Hesitant Fuzzy Maximal and Minimal Clopen Sets

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ABSTRACT. This article is to study the notions of hesitant fuzzy minimal clopen and hesitant fuzzy maximal clopen sets. In this article, we have proved hesitant fuzzy maximal and minimal clopen sets are independent to hesitant fuzzy minimal and maximal open sets (resp. closed sets). Using hesitant fuzzy disconnectedness certain properties of both hesitant fuzzy minimal and maximal clopen sets are discussed.

1. INTRODUCTION


In this paper, we have defined both hesitant fuzzy minimal (resp. maximal) clopen sets of hesitant fuzzy topological space and study certain properties of it.

2. PRELIMINARIES

Definition 2.1. [5] A hesitant fuzzy set $h$ in $X$ is a function $h : X \rightarrow P[0,1]$, where $P[0,1]$ represents the power set of $[0,1]$.

We define the hesitant fuzzy empty set $h^0$ (resp. whole set $h^1$) is a hesitant fuzzy set in $X$ as follows: $h^0(x) = \phi$ (resp. $h^1(x) = [0,1]$), $\forall x \in X$. $HS(X)$ stands for collection of hesitant fuzzy set in $X$.

Definition 2.2. [3] Two hesitant fuzzy set $h_1, h_2 \in HS(X)$ such that $h_1(x) < h_2(x), \forall x \in X$, then $h_1$ is contained in $h_2$.

Definition 2.3. [3] Two hesitant fuzzy set $h_1$ and $h_2$ of $X$ are said to be equal if $h_1 < h_2$ and $h_2 < h_1$.

Definition 2.4. [5] Let $h \in HS(X)$ for any nonempty set $X$. Then $h^c$ is the complement of $h$ which is hesitant fuzzy set in $X$ such that $h^c(x) = [h(x)]^c = [0,1] \setminus h(x)$.

Definition 2.5. [6] Let $X$ be a nonempty set. A hesitant fuzzy topology $\tau$ of subsets $X$ is said to be hesitant fuzzy topology on $X$ if

(i) $h_0, h^1 \in \tau$.
(ii) $\bigcup_{i \in J} h_i \in \tau$ for each $(h_i)_{i \in J} \in \tau$.
(iii) $h_1 \cap h_2 \in \tau$ for any $h_1, h_2 \in \tau$.

“The pair $(X, \tau)$ is called hesitant fuzzy topology. The members of $\tau$ are called hesitant fuzzy open sets in $X$. A hesitant fuzzy set $h$ in $X$ is hesitant fuzzy closed set (in short hesitant fuzzy closed) in $(X, \tau)$ if $h^c \in \tau$.”
Definition 2.6. ([10]) A proper hesitant fuzzy open set \( h_1 \) of \( X \) is an hesitant fuzzy maximal open set if \( h_2 \) is an hesitant fuzzy open set such that \( h_1 < h_2 \), then \( h_1 = h_2 \) otherwise \( h_2 = h_1 \).

Definition 2.7. ([10]) A proper hesitant fuzzy open set \( h_1 \) of \( X \) is an hesitant fuzzy minimal open set if \( h_2 \) is an hesitant fuzzy open set such that \( h_2 < h_1 \), then \( h_1 = h_2 \) otherwise \( h_2 = h_0 \).

Definition 2.8. ([10]) A proper hesitant fuzzy closed set \( h_1 \) of \( X \) is an hesitant fuzzy maximal closed set if \( h_2 \) is an hesitant fuzzy closed set such that \( h_2 < h_1 \), then \( h_1 = h_2 \) otherwise \( h_2 = h_0 \).

Definition 2.9. ([10]) A proper hesitant fuzzy closed set \( h_1 \) of \( X \) is an hesitant fuzzy maximal closed set if \( h_2 \) is an hesitant fuzzy closed set such that \( h_1 < h_2 \), then \( h_1 = h_2 \) otherwise \( h_2 = h_1 \).

