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Some fixed point results for almost contraction on orthogonal metric space

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ABSTRACT. In this paper, we consider almost type F-contraction on orthogonal metric space and we establish the existence and uniqueness of fixed point of such mapping. At the end, we give an illustrative example.

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory has been studied by three different aspects: topological, metrical and discrete fixed point theory. Generally, the boundary lines between the three areas were defined by the discovery of three major theorems: Brouwer, Banach and Tarski's fixed point theorems. In 1922, Banach proved a remarkable result called Banach contraction principle. Because of its application, this result has been generalized in many ways. For instance see [1, 4, 5, 6, 7, 9, 10, 12, 14, 18, 19, 22, 24]. One of these generalizations is done using $F-$contraction presented by Wardowski [23].

Let $\mathcal{F}$ be the set of all functions $F : (0, \infty) \to \mathbb{R}$. For the sake of completeness, we will consider the following conditions:

(F1) F is strictly increasing, i.e., for all $a, b \in (0, \infty)$ such that $a < b, F(a) < F(b)$,

(F2) For each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$,

(F3) There exists $s \in (0, 1)$ such that $\lim_{a\to 0^+} a^s F(a) = 0$.

Definition 1.1 ([23]). Let (X, d) be a metric space and $f : X \to X$ be a mapping. Given $F \in \mathcal{F}$, we say that f is F-contraction, if there exists $\tau > 0$ such that

$$x, y \in X,\ d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leq F(d(x, y)). \tag{1.1}$$

In the following, you can see some examples of such functions.

Example 1.1 ([23]). *Let $F_a : (0, \infty) \to \mathbb{R}$ be given by $F_a(a) = \ln a$. It is clear that $F_a \in \mathcal{F}$.*

Example 1.2 ([23]). *Let $F_b : (0, \infty) \to \mathbb{R}$ be given $F_b(b) = b + \ln b$. It is clear that $F_b \in \mathcal{F}$.*

Example 1.3 ([23]). *Let $F_c : (0, \infty) \to \mathbb{R}$ be given $F_c(c) = -\frac{1}{\sqrt{c}}$. It is clear that $F_c \in \mathcal{F}$.*

Theorem 1.1 ([23]). *Let (X, d) be a complete metric space and let $f : X \to X$ be an F-contraction. Then f has an unique fixed point in X.*

Also, many articles on $F-$contraction are available in the literature ([2, 16, 20, 21]).

Motivated by the mentioned works, we modify the concept of $F-$contraction mappings to orthogonal sets and prove some fixed point theorems for almost $F-$contraction in orthogonally complete metric spaces. But first of all, let's remember some information about orthogonal metric spaces.

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Definition 1.2 ([8]). Let X be a non-empty set and $\perp$ be a binary relation defined on X. If binary relation $\perp$ fulfills the following criteria:

$$\exists x_0(\forall y \in X, y \perp x_0) \text{ or } (\forall y \in X, x_0 \perp y),$$

then pair, $(X, \perp)$ known as an orthogonal set. The element x_0 is called an orthogonal element. We denote this O-set or orthogonal set by $(X, \perp)$.

Example 1.4 ([8]). Let X be the set of all peoples in the word. Define the binary relation $\perp$ on X by $x \perp y$ if x can give blood to y. According to the following table, if x_0 is a person such that his (her) blood type is $O-$, then we have $x_0 \perp y$ for all $y \in X$. This means that $(X, \perp)$ is an O-set. In this O-set, x_0 (in definition) is not unique.

$type$	$you\ can\ give\ blood\ to$	$you\ can\ receive\ bood\ from$
$A+$	$A+AB+$	$A+A-O+O-$
$O+$	$O+A+B+AB+$	$O+O-$
$B+$	$B+AB+$	$B+B-O+O-$
$AB+$	$AB+$	$Everyone$
$A-$	$A+A-AB+AB-$	$A-O-$
$O-$	$Everyone$	$O-$
$B-$	$B+B-AB+AB-$	$B-O-$
$AB-$	$AB+AB-$	$O-A-B-AB-$

Example 1.5 ([8]). Let $X = \mathbb{Z}$. Define the binary relation $\perp$ on X by $x \perp y$ if there exists $k \in \mathbb{Z}$ such that $x = ky$. It is easy to see that $0 \perp y$ for all $y \in \mathbb{Z}$. Hence, $(X, \perp)$ is an O-set.

Definition 1.3 ([8]). Let $(X, \perp)$ be an orthogonal set (O-set). Any two elements $x, y \in X$ such that $x \perp y$, then $x, y \in X$ are said to be orthogonally related.

