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Durrmeyer variant of q-Favard-Szász operators based on Appell polynomials

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ABSTRACT. Karaisa [Karaisa, A., *Approximation by Durrmeyer type Jakimoski Leviatan operators*, Math. Method. Appl. Sci., DOI: 10.1002/mma.3650 (2015)] introduced the Durrmeyer type variant of Jakimovski-Leviatan operators based on Appell polynomials and studied some approximation properties. The aim of the present paper is to define the q analogue of these operators and establish the rate of convergence for a Lipschitz type space and a Lipschitz type maximal function for the Durrmeyer type variant of these operators. Also, we study the degree of approximation of these operators in a weighted space of polynomial growth and by means of weighted modulus of continuity.

1. INTRODUCTION

For $f \in C^*[0,\infty) := \{f \in C[0,\infty) : |f(x)| < Me^{Ax}, \text{ for some M} > 0, \text{A} \in \mathbb{R}\}$ and $0 \leq \alpha \leq \beta$, Karaisa [7] introduced a Stancu type generalization of the q-Favard-Szász operators as follows:

$$T_{n,t}^{\alpha,\beta}(f;q,x) = \frac{E_q^{-[n]_q t}}{g(1)} \sum_{k=0}^{\infty} \frac{P_k(q;[n]_q t)}{[k]_q!} f\left(x + \frac{[k]_q + \alpha}{[n]_q + \beta}\right),$$

where $P_k(q;.)$ for each k is a q Appell polynomial generated by

$$g(u)e_q^{[n]_q tu} = \sum_{k=0}^{\infty} \frac{P_k(q;[n]_q t)u^k}{[k]_q!}$$

and $g(u)$ is defined by

$$g(u) = \sum_{k=0}^{\infty} a_k u^k$$

and studied Korovkin-type statistical approximation properties and rate of convergence using modulus of continuity. He also obtained some local approximation results for these operators.

For a real valued bounded and continuous function on $[0,\infty)$, Karaisa [8] proposed a Durrmeyer type variant of Jakimovski Leviatan operators as follows:

$$L_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{P_k(nx)}{B(n+1,k)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f(t)dt, \ x \geq 0. \tag{1.1}$$

and studied some direct theorems.

For a given $\gamma > 0$, let $C_\gamma[0,\infty) := \{f \in C[0,\infty) : |f(t)| \leq K_f(1+t^\gamma), \ as\ t \to \infty\}$,

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endowed with the norm $||f||_\gamma = \sup_{0\leq x<\infty} \frac{|f(x)|}{(1+x^\gamma)}$, then for a function $f \in C_\gamma[0,\infty)$, we define the q analogue of the operators (1.1) as follows:

$$L_n(f;x) = \frac{E_q^{-[n]_q x}}{g(1)} \sum_{k=1}^{\infty} \frac{P_k(q;[n]_q x)}{[k]_q! B_q(n+1,k)} q^{\frac{k(k-1)}{2}} \int_0^\infty \frac{t^{k-1}}{(1+t)_q^{n+k+1}} f(q^k t) d_q t + \frac{E_q^{-[n]_q x}}{g(1)} a_0 f(0),\ x \geq 0, \tag{1.2}$$

and obtain the rate of convergence in terms of the weighted modulus of continuity and a Lipschitz type maximal function for these operators. Also, we study the rate of approximation of these operators in a weighted space.

2. Preliminaries

Lemma 2.1. *For the operators* (1.2)*, the estimates of moments are obtained as follows:*

(i) $L_n(1;q,x) = 1$*;*

(ii) $L_n(t;q,x) = x + R\frac{D_q g(1)}{[n]_q}$*;*

(iii) $L_n(t^2;q,x) = \frac{q[n]_q x^2}{[n-1]_q} + \frac{1}{q[n-1]_q}\left((1+q) + Rq^2 D_q g(1) + Rq^3 D_q g(q)\right)x + \frac{R([2]_q D_q g(1) + q^2 D_q^2 g(1))}{q[n-1]_q [n]_q}$,

where $R = \frac{E_q^{-[n]_q x} e_q^{q[n]_q x}}{g(1)}$.

Remark 2.1. From part (ii) of Lemma 2.1, $L_n(t;q,0) = R\frac{D_q g(1)}{[n]_q}$, which implies that $D_q(1) \geq 0$, since $R > 0$ and $L_n(f;q,x)$ is a linear positive operator.