Theorem 2.1. ([10]) Let \( h_1 \) and \( h_2 \) be hesitant fuzzy minimal open and hesitant fuzzy maximal open sets respectively in a hesitant fuzzy topology \( X \) with \( h_2 \nsubseteq h_1 \) then \( h_1 = h^1 - h_2 \).

3. HESITANT FUZZY MINIMAL AND MAXIMAL CLOPEN SETS

We now introduce hesitant fuzzy minimal clopen and maximal clopen sets:

Definition 3.10. A proper hesitant fuzzy clopen \( h_1 \) of a hesitant fuzzy topology \( X \) is said to be hesitant fuzzy minimal clopen set if any other hesitant fuzzy clopen set contained in \( h_1 \) is either \( h_0 \) or itself.

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Example 3.1. Let \( X = \{a, b, c\} \) and \( \tau = \{h^0, h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h^1\} \) where

\[
\begin{align*}
    h_1(a) &= 0.4, \quad h_1(b) = 0.8, \quad h_1(c) = 0.3, \\
    h_2(a) &= (0.4, 1) \vee \{0\}, \quad h_2(b) = [0.8, 1], \quad h_2(c) = [0, 0.3] \vee (0.9, 1) \\
    h_3(a) &= (0.3, 1) \vee \{0\}, \quad h_3(b) = (0, 0.7), \quad h_3(c) = (0.3, 0.9) \\
    h_4(a) &= (0, 0.3) \vee \{0\}, \quad h_4(b) = [0.7, 1], \quad h_4(c) = [0, 0.3] \vee (0.9, 1) \\
    h_5(a) &= (0, 0.3) \vee \{0\}, \quad h_5(b) = (0, 1), \quad h_5(c) = (0.3, 1) \\
    h_6(a) &= (0, 0.3) \vee \{0.4, 1\}, \quad h_6(b) = (0, 0.7) \vee [0.8, 1], \quad h_6(c) = (0.3, 1) \\
    h_7(a) &= (0, 0.4), \quad h_7(b) = (0, 0.8), \quad h_7(c) = (0, 0.9) \\
    h_8(a) &= (0, 0.3), \quad h_8(b) = (0, 0.7), \quad h_8(c) = (0, 0.9) \\
    h_9(a) &= (0, 0.3) \vee \{0.4, 1\}, \quad h_9(b) = (0, 0.7) \vee [0.8, 1], \quad h_9(c) = (0, 1).
\end{align*}
\]

Here \( \{h_1, h_2, h_3, h_4\} \) are hesitant fuzzy clopen sets. Obviously \( h_3 \) is hesitant fuzzy minimal clopen set and \( h_4 \) is hesitant fuzzy maximal clopen set. Also, \( h_3 \) is hesitant fuzzy minimal open and \( h_5 \) is hesitant fuzzy maximal open. Hence, hesitant fuzzy minimal closed, hesitant fuzzy maximal closed, hesitant fuzzy minimal open, hesitant fuzzy maximal open, and hesitant fuzzy minimal clopen, hesitant fuzzy maximal clopen notions are independent.

It is evident that a hesitant fuzzy maximal open or a hesitant fuzzy maximal closed may not be a hesitant fuzzy maximal clopen set. Therefore, the notion of hesitant fuzzy maximal clopen sets is independent to the notions of hesitant fuzzy maximal open and hesitant fuzzy maximal closed as well as sets. It is also easy to see that if \( h_4 \) is both hesitant fuzzy maximal open and hesitant fuzzy maximal closed, then \( h_4 \) is hesitant fuzzy
maximal clopen. In fact, a hesitant fuzzy clopen set is hesitant fuzzy maximal clopen if it is either hesitant fuzzy maximal open or hesitant fuzzy maximal closed. In [10], we observed that if a hesitant fuzzy topology with a single proper hesitant fuzzy open set \( h_1 \), then \( h_1 \) is both hesitant fuzzy maximal open and hesitant fuzzy minimal open. Also, we have observe \( h_1 \) is neither hesitant fuzzy maximal closed nor hesitant fuzzy minimal closed. If \( h_1, h_2 \) are the only proper hesitant fuzzy open sets such that one is not contained in other, then both are hesitant fuzzy maximal closed, hesitant fuzzy minimal closed sets. In addition, hesitant fuzzy maximal closed or hesitant fuzzy minimal closed sets can exist only in a hesitant fuzzy disconnected space. Clearly theorems 3.2 to corollary 3.2 are obvious, the proofs of them are omitted.