Definition 1.4 ([8]). A sequence $\{x_n\}$ is called an orthogonal sequence (briefly O-sequence) if

$$(\forall n \in \mathbb{N}, x_n \perp x_{n+1}) \text{ or } (\forall n \in \mathbb{N}, x_{n+1} \perp x_n).$$

Definition 1.5 ([8]). *Let $(X, \perp)$ be an orthogonal set and d be a metric on X. Then $(X, \perp, d)$ is called an orthogonal metric space (shortly O-metric space).*

Definition 1.6 ([8]). Let $(X, \perp, d)$ be an orthogonal metric space. Then X is said to be a O-complete if every Cauchy O-sequence is converges in X.

Remark 1.1 ([8]). Every complete metric space is O-complete and the converse is not true.

Definition 1.7 ([8]). Let $(X, \perp, d)$ be an orthogonal metric space. A function $f : X \to X$ is said to be orthogonally continuous ($\perp$-continuous) at x if for each O-sequence $\{x_n\}$ converging to x implies $f(x_n) \to f(x)$ as $n \to \infty$. Also f is $\perp$-continuous on X if f is $\perp$-continuous at every $x \in X$.

It is easy to see that every continuous mapping is $\perp$-continuous. You can see it in [8] Example 3.3.

Definition 1.8 ([8]). Let a pair $(X, \perp)$ be an O-set, where $X(\neq \emptyset)$ be a non-empty set and $\perp$ be a binary relation on set X. A mapping $f : X \to X$ is said to be $\perp$-preserving if $f(x) \perp f(y)$ whenever $x \perp y$ and weakly $\perp$-preserving if $f(x) \perp f(y)$ or $f(y) \perp f(x)$ whenever $x \perp y$.

Remark 1.2 ([8]). It is easy to see that every $\perp$-preserving mapping is weakly $\perp$-preserving. But the converse is not true.

2. Main results

In this section, we give a definition of generalized orthogonal almost type $F-$contraction and we aim to obtain some results on O-complete orthogonal metric space $(X, \perp, d)$.

Definition 2.9. Let $(X, \perp, d)$ be an orthogonal metric space. A mapping $f : X \to X$ is called generalized orthogonal almost $F-$contraction if there are $F \in \mathcal{F}$, $L > 0$ and $\tau > 0$ such that the following condition holds: $\forall x, y \in X$ with $x \perp y$

$$[d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leq F(M(x, y) + LN(x, y)], \tag{2.2}$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\right\}$$

and

$$N(x, y) \quad = \quad \min\{d(x, fy), d(y, fx)\}.$$

Theorem 2.2. *Let $(X, \perp, d)$ be an O-complete orthogonal metric space with an orthogonal elements x_0 and f be a self mapping on X satisfying the following conditions:*
(i) f is $\perp$-preserving,
(ii) f is a generalized orthogonal almost type $F-$contraction,
(iii) f is $\perp$-continuous.
Then, f has a fixed point in X.

Proof. From the definition of the orthogonality, it follows that $x_0 \perp f(x_0)$ or $f(x_0) \perp x_0$. Let

$$x_1 := fx_0, x_2 := fx_1 = f^2x_0, \cdots, x_n := fx_{n-1} = f^n x_0$$

for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n^*} = x_{n^*+1}$ then for some $n^* \in \mathbb{N} \cup \{0\}$, then x_{n^*} is a fixed point of f and so the proof is completed. So, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, we have $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since f is $\perp$ $-$preserving, we have

$$x_n \perp x_{n+1} \text{ or } x_{n+1} \perp x_n.$$

This implies that $\{x_n\}$ is an O-sequence. Since f is a generalized orthogonal almost type $F-$contraction, we have

$$\begin{aligned}
& F(d(x_n, x_{n+1})) \\
= \quad & F(d(fx_{n-1}, fx_n)) \\
\leq \quad & F(M(x_{n-1}, x_n) + LN(x_{n-1}, x_n)) - \tau \\
= \quad & F\left(\begin{array}{l} \max\left\{\begin{array}{l} d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}), d(x_n, fx_n), \\ \frac{d(x_{n-1}, fx_{n+1}) + d(x_n, fx_{n-1})}{2} \end{array}\right\} \\ +L\min\{d(x_{n-1}, fx_{n+1}), d(x_n, fx_{n-1})\} \end{array}\right) - \tau \\
\leq \quad & F(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) - \tau \\
\leq \quad & F(d(x_{n-1}, x_n)) - \tau
\end{aligned} \tag{2.3}$$

for all $n \in \mathbb{N}$. Taking $a_n := d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and using (2.3) we have

$$F(a_n) \leq F(a_{n-1}) - \tau \leq F(a_{n-2}) - 2\tau \leq \cdots \leq F(a_0) - n\tau. \tag{2.4}$$