By a simple computation and reasoning, it follows

Corollary 2.1. *We have*

$$L_n((t-x)^2;q,x) \leq \frac{C}{q[n-1]_q}\left(\phi^2(x) + \frac{1}{[n]_q}\right),$$

where C is independent of x and $\phi(x) = \sqrt{x(x+1)}$. Through this paper, let

$$L_n((t-x)^2;q,x) = \gamma_{n,q}(x)$$

and C denotes a constant not necessarily the same at each occurrence.

3. Main results

Theorem 3.1. *Let $0 < q_n < 1$ and $A > 0$.Then for each $f \in C_\gamma[0,\infty)$, the sequence $L_{n,q_n}(f;x)$ converges to f uniformly on $[0,A]$ if and only if $\lim_{n\to\infty} q_n = 1$.*

Proof. The proof of the theorem follows along the lines of the proof of Theorem 1 in [1]. Hence the details are omitted. □

Now the Lipschitz-type space [10] is defined as:

$$Lip_M^*(r) := \left\{ f \in C[0,\infty) : |f(t) - f(x)| \leq M\frac{|t-x|^r}{(t+x)^{\frac{r}{2}}}; x,t \in (0,\infty)\right\},$$

for some $M > 0$ and each $0 < r \leq 1$.
In what follows, let $0 < q_n < 1,\ q_n \to 1\ and\ q_n^n \to a\ (0 \leq a < 1),\ as\ n \to \infty.$

Theorem 3.2. *Let $0 < r \leq 1$ and $f \in Lip_M^*(r)$. Then for all $x > 0$ and $n > 2$, we have*

$$|L_n(f; q_n, x) - f(x)| \leq M\left(\frac{\gamma_{n,q_n}(x)}{x}\right)^{\frac{r}{2}}.$$

Proof. Applying Hölder's inequality with $p = \frac{2}{r}$ and $q = \frac{2}{2-r}$, the theorem is easily proved.

□

Definition 3.1. For $f \in C_B[0, \infty)$, the space of all bounded and continuous functions on $[0, \infty)$, the Lipschitz-type maximal function of order τ given by Lenze [9] is defined as follows:

$$\tilde{\omega}_\tau(f, x) = \sup_{t \neq x, t \in [0,\infty)} \frac{|f(t) - f(x)|}{|t - x|^\tau},\quad x \in [0, \infty)\ and\ \tau \in (0, 1].$$

In the next result we obtain an estimate of the error for a Lipschitz type maximal function.

Theorem 3.3. *Let $f \in C_B[0, \infty)$ and $0 < \tau \leq 1$. Then for all $x \in [0, \infty)$, we get*

$$|L_n(f; q_n, x) - f(x)| \leq \tilde{\omega}_\tau(f, x)\gamma_{n,q_n}^{\tau/2}(x).$$

Proof. By the definition of $\tilde{\omega}_\tau(f, x)$ and applying the Hölder's inequality with $p = \frac{2}{\alpha}$ and $\frac{1}{q} = 1 - \frac{1}{p}$, the proof easily follows. □

3.1. **Local Approximation Theorem.**

Definition 3.2. Let $\tilde{C}_B[0, \infty)$ be the space of all real valued bounded and uniformly continuous functions on $[0, \infty)$ having the norm

$$||f|| = sup_{x\in[0,\infty)}|f(x)|.$$

Definition 3.3. For $\delta > 0$ and $W^2 = \{h \in \tilde{C}_B[0, \infty) : h'' \in \tilde{C}_B[0, \infty)\}$, let us consider the following K-functional:

$$K_2(f, \delta) = \inf_{h \in W^2}\{||f - h|| + \delta||h''||\}. \tag{3.3}$$

Consequently, from [2], there exists an absolute constant $C > 0$ such that

$$K_2(f, \delta) \leq C\omega_2(f, \sqrt{\delta}), \tag{3.4}$$

where

$$\omega_2(f, \sqrt{\delta}) = \sup_{0<t\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x + 2t) - 2f(x + t) + f(x)|$$

is the second order modulus of continuity of f. In order to prove local approximation theorem, let us define an auxiliary operator as

$$\widetilde{L}_n(f; q, x) = L_n(f; q, x) + f(x) - f\left(x + \frac{RD_q g(1)}{[n]_q}\right),$$

and then $\widetilde{L}_n(1; q, x) = 1$; and $\widetilde{L}_n(t; q, x) = x$.

