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Composition operators between different weighted Besov spaces

EBRAHIM ZAMANI and HAMID VAEZI

ABSTRACT. Let D denote the open unit disc in the complex plane and let dA be the normalized Lebesgue area measure on D. The weighted Besov space $B_p(\sigma)(p > 1)$ is the space of analytic functions f on D such that $\int_D |f'(z)|^p \sigma(z) dA(z) < \infty$, where σ is a weight function on D.
In this article we study the boundedness of composition operators on weighted Besov spaces with admissible Bekolle weights.

1. INTRODUCTION AND PRELIMINARIES

Given a nonegative integrable function σ on the unite disc, the weighted Besov space $B_p(\sigma)$, $p > 1$, is the space of analytic functions f on D with

$$\|f\|^p_{B_p(\sigma)} = \int_D |f'(z)|^p \sigma(z) dA(z) < \infty.$$

In the special case $p = 2$, $B_p(\sigma)$ is denoted by H_σ (see [9]).

We denote by $H(D)$ the space of all holomorphic functions on D. Given a self map ϕ on D, the linear composition operator C_φ on $H(D)$, is defined by

$$(C_\phi f)(z) = f(\phi(z)), \text{ for all } z \in D.$$

Let $\lambda \in D$ be given. The basic conformal automorphism is defined by

$$\alpha_\lambda(z) = \frac{a - z}{1 - \overline{a}z}, \quad |a| < 1.$$

Recall that $\rho(z, \lambda) = |\alpha_\lambda(z)|$ (pseudohyperbolic metric) and

$$E(\lambda, r) = \{z \in D : \rho(z, \lambda) < r\},$$

for $\lambda \in D$ and $r \in (0, 1)$.

Definition 1.1. The weighted Bergman space $A^p(\sigma)$ ($p > 0$), is the the space of analytic functions f on D with

$$\|f\|^p_{A^p(\sigma)} = \int_D |f(z)|^p \sigma(z) dA(z) < \infty.$$

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Definition 1.2. For a given analytic self-map ϕ of D, the classical Nevanlinna counting function N_ϕ is defined by

$$N_\phi(z) = \Sigma_{w \in \phi^{-1}(z)} log \frac{1}{|z|} \qquad (z \in D \backslash \{\phi(0)\}).$$

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Corresponding author: Ebrahim Zamani; E. Zamani@tabrizu.ac.ir

Definition 1.3. Let ϕ be an analytic self-map of D and σ a weight function on D. We define the function

$$\aleph_{\phi,\sigma}(z) = \sum_{w\in\phi^{-1}(z)} \sigma(w) \ \ (z \in D\backslash\{\phi(0)\}).$$

As in the case of the classical Nevanlinna counting function N_ϕ, we understand that $\aleph_{\phi,\sigma}(z) = 0$ for $z \notin \phi(D)$ and $w \in \phi^{-1}(z)$ is repeated according to the multiplicity of zeros of $\phi - z$. Conventionally, we consider that $\aleph_{\phi,\sigma}(z) = 0$ if $z = \phi(0)$. We call $\aleph_{\phi,\sigma}$ a generalized Nevanlinna counting function.

The composition operators on different spaces of (analytic) functions are studied by many authors, see for example [2], [3] and [8] . The books [6], [11] and [13] are good resource in this context. Tjani give a Carleson measure characterization of the compact composition operators on Besov spaces. She solved the boundedness and compactness problem for composition operators on Besov spaces in [16]. In [10], Kumar and Sharma give a unified approach to some known and some new creteria for bounded and compact composition operators between Besov spaces. In [17] we gave results on composition operators on Besov type space. In this paper we characterize the bounded and compact composition operators from $B_p(\sigma_1)$ into $B_q(\sigma_2)$ in terms of the behavior of the generalized Nevanlinna counting function, where σ_1 and σ_2 are weight functions. Throughout this paper, the notation $a \preceq b$ means that there exists a positive constant C such that $a \le Cb$. Moreover, if both$a \preceq b$ and $b \preceq a$ hold, then one says that $a \approx b$.