**Theorem 3.2.** Let \( h_1 \) be a hesitant fuzzy minimal clopen set. Then either \( h_1 \land h_2 = h^0 \) or \( h_1 < h_2 \) for any hesitant fuzzy clopen set \( h_2 \) in \( X \).

**Corollary 3.1.** Let \( h_1, h_2 \) be distinct hesitant fuzzy minimal clopen sets in \( X \). Then \( h_1 \land h_2 = h^0 \).

**Theorem 3.3.** Let \( h_1 \) be a hesitant fuzzy maximal clopen set. Then either \( h_1 \lor h_2 = h^1 \) or \( h_2 < h_1 \) for any hesitant fuzzy clopen set \( h_2 \) in \( X \).

**Corollary 3.2.** Let \( h_1 \) and \( h_2 \) be distinct hesitant fuzzy maximal clopen sets in \( X \). Then \( h_1 \lor h_2 = h^1 \).

**Lemma 3.1.** If \( h_1 \) is hesitant fuzzy minimal clopen in a hesitant fuzzy topology \( (X, \tau) \), then \( h_1^c \) is hesitant fuzzy maximal clopen in \( X \) and conversely.

**Proof.** For any two proper hesitant fuzzy clopen sets \( h_1 \) and \( h_2 \), then \( h_1^c < h_2 \) implies that \( h_1^c < h_1 \). Since \( h_1 \) is hesitant fuzzy minimal clopen set, we have \( h_1^c = h_1 \) otherwise \( h_1^c = h_2 \) which gives \( h_1^c = h_2 \) or \( h_2 = h^1 \). Therefore \( h_1^c \) is a hesitant fuzzy maximal clopen set. Similarly follows the converse. \( \Box \)

**Theorem 3.4.** If \( h_1 \) is a hesitant fuzzy minimal clopen and \( h_2 \) is a hesitant fuzzy maximal clopen set in \( (X, \tau) \), then \( h_1 < h_2 \) or \( h_1 < h_2^c \).

**Proof.** Proof is similar to that of theorem 3.1 in [8]. \( \Box \)

**Theorem 3.5.** If \( h_1 \) is both hesitant fuzzy minimal clopen and hesitant fuzzy maximal clopen of a hesitant fuzzy topology \( (X, \tau) \), then

(i) \( h_1 \) and \( h_1^c \) are the only hesitant fuzzy sets in \( X \) which are both hesitant fuzzy maximal clopen and hesitant fuzzy minimal clopen.

(ii) The only proper hesitant fuzzy clopen sets in \( X \) are \( h_1 \) and \( h_1^c \).

**Proof.** (i) Since lemma 3.1, for any other hesitant fuzzy minimal clopen and hesitant fuzzy maximal clopen set \( h_3 \) in \( X \), \( h_3^c \) is also both hesitant fuzzy maximal clopen and hesitant fuzzy minimal clopen in \( X \). Let \( h_2 \) be any other hesitant fuzzy maximal clopen and hesitant fuzzy minimal clopen distinct from \( h_1 \) in \( X \). By deploying lemma 3.1, \( h_2^c \) is also both hesitant fuzzy maximal clopen and hesitant fuzzy minimal clopen in \( X \). Since \( h_1 \) and \( h_2 \) are hesitant fuzzy minimal clopen and hesitant fuzzy maximal clopen set, by corollary 3.1 and corollary 3.2, we have \( h_1 \land h_2 = h^0 \) and \( h_1 \lor h_2 = (h)^1 \). As for any \( h_1 \neq h_2 \), \( h_2 \) and \( h_2^c \) are identical to \( h_1 \) and \( h_1^c \) respectively. This completes the proof for all possible combinations of \( h_1, h_2, h_1^c, h_2^c \).