Theorem 3.4. *Let $f \in \tilde{C}_B[0,\infty)$. Then for all $x \geq 0$, the following inequality holds:*

$$|\widetilde{L}_n(f;q_n,x) - f(x)| \leq C\omega_2(f,\psi_{n,q_n}(x)) + \omega\left(f;\frac{RD_{q_n}g(1)}{[n]_{q_n}}\right),$$

where $\psi_{n,q_n}(x) = \left(L_n((t-x)^2;q_n,x) + \left(\frac{RD_{q_n}g(1)}{[n]_{q_n}}\right)^2\right)$.

Proof. Let $h \in W^2$ and $t \in [0,\infty)$. By Taylor's expansion we have

$h(t) = h(x) + (t-x)h'(x) + \int_x^t (t-u)h''(u)du.$
Thus,

$$|\widetilde{L}_n(h;q_n,x) - h(x)| \leq L_n\left(\left|\int_x^t |t-u||h''(u)|du\right|;q_n,x\right)du + \left|\int_x^{\left(x+\frac{RD_{q_n}g(1)}{[n]_{q_n}}\right)}\left|x+\frac{RD_{q_n}g(1)}{[n]_{q_n}}-u\right||h''(u)|du\right|.$$

Obviously, $|\int_x^t (t-u)h''(u)du| \leq (t-x)^2||h''||$, therefore

$$|\widetilde{L}_n(h;q_n,x) - h(x)| \leq \left(L_n((t-x)^2;q_n,x) + \left(\frac{RD_{q_n}g(1)}{[n]_{q_n}}\right)\right)^2 ||h''|| = \psi_{n,q_n}(x)||h''||.$$

Since $|L_{n,q}(f;x)| \leq ||f||$, we get $|\widetilde{L}_n(f;q_n,x)| \leq 3||f||$. Thus

$$\begin{aligned}|L_n(f;q_n,x) - f(x)| &\\ &\leq |\widetilde{L}_n(f-h;q_n,x)| + |(f-h)(x)| + |\widetilde{L}_n(h;q_n,x) - h(x)| + \\ &+\left|f\left(x+\frac{RD_{q_n}g(1)}{[n]_{q_n}}\right) - f(x)\right| \\ &\leq 4||f-h|| + \psi_{n,q_n}(x)||h''|| + \omega\left(f;\frac{RD_{q_n}g(1)}{[n]_{q_n}}\right).\end{aligned}$$

Finally, taking the infimum over all $h \in W^2$ on the right side of above inequality and using (3.3)-(3.4), we obtain the desired result. □

Definition 3.4. For $f \in C_B[0,\infty)$ and $\delta > 0$, the second order Ditzian-Totik modulus of smoothness is defined by

$$\omega_\phi^2(f,\delta) = \sup_{0\leq t\leq\delta}\ \sup_{x\pm t\phi(x)\in[0,\infty)} |f(x+t\phi(x)) - 2f(x) + f(x-t\phi(x))|,$$

where $\phi(x) = \sqrt{x(x+1)},\ x \geq 0$.

Definition 3.5. The appropriate K-functional is given by

$$K_{2,\phi}(f,\delta^2) = \inf_{h\in W_\infty^2(\phi)}\{||f-h|| + \delta^2||\phi^2 h''||\},$$

where $W_\infty^2 = \{h \in C_B[0,\infty) : h' \in AC_{loc}[0,\infty) : \phi^2 h'' \in C_B[0,\infty)\}$ and $AC_{loc}[0,\infty)$ denotes the space of locally absolutely continuous functions on $[0,\infty)$.

Consequently, from ([3],Theorem 2.1.1) we have

$$C^{-1}\omega_\phi^2(f,\delta) \leq K_{2,\phi}(f,\delta^2) \leq C\omega_\phi^2(f,\delta),$$

for some positive constant C. Also, the Ditzian-Totik modulus of the first order is given by

$$\overrightarrow{\omega}_{\phi}(f,\delta)=\sup_{0\leq|t|\leq\delta}\ \sup_{x\pm t\phi(x)\in[0,\infty)}|f(x+t\phi(x))-f(x)|,$$

where ϕ is an admissible step-function on $[0,\infty)$.