From (2.4), we get $\lim_{n\to\infty} F(a_n) = -\infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} a_n = 0. \tag{2.5}$$

By the property (F3), there exists $s \in (0,1)$ such that

$$\lim_{n\to\infty} a_n^s F(a_n) = 0. \tag{2.6}$$

By (2.4), the following holds for all $n \in \mathbb{N}$

$$a_n^s F(a_n) - a_n^s F(a_0) \leq -a_n^s n\tau \leq 0. \tag{2.7}$$

Letting $n \to \infty$ in (2.7) and using (2.5) and (2.6) we obtain that

$$\lim_{n\to\infty} n a_n^s = 0. \tag{2.8}$$

From (2.8), there exits $n_1 \in \mathbb{N}$ such that $n a_n^s \leq 1$ for all $n \geq n_1$. So, we have

$$a_n \leq \frac{1}{n^{1/s}}. \tag{2.9}$$

for all $n \geq n_1$. In order to show that $\{x_n\}$ is a Cauchy O-sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (2.9), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= a_n + a_{n+1} + \cdots + a_{m-1} \\ &= \sum_{i=n}^{m-1} a_i \\ &\leq \sum_{i=n}^{m-1} \frac{1}{i^{1/s}}. \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/s}}$, it follows that $\{x_n\}$ is a Cauchy O-sequence in X. Since X is O-complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since f is $\perp$ −continuous, we have

$$fx^* = f(\lim_{n\to\infty} fx_n) = \lim_{n\to\infty} x_{n+1} = x^*$$

and so x^* is a fixed point of f. □

Theorem 2.3. *Let $(X, \perp, d)$ be an O-complete orthogonal metric space with an orthogonal elements x_0 and f be a self mapping on X satisfying the following conditions:*

(i) f is $\perp$ −preserving,

(ii) f is a generalized orthogonal almost type $F-$contraction,

(iii) f is $\perp$ −continuous.

Also f satisfies the following condition: there exist $F_1 \in \mathcal{F}$, $L_1 \geq 0$ and $\tau_1 > 0$ such that $\forall x, y \in X$ with $x \perp y$

$$[d(fx, fy) > 0 \Rightarrow \tau_1 + F_1(d(fx, fy)) \leq F_1(M(x, y) + L_1 d(y, fy)]$$

Then, f has an unique fixed point in X.

Proof. Suppose that, there exist two distinct fixed points x^* and y^*. If $x_n \to y^*$ as $n \to \infty$, we have $x^* = y^*$. If x_n does not convergence to y^* as $n \to \infty$, there is a subsequence $\{x_{n_k}\}$ such that $fx_{n_k} \neq y^*$ for all $k \in \mathbb{N}$. By the choice of x_0 in the first of the proof, we have

$$(x_0 \perp y^*) \text{ or } (y^* \perp x_0).$$

Since f is $\perp$ −preserving and $f^n y^* = y^*$ for all $n \in \mathbb{N}$, we have

$$(f^n x_0 \perp y^*) \text{ or } (y^* \perp f^n x_0)$$

for all $n \in \mathbb{N}$. From (2.2), we get

$$\begin{aligned}
& F(d(f^{n_k}x_0, y^*)) \\
= \; & F(d(f^{n_k}x_0, f^{n_k}y^*)) \\
\leq \; & F(M(f^{n_{k-1}}x_0, f^{n_{k-1}}y^*) + Ld(f^{n_{k-1}}y^*, ff^{n_{k-1}}y^*)) - \tau \\
= \; & F(M(f^{n_{k-1}}x_0, f^{n_{k-1}}y^*)) - \tau \\
= \; & F\left(\max\left\{\begin{array}{l} d(f^{n_{k-1}}x_0, f^{n_{k-1}}y^*), d(f^{n_{k-1}}x_0, ff^{n_{k-1}}x_0), \\ d(f^{n_{k-1}}y^*, ff^{n_{k-1}}y^*), \\ \frac{d(f^{n_{k-1}}x_0, ff^{n_{k-1}}y^*) + d(f^{n_{k-1}}y^*, ff^{n_{k-1}}x_0)}{2} \end{array}\right\}\right) - \tau \\
= \; & F\left(\max\left\{\begin{array}{l} d(f^{n_{k-1}}x_0, f^{n_{k-1}}y^*), d(f^{n_{k-1}}x_0, f^{n_k}x_0), \\ \frac{d(f^{n_{k-1}}x_0, f^{n_k}y^*) + d(f^{n_{k-1}}y^*, f^{n_k}x_0)}{2} \end{array}\right\}\right) - \tau \\
\leq \; & F\left(d(f^{n_{k-1}}x_0, f^{n_{k-1}}y^*)\right) - \tau \\
& \vdots \\
\leq \; & F(d(x_0, y^*)) - n_k\tau
\end{aligned}$$