(ii) Let \( h_3 \) be a hesitant fuzzy clopen in \( X \). Clearly, either \( h_1 \lor h_3 = h^1 \) or \( h_3 < h_1 \) for a hesitant fuzzy maximal clopen set \( h_1 \). Again, \( h_1 \land h_3 = h^0 \) or \( h_1 < h_3 \) for a hesitant fuzzy minimal clopen set \( h_1 \). Therefore, if \( h_1 \lor h_3 = h^1 \) and \( h_1 \land h_3 = h^0 \), \( h_3 = h_1^c \).

\[
\begin{align*}
&h_1 \lor h_3 = h^1 \quad \text{and} \\
&h_1 \land h_3 = h^0 \quad \Rightarrow \quad h_3 = h_1^c.
\end{align*}
\]

If \( h_1 < h_3 \), \( h_3 = h^1 \) and \( h_3 < h_1 \) implies that \( h_3 = h^0 \). This completes the proof. \( \Box \)
Theorem 3.6. In a hesitant fuzzy topology \((X, \tau)\), hesitant fuzzy maximal clopen and hesitant fuzzy minimal clopen sets appears in pairs.

Proof. By theorem 3.5, if \(h_1\) is both hesitant fuzzy minimal clopen and hesitant fuzzy maximal clopen in \(X\), then \(h_1^c\) is also hesitant fuzzy minimal clopen and hesitant fuzzy maximal clopen and these pairs of sets in \(X\) are unique. By lemma 3.1, if \(h_1^c\) is hesitant fuzzy minimal(resp.max)clopen in \(X\), then \(h_1^c\) is a hesitant fuzzy maximal(resp.min)clopen in \(X\).

\[\square\]

Theorem 3.7. If \(h_1\) is a hesitant fuzzy maximal open and \(h_2\) is a hesitant fuzzy minimal open set of a hesitant fuzzy topology \((X, \tau)\) with \(h_1 \not< h_2\), then \(h_1\) is a hesitant fuzzy maximal clopen and \(h_2\) is a hesitant fuzzy minimal clopen set.

Proof. In accordance with fuzzy maximality of \(h_1\) by theorem 2.1, \(h_1 = (h_2)^c\). Hence \(h_1\) and \(h_2\) are hesitant fuzzy clopen set. Clearly \(h_1\) is hesitant fuzzy maximal clopen because it is both hesitant fuzzy clopen and hesitant fuzzy maximal open set. Similarly we can prove that \(h_1\) is hesitant fuzzy minimal clopen set.

\[\square\]

Theorem 3.8. If \(h_3\) is a hesitant fuzzy maximal clopen in \(X\), then \(h_3 \lor h_4\) is not a proper hesitant fuzzy clopen set distinct from \(h_3\) for any proper hesitant fuzzy open(or hesitant fuzzy closed) set \(h_4\) in \(X\).

Proof. Suppose that \(h_3 \lor h_4\) is a proper hesitant fuzzy clopen set in \(X\). Clearly either \(h_3 \lor h_4 = (h_1^1)\) or \(h_3 \lor h_4 = h_3\) as \(h_3\) is a hesitant fuzzy maximal clopen in \(X\). Hence \(h_4 < h_3\) as \(h_3 \lor h_4 = h_3\).

\[\square\]

Theorem 3.9. If \(h_3\) is hesitant fuzzy minimal clopen in \(X\), then \(h_3 \land h_4\) is not a proper hesitant fuzzy clopen set distinct from \(h_3\) for any proper hesitant fuzzy open(or hesitant fuzzy closed) set \(h_4\) in \(X\).

Proof. Obvious.