Theorem 3.5. *If $f\in C_B[0,\infty)$ and $n\in\mathbb{N}$, then*

$$|L_n(f,q_n,x)-f(x)|\leq C\omega_{\phi}^2\bigg(f,1/\sqrt{[n]_{q_n}}\bigg)+\overrightarrow{\omega}_{\phi}\bigg(f,\frac{RD_{q_n}g(1)}{[n]_{q_n}\sqrt{x(x+1)}}\bigg).$$

Proof. For any $h\in W_{\infty}^2$, we have

$$\begin{aligned}|\widetilde{L}_n(h;q_n,x)-h(x)| \quad &\leq L_n\bigg(\bigg|\int_x^t|t-u||h''(u)|du\bigg|;q_n,x\bigg)du\\ &+\bigg|\int_x^{\left(x+\frac{RD_{q_n}g(1)}{[n]_{q_n}}\right)}\bigg|x+\frac{RD_{q_n}g(1)}{[n]_{q_n}}-u\bigg||h''(u)|du\bigg|\\ &\leq||\phi^2h''||\frac{L_n((t-x)^2;q_n,x)}{x(x+1)}\\ &+||\phi^2h''||\bigg|\int_x^{\left(x+\frac{RD_{q_n}g(1)}{[n]_{q_n}}\right)}\bigg|\frac{x+\frac{RD_{q_n}g(1)}{[n]_{q_n}}-u}{x(x+1)}\bigg|du\bigg|,\end{aligned}$$

since

$$\frac{|t-u|}{\phi^2(u)}\leq\frac{|t-x|}{\phi^2(x)},$$

for u between t and x. Hence

$$\begin{aligned}|\widetilde{L}_n(h;q_n,x)-h(x)| \quad &\leq||\phi^2h''||\frac{C}{q_n[n-1]_{q_n}}\bigg(1+\frac{1}{[n]_{q_n}}\phi^2(x)\bigg)+\\ &+||\phi^2h''||\frac{\left(\frac{RD_{q_n}g(1)}{[n]_{q_n}}\right)^2}{x(x+1)}\leq\frac{C}{[n]_{q_n}}||\phi^2h''||.\end{aligned}$$

Now for $f\in C_B[0,\infty)$ and any $h\in W_{\infty}^2$

$$\begin{aligned}|L_n(f;q_n,x)-f(x)|&\leq|\widetilde{L}_n(f-h;q_n,x)-(f-h)(x)|+|\widetilde{L}_n(h;q_n,x)-h(x)|\\ &+\bigg|f\bigg(x+\frac{RD_{q_n}g(1)}{[n]_{q_n}}\bigg)-f(x)\bigg|\\ &\leq4||f-h||+\frac{C}{q_n[n]_{q_n}}||\phi^2h''||+\\ &\bigg|f\bigg(x+\phi(x)\frac{RD_{q_n}g(1)}{[n]_{q_n}\sqrt{x(x+1)}}\bigg)-f(x)\bigg|\\ &\leq C\bigg(||f-h||+\frac{||\phi^2h''||}{[n]_{q_n}}\bigg)+\overrightarrow{\omega_{\phi}}\bigg(f,\frac{RD_{q_n}g(1)}{[n]_{q_n}\sqrt{x(x+1)}}\bigg).\end{aligned}$$

Now taking the infimum on the right hand side of the above inequality over all $h\in W_{\infty}^2$ and using the equivalence between $K_{2,\phi}(f,\delta^2)$ and $\omega_{\phi}^2(f,\delta)$, we get the desired result. □

Definition 3.6. For any $b > 0$, the usual modulus of continuity on the interval $[0, b]$ is defined as

$$\omega_b(f;\delta) = \sup_{|t-x|\leq\delta} \sup_{x,t\in[0,b]} |f(t) - f(x)|.$$

3.2. **Weighted Approximation.**

Theorem 3.6. *If $f \in C_2[0,\infty)$, then for every $x \in [0, b]$ and $n \in \mathbb{N}$*

$$|L_n(f;q_n,x) - f(x)| \leq 4K_f(1+x^2)\gamma_{n,q_n}(x) + 2\omega_{b+1}\Big(f;\sqrt{\gamma_{n,q_n}(x)}\Big).$$

Proof. From [5], for $x \in [0, b]$ and $t \in [0,\infty)$

$$|f(t) - f(x)| \leq 4K_f(1+x^2)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right)\omega_{b+1}(f;\delta).$$