for all $n \in \mathbb{N}$. Thus $F(d(f^{n_k}x_0, y^*)) \to -\infty$ as $k \to \infty$ and so $d(f^{n_k}x_0, y^*) \to 0$ as $k \to \infty$. This yields that $x_n \to y^*$as $n \to \infty$, which is a contradiction. Hence f has an unique fixed point. □

Example 2.6. Let $X = [0, \infty)$ and $d : X \times X \to [0, \infty)$ be a mapping defined by

$$d(x, y) = |x - y|$$

for all $x, y \in X$. Consider the sequence $\{S_k\}_{k\in\mathbb{N}}$ defined as

$$S_k = \frac{k(k+1)}{2}, \forall k \in \mathbb{N} \cup \{0\}.$$

Define a relation $\perp$ on X by

$$x \perp y \Longleftrightarrow xy \in \{x, y\} \subseteq \{S_k\}.$$

Thus $(X, \perp, d)$ is an O-complete metric space. Now, we will define a mapping $f : X \to X$ by

$$fx = \begin{cases} S_0 & if \quad S_0 \leq x \leq S_1 \\ S_{k-1} & if \quad S_k \leq x \leq S_{k+1}, \forall k \geq 1 \end{cases}.$$

Then f is $\perp$ −continuous and X is $\perp$ −preserving. Let $F \in \mathcal{F}$ be a function defined by $F(\alpha) = \alpha + \ln \alpha$ for al $\alpha > 0$. We claim that f is a generalized orthogonal almost type $F-$contraction with $\tau = 1$ and $L = 1$. To see this, we consider the following cases. First, observe that, let $x, y \in X$ with $x \perp y$ and $d(fx, fy) > 0$. Without loss of generality, we may assume that $x < y$. So, $x \in \{S_0, S_1\}$ and $y = S_k$ for some $k \geq 2$. Then,
Case 1. $x = S_0$, $y = S_k$, $k \geq 2$, we have

$$\begin{aligned} & \frac{d(fx, fy)}{M(x, y) + LN(x, y)} e^{d(fx,fy)-(M(x,y)+LN(x,y))} \\ = & \frac{\frac{k^2-k}{2}}{\frac{k^2+k}{2} + \frac{k^2-k}{2}} e^{\frac{k^2-k}{2} - \left(\frac{k^2+k}{2} + \frac{k^2-k}{2}\right)} < e^{-\left(\frac{k^2+k}{2}\right)} < e^{-1}. \end{aligned}$$

Case 2. $x = S_1$, $y = S_k$, $k \geq 2$, we have

$$\begin{aligned} & \frac{d(fx, fy)}{M(x, y) + LN(x, y)} e^{d(fx,fy)-(M(x,y)+LN(x,y))} \\ = & \frac{\frac{k^2-k}{2}}{\frac{k^2+k-2}{2} + \frac{k^2-k-2}{2}} e^{\frac{k^2-k}{2} - \left(\frac{k^2+k-2}{2} + \frac{k^2-k-2}{2}\right)} \\ = & \frac{k^2-k}{2k^2-4} e^{\frac{-k^2-k+4}{2}} < e^{-1}. \end{aligned}$$

Hence, all the conditions of Theorem 2.2 are satisfied and so f has a fixed point. Also, all the contidions of Theorem 2.3 are satisfied with $F_1(\alpha) = \alpha + \ln \alpha$ for al $\alpha > 0$, $\tau_1 = 1$ and $L_1 = 1$, then f has an unique fixed point.

Corollary 2.1. *Let $(X, \perp, d)$ be an O-complete orthogonal metric space with an orthogonal elements x_0 and f be a self mapping on X satisfying the following conditions:*
(i) *f is $\perp$ −preserving,*
(ii) *f is an orthogonal almost $F-$contraction such that*

$$\forall x, y \in X \text{ with } x \perp y \, [d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leq F(d(x, y) + LN(x, y))]$$

(iii) *f is $\perp$ −continuous.*
Then, f has a fixed point in X.

Corollary 2.2. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping such that, for some $\alpha \in (0, 1]$,*

$$d(fx, fy) \leq \alpha d(x, y)$$

for all $x, y \in X$. Then f has an unique fixed point in X.

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