According to theorem 3.8 and theorem 3.9, the form of hesitant fuzzy clopen sets in hesitant fuzzy topology contains either hesitant fuzzy maximal clopen or hesitant fuzzy minimal clopen set.

\[\square\]

Theorem 3.10. If \(J\) is a collection of distinct hesitant fuzzy maximal clopen sets and \(h_1 \in J\), then

\[\bigwedge_{h_2 \in J - \{h_1\}} h_2 \neq h_0.\]

Then

\[\bigwedge_{h_2 \in J - \{h_1\}} h_2\]

is a hesitant fuzzy minimal clopen iff \(h_1^c = \bigwedge_{h_2 \in J - \{h_1\}} h_2\).\]

Proof. Obviously, \(h_2^c\) is hesitant fuzzy minimal clopen set as \(h_2 \in J - \{h_1\}\) is a hesitant fuzzy maximal clopen set. By deploying theorem 3.4, \(h_2^c < h_1\). Hence \((\bigwedge_{h_2 \in J - \{h_1\}} h_2)^c < h_1\) implies that \(h_1 = h_1^1\) if \((\bigwedge_{h_2 \in J - \{h_1\}} h_2) = h_0^0\), a contradiction to the assumption that \(h_1\) is a hesitant fuzzy maximal clopen set. Hence \(\bigwedge_{h_2 \in J - \{h_1\}} h_2 \neq h_0^0\).

Conversely let us assume that \(\bigwedge_{h_2 \in J - \{h_1\}} h_2\) is hesitant fuzzy minimal clopen. Consider

\[\bigwedge_{h_2 \in J - \{h_1\}} h_2\] is a hesitant fuzzy clopen set. As \(J\) is finite family of hesitant fuzzy clopen set, then \(\bigwedge_{h_2 \in J - \{h_1\}} h_2\) is a hesitant fuzzy clopen set. Since \(h_1^1 - \bigwedge_{h_2 \in J - \{h_1\}} h_2 < h_1\), we have \(h_1^1 - h_1 < \bigwedge_{h_2 \in J - \{h_1\}} h_2\). As \(h_1\) is hesitant fuzzy maximal clopen set, then \(h_1^1 - h_1\) is
hesitant fuzzy minimal clopen. If \( \bigwedge_{h_2 \in J - \{h_1\}} h_2 \) is a hesitant fuzzy minimal clopen distinct from \( h^1 - h_1 \), then by corollary 3.1 we have \( \left( \bigwedge_{h_2 \in J - \{h_1\}} h_2 \right) \wedge (h^1 - h_1) = h^0 \). This gives that \( \bigwedge_{h_2 \in J - \{h_1\}} h_2 < h_1 \). Therefore \( h^1 - h_1 < \bigwedge_{h_2 \in J - \{h_1\}} h_2 < h_1 \) which is wrong. Hence we obtain \( \bigwedge_{h_2 \in J - \{h_1\}} h_2 \).

**Theorem 3.11.** If \( J \) is a collection of distinct hesitant fuzzy maximal clopen sets and \( h_1 \in J \), then \( \bigvee_{h_2 \in J - \{h_1\}} h_2 \neq h^1 \). Then \( \bigvee_{h_2 \in J - \{h_1\}} h_2 \) is a hesitant fuzzy minimal clopen iff \( h_1 = \bigvee_{h_2 \in J - \{h_1\}} h_2 \).

**Proof.** Proof is similar to theorem 3.10. \( \Box \)

If \( h_3 \) is a hesitant fuzzy clopen in hesitant fuzzy topology \((X, \tau)\), then \( h_1 \wedge h_3 \) is hesitant fuzzy clopen in \((h_1, \tau_{h_1})\). Also, if \( h_1 \) is hesitant fuzzy clopen in \((X, \tau)\) then a hesitant fuzzy clopen set in \((h_1, \tau_{h_1})\) is also hesitant fuzzy clopen in \((X, \tau)\).

**Theorem 3.12.** Let \( h_1, h_4 \) be hesitant fuzzy clopen sets in \( X \) such that \( h_1 \wedge h_4 \neq h^0 \). Then \( h_1 \wedge h_4 \) is a hesitant fuzzy minimal clopen in \((h_1, \tau_{h_1})\) if \( h_4 \) is a hesitant fuzzy minimal clopen in \((X, \tau)\).

**Proof.** Proof is similar to Theorem 4.11 in [8]. \( \Box \)

**REFERENCES**