Applying $L_n(.;q_n,x)$ and Cauchy-Schwarz inequality to the above inequality, we obtain

$$\begin{aligned}|L_n(f;q_n,x) - f(x)| \quad &\leq 4K_f(1+x^2)L_n((t-x)^2;q_n,x) + \\ &\omega_{b+1}(f;\delta)\left(1 + \frac{L_n(|t-x|;q_n,x)}{\delta}\right) \\ &\leq 4K_f(1+x^2)\gamma_{n,q_n}(x) + \omega_{b+1}(f;\delta)\left(1 + \frac{\sqrt{\gamma_{n,q_n}(x)}}{\delta}\right).\end{aligned}$$

Taking $\delta = \sqrt{\gamma_{n,q_n}(x)}$, the required result follows. □

Definition 3.7. *The space $C_2^*[0,\infty)$ is defined by*

$$C_2^*[0,\infty) := \{f \in C_2[0,\infty) : lim_{x\to\infty}\frac{|f(x)|}{1+x^2} < \infty\}.$$

Theorem 3.7. *For $f \in C_2^*[0,\infty)$*

$$\lim_{n\to\infty} ||L_n(f;q_n,x) - f||_2 = 0. \tag{3.5}$$

Proof. From[4], it is sufficient to verify the following:

$$\lim_{n\to\infty} ||L_n(t^k;q_n,x) - x^k||_2 = 0,\ k = 0, 1, 2.$$

The desired result is obvious for $k = 0$, in view of Lemma 2.1.
Next, again using 2.1, we have

$$\begin{aligned}||L_n(t;q_n,x) - x||_2 \quad &= \frac{1}{g(1)[n]_{q_n}} \sup_{x\in[0,\infty)} \frac{E^{-[n]_{q_n}x}e^{q_n[n]_{q_n}x}|D_{q_n}g(1)|}{1+x^2} \\ &\leq \frac{|D_{q_n}g(1)|}{g(1)[n]_{q_n}},\end{aligned}$$

and hence the condition (3.5) holds for $k = 1$.

Lastly, applying Lemma 2.1 once again, we get

$$||L_n(t^2;q_n,x)-x^2||_2$$
$$\leq \left(\frac{q_n[n]_{q_n}}{[n-1]_{q_n}}-1\right)\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}$$
$$+\frac{(q_n^2[n]_{q_n}|D_{q_n}g(1)|+q_n^3|D_{q_n}g(q_n)|[n]_{q_n}+(1+q_n)[n]_{q_n}g(1))}{g(1)q_n[n]_{q_n}[n-1]_{q_n}}$$
$$\sup_{x\in[0,\infty)}\frac{xE^{-[n]_{q_n}x}e^{q_n[n]_{q_n}x}}{1+x^2}$$
$$+\frac{((1+q_n)|D_{q_n}g(1)|+q_n^2|D_{q_n}^2g(1)|}{q_n[n]_{q_n}[n-1]_{q_n}}$$
$$\leq \left(\frac{q_n[n]_{q_n}}{[n-1]_{q_n}}-1\right)$$
$$+\frac{(q_n^2[n]_{q_n}|D_{q_n}g(1)|+q_n^3|D_{q_n}g(q_n)|[n]_{q_n}+(1+q_n)[n]_{q_n}g(1))}{g(1)q_n[n]_{q_n}[n-1]_{q_n}}$$
$$+\frac{((1+q_n)|D_{q_n}g(1)|+q_n^2|D_{q_n}^2g(1)|}{q_n[n]_{q_n}[n-1]_{q_n}},$$

which tends to zero as $n\to\infty$, thus the required result is also true for $k=2$.

