta,\alpha), \tag{2.18}$$

where $p(\delta,\alpha)$, $q(\delta,\alpha)$ and $r(\delta,\alpha)$ are defined in (2.7), (2.8) and (2.9) respectively.
Hence, using Theorem 2.1, Theorem 2.5, (2.16), (2.17) and (2.18) and applying triangle inequality in (1.3), the result (2.6) is obvious.

□

For $\delta = 1, n = 0$, Theorem 2.6 gives the following result due to Singh et al. [32]:

Corollary 2.13. *If $f \in \mathcal{R}(\alpha)$, then*

$$|H_4(1)| \leq \begin{cases} 152.0866 & \text{if } \alpha = 0, \\ \dfrac{8}{(1+2\alpha)(1+6\alpha)}\left[\dfrac{2}{(1+2\alpha)^2}+\dfrac{1}{1+4\alpha}+\dfrac{(1+3\alpha+6\alpha^2)^{\frac{3}{2}}}{3\sqrt{6}\alpha(1+\alpha)(1+3\alpha)^2}\right] & \\ +\dfrac{2}{(1+5\alpha)}p(\alpha)+\dfrac{2}{(1+4\alpha)}q(\alpha)+\dfrac{2}{(1+3\alpha)}r(\alpha) & \text{if } 0 < \alpha \leq 1, \end{cases}$$

where

$$p(\alpha) = 4\left[\frac{1}{(1+\alpha)^2(1+5\alpha)}+\frac{1}{(1+3\alpha)(1+2\alpha)^2}+\frac{1}{(1+\alpha)(1+3\alpha)^2}\right]+\frac{29}{4(1+\alpha)(1+2\alpha)(1+4\alpha)},$$

$$q(\alpha) = 4\left[\frac{63}{50(1+\alpha)(1+2\alpha)(1+5\alpha)}+\frac{9}{5(1+4\alpha)(1+2\alpha)^2}+\frac{76}{75(1+2\alpha)(1+3\alpha)^2}\right]$$

and

$$r(\alpha) = 4\left[\frac{1}{(1+2\alpha)^2(1+5\alpha)}+\frac{1}{(1+\alpha)(1+3\alpha)(1+5\alpha)}+\frac{2}{(1+3\alpha)^3}+\frac{1}{(1+\alpha)(1+4\alpha)^2}\right]$$
$$+\frac{68}{16(1+2\alpha)(1+3\alpha)(1+4\alpha)}+\frac{1}{(1+\alpha)(1+2\alpha)^2(1+3\alpha)(1+4\alpha)^2(1+5\alpha)}.$$

For $\delta = 1, n = 1$, Theorem 2.6 agrees with the following result due to Singh et al. [32]:

Corollary 2.14. *If $f \in \mathcal{R}'(\alpha)(0 \leq \alpha \leq \frac{1}{2})$, then*

$$|H_4(1)| \leq \frac{2}{21(1+2\alpha)(1+6\alpha)}\left[\frac{8}{9(1+2\alpha)^2}+\frac{4}{5(1+4\alpha)}+\frac{(5+15\alpha+18\alpha^2)^{\frac{3}{2}}}{12(1+\alpha)(1+3\alpha)^2\sqrt{3(1+3\alpha+6\alpha^2)}}\right]$$
$$+\frac{1}{3(1+5\alpha)}u(\alpha)+\frac{2}{5(1+4\alpha)}v(\alpha)+\frac{1}{2(1+3\alpha)}w(\alpha),$$

where

$$u(\alpha) = \frac{1}{6(1+\alpha)^2(1+5\alpha)}+\frac{1}{9(1+3\alpha)(1+2\alpha)^2}+\frac{1}{8(1+\alpha)(1+3\alpha)^2}+\frac{29}{120(1+\alpha)(1+2\alpha)(1+4\alpha)},$$

$$v(\alpha) = \frac{7}{50(1+\alpha)(1+2\alpha)(1+5\alpha)}+\frac{4}{25(1+4\alpha)(1+2\alpha)^2}+\frac{19}{225(1+2\alpha)(1+3\alpha)^2}$$

and

$$w(\alpha) = \frac{2}{27(1+2\alpha)^2(1+5\alpha)}+\frac{1}{12(1+\alpha)(1+3\alpha)(1+5\alpha)}+\frac{1}{8(1+3\alpha)^3}+\frac{2}{25(1+\alpha)(1+4\alpha)^2}$$
$$+\frac{17}{240(1+2\alpha)(1+3\alpha)(1+4\alpha)}+\frac{1}{10800(1+\alpha)(1+2\alpha)^2(1+3\alpha)(1+4\alpha)^2(1+5\alpha)}.$$

For $\delta = 1, n = 0, \alpha = 1$, Theorem 2.6 coincides with the following result due to Singh et al. [32]:

Corollary 2.15. *If $f \in \mathcal{R}$, then*

$$|H_4(1)| \leq 0.7973.$$

3. Conclusion

In this paper, we have defined a unified class of analytic functions with the help of generalized Sãlãgean operator and established the upper bound of the fourth Hankel determinant for this class. Many known results follow as special cases by giving particular values to the parameters involved in the results of this paper. Till now, only a few researchers became successful in establishing the bound for the fourth Hankel determinant of some standard classes and no one has studied the fourth Hankel determinant for a unified class. So, this paper will pave the way for other researchers to study the fourth Hankel determinant problems for some more unifying classes.

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