This completes the proof. □

Theorem 3.8. *For each $f\in C_2^*[0,\infty)$ and $\alpha>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|L_n(f;q_n,x)-f(x)|}{(1+x^2)^{1+\alpha}}=0.$$

Proof. Let $x_0\in[0,\infty)$ be arbitrary but fixed. Then

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|L_n(f;q_n,x)-f(x)|}{(1+x^2)^{1+\alpha}} &\leq \sup_{x\leq x_0}\frac{|L_n(f;q_n,x)-f(x)|}{(1+x^2)^{1+\alpha}}+\sup_{x>x_0}\frac{|L_n(f;q_n,x)-f(x)|}{(1+x^2)^{1+\alpha}}\\ &\leq ||L_n(f;q_n)-f||_{C[0,x_0]}+||f||_2\sup_{x>x_0}\frac{|L_n(1+t^2;q_n,x)|}{(1+x^2)^{1+\alpha}}\\ &\quad+\sup_{x>x_0}\frac{|f(x)|}{(1+x^2)^{1+\alpha}}\\ &= I_1+I_2+I_3, say. \end{aligned}\tag{3.6}$$

Since $|f(x)|\leq ||f||_2(1+x^2)$, we have $sup_{x>x_0}\frac{|f(x)|}{(1+x^2)^{1+\alpha}}\leq\frac{||f||_2}{(1+x_0^2)^\alpha}$.

Let $e>0$ be arbitrary. We can choose x_0 to be so large that

$$\frac{||f||_2}{(1+x_0^2)^\alpha}<\frac{\epsilon}{6}.\tag{3.7}$$

In view of Theorem 3.1, there exists a $n_1\in\mathbb{N}$ such that

$$||f||_2\frac{|L_n(1+t^2;q_n,x)|}{(1+x^2)^{1+\alpha}}<\frac{(1+x^2)||f||_2}{(1+x^2)^{1+\alpha}}+\frac{\epsilon}{3},\ \forall n\geq n_1.$$

Hence

$$||f||_2 \sup_{x>x_0} \frac{|L_n(1+t^2;q_n,x)|}{(1+x^2)^{1+\alpha}} < \frac{||f||_2}{(1+x_0^2)^{\alpha}} + \frac{\epsilon}{3},\ \forall n \geq n_1. \tag{3.8}$$

Thus, combining (3.7) and (3.12)

$$I_2 + I_3 < \frac{\epsilon}{6} + \frac{\epsilon}{3} + \frac{\epsilon}{6} = \frac{2\epsilon}{3},\ \forall n \geq n_1 \tag{3.9}$$

Using Theorem 3.2, we can see that the first term of the inequality (3.6) implies that

$$||L_n(f;q_n) - f||_{C[0,x_0]} < \frac{\epsilon}{3}, \forall n \geq n_2. \tag{3.10}$$

Let $n_0 = max(n_1, n_2)$. Then, combining (3.6), (3.9) and (3.10) we get the desired result. □

If $f \in C_2^*[0,\infty)$, then the weighted modulus of continuity is defined by

$$\Omega(f;\delta) = \sup_{0\leq|h|<\delta, x\in[0,\infty)} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}. \tag{3.11}$$

We have the following basic properties of the weighted modulus of continuity $\Omega(f;\delta)$:

Lemma 3.2. [6]. *For the function $\Omega(f,\delta)$, we have*

(1) $\Omega(f,\delta)$ is a monotone increasing function of δ,
(2) $lim_{\delta\to 0^+}\Omega(f,\delta) = 0$;
(3) for any $\lambda \in [0,\infty), \Omega(f,\lambda\delta) \leq 2(1+\lambda)(1+\delta^2)\Omega(f,\delta)$.

Theorem 3.9. *For $f \in C_2^*[0,\infty)$ there exists a positive constant A such that*

$$\sup_{x\in[0,\infty)} \frac{|L_n(f;q_n,x) - f(x)|}{(1+x^2)^{5/2}} \leq A\Omega\Big(f; 1/\sqrt{[n]_{q_n}}\Big).$$

Proof. We have

$$|L_n(f;q_n,x) - f(x)| \leq L_n(|f(t) - f(x)|; q_n, x)$$

By using (3.11) and Lemma 3.2. for $f \in C_2[0,\infty)$, we have

$$|f(t) - f(x)| \leq (1+(t-x)^2)(1+x^2)\Omega(f;|t-x|) \tag{3.12}$$

$$\leq 2\left(1+\frac{|t-x|}{\delta}\right)(1+\delta^2)\Omega(f;\delta)(1+(t-x)^2)(1+x^2). \tag{3.13}$$

Hence

$$\begin{aligned}|L_n(f;q_n,x) - f(x)| &\leq 2(1+\delta^2)\Omega(f;\delta)(1+x^2)\left[L_n\left(\left(1+\frac{|t-x|}{\delta}\right)(1+(t-x)^2);q_n,x\right)\right] \\ &\leq 2(1+\delta^2)\Omega(f;\delta)(1+x^2)\Big\{L_n(1;q_n,x) + L_n((t-x)^2;q_n,x) \\ &+\frac{1}{\delta}L_n(|t-x|;q_n,x) + \frac{1}{\delta}L_n(|t-x|(t-x)^2;q_n,x)\Big\}.\end{aligned}$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned}|L_n(f;q_n,x) - f(x)| &\leq 2(1+\delta^2)\Omega(f;\delta)(1+x^2)\Big\{L_n(1;q_n,x) + L_n((t-x)^2;q_n,x) \\ &+ \frac{1}{\delta}\sqrt{L_n((t-x)^2;q_n,x)} + \frac{1}{\delta}\sqrt{L_n((t-x)^4;q_n,x)}\sqrt{L_n((t-x)^2;q_n,x)}\Big\}.\end{aligned}$$

There exist positive constants A_1 and A_2 such that

$$L_n((t-x)^2;q_n,x) \leq A_1 \frac{(1+x^2)}{[n]_{q_n}}, L_n((t-x)^4;q_n,x) \leq A_2(1+x^2)^2$$

and

$$\left(L_n\left(\frac{(t-x)^2}{\delta^2};q_n,x\right)\right)^{1/2} \leq \frac{\sqrt{A_1}}{\delta[n]_{q_n}^{1/2}}(1+x^2)^{\frac{1}{2}}.$$

So, we have

$$\begin{aligned} |L_n(f;q_n,x)-f(x)| &\leq 2(1+\delta^2)\Omega(f;\delta)(1+x^2)\Bigg\{1+A_1(1+x^2) \\ &+\frac{\sqrt{A_1}}{\delta[n]_{q_n}^{1/2}}(1+x^2)^{\frac{1}{2}}+\sqrt{A_1A_2}\frac{(1+x^2)}{\delta[n]_{q_n}^{1/2}}(1+x^2)^{\frac{1}{2}}\Bigg\}. \end{aligned}$$

Choosing $\delta = \frac{1}{[n]_{q_n}^{1/2}}$, we obtain

$$\begin{aligned} |L_n(f;q_n,x)-f(x)| &\leq 2\left(1+\frac{1}{[n-1]_q}\right)\Omega(f;1\sqrt{([n]_{q_n})}(1+x^2)\Bigg\{1+A_1(1+x^2) \\ &+\sqrt{A_1}(1+x^2)^{\frac{1}{2}}+\sqrt{A_1A_2}(1+x^2)(1+x^2)^{\frac{1}{2}}\Bigg\}. \end{aligned}$$

Taking $A = 4(1+A_1+\sqrt{A_1}+\sqrt{A_1A_2})$, we have the desired result. □

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REFERENCES

[1] Agrawal, P. N., Karsli, H. and Goyal, M., *Szász-Baskakov type operators based on q-integers*, J. Inequal. Appl., 2014 (2014), 1–18

[2] DeVore, R. A. and Lorentz, G. G., *Constructive Approximation*, Springer, Berlin (2013)

[3] Ditzian, Z. and Totik, V., *Moduli of Smoothness*, Springer-Verlag, New York, 1987

[4] Gadjiev, A. D. and On, P. P. *Korovkin type theorems*, Math. Zametki, **20** (1976), No. 5, 781–786

[5] İbikli, E. and Gadjieva, E. A., *The order of approximation of some unbounded functions by the sequence of positive linear operators*, Turkish J. Math., **19** (1995), No. 3, 331–337

[6] Ispir, N., *On modified Baskakov operators on weighted spaces*, Turk. J. Math., **26** (2001), No. 3, 355–365

[7] Karaisa, A., Tollu, D. T. and Asar, Y., *Stancu type generalization of q-Favard-Szász operators*, Appl. Math. Comput. **264** (2015), 249–257

[8] Karaisa, A., *Approximation by Durrmeyer type Jakimoski Leviatan operators*, Math. Method. Appl. Sci., DOI: 10.1002/mma.3650 (2015)

[9] Lenze, B., *On Lipschitz-type maximal functions and their smoothness spaces*, Nederl. Akad. Wetensch. Indag. Math. **50** (1988), No. 1, 53–63

[10] Özarslan, M. A. and Duman, O., *Local approximation behaviour of modified SMK operators*, Miskolc Math. Notes, **11** (2010), No. 1, 87–99

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