Definition 1.4. For each $\alpha > -1$, let dA_α denote the normalized measure on D. For $p > 1$ and $\alpha > -1$, the class $B_p(\alpha)$ consists of all weight functions σ with the property that there is a constant $C > 0$ such that for every

$$S(a) = \{\alpha_\lambda(z) : Re(z\overline{a}) \le 0\}, \ \ a \in D,$$

we have

$$(\int_{S(a)} \sigma dA_\alpha)(\int_{S(a)} \sigma^{\frac{-p'}{p}} dA_\alpha)^{\frac{p}{p'}} \le C\{A_\alpha(S(a))\}^p,$$

where p' is the conjugate of p. Note that we put $S(0) = D$.

In this paper, σ denotes a non-negative continuous function on $[0, 1)$ such that $\sigma(r) \le 1$ for $r \in [0, 1)$.

For $z \in D$, we write $\sigma(z) = \sigma(|z|)$ and call such σ a weight function on D.

Definition 1.5. Let the weight σ satisfies the following three properties:

(i) σ is non-increasing;

(ii) $\dfrac{\sigma(r)}{(1-r)^{1+\delta}}$ is non-decreasing for some $\delta > 0$;

(iii) $\lim\limits_{r\to 1} \sigma(r) = 0$.

If σ also satisfies one of the following properties:

(iv) σ is convex and $\lim\limits_{r\to 1} \sigma(r) = 0$; or

(v) σ is concave,

then such a weight function is called admissible (see [9]). If σ satisfies conditions (i), (ii), (iii) and (iv) then it is said that σ is I-admissible. If σ satisfies conditions (i), (ii), (iii) and (v) then it is said that σ is II-admissible. The weight σ is called admissible if it is I-admissible or II-admissible.

For the Bekolle weights we refer to [1].

Definition 1.6. A weight function σ is called an admissible Bekolle weight if σ satisfies

(i) $\frac{\sigma(z)}{1-|z|^2)^\alpha} \in B_{p_0}(\alpha)$ for some $p_0 > 1$ and $\alpha > -1$,

(ii) σ is non-increasing on [0,1),

(iii) $\frac{\sigma(r)}{(1-r^2)^{1+\delta}}$ is non-decreasing on [0,1) for some $\delta > 0$.

Examples: Let $z \in D$.

- The weight $\sigma(z) = 1$ is not admissible weight.
- The weight $\sigma(z) = (1-|z|^2)^\alpha$ is Bekolle weight for $-1 < \alpha$.
- The weight $\sigma(z) = (1-|z|^2)^\alpha$ is not admissible Bekolle weight for $-1 < \alpha < 0$.
- The weight $\sigma(z) = (1-|z|^2)^\alpha$ is admissible Bekolle weight for $0 < \alpha$.

We need the following results in our proofs.

Lemma 1.1. [12] *Let $p > 0$ and σ be an admissible Bekolle weight function.Then for each $f \in A^p(\sigma)$,*

(i) $|f(z)| \preceq \dfrac{\|f\|_{A^p(\sigma)}}{\sigma(z)^{\frac{1}{p}}(1-|z|^2)^{\frac{2}{p}}}$,

(ii) $|f'(z)| \preceq \dfrac{\|f\|_{A^p(\sigma)}}{\sigma(z)^{\frac{1}{p}}(1-|z|^2)^{1+\frac{2}{p}}}$.

Lemma 1.2. [4] *Let $r \in (0,1), \beta \in N, p > 0, \gamma > -1$ and ω is a positive weight function . Then there exists $k > 1$ independent of $\lambda \in D$ such that*

$$\frac{(\int_{D_{\lambda,r}} \omega(z)dA(z))^{\frac{1}{p}}}{k(1-|\lambda|)^{\gamma+\beta+2}} \le \|\overline{\partial}^\beta K_\lambda^\gamma|\|_{A^p(\omega)} \le k\frac{(\int_{D_{\lambda,r}} \omega(z)dA(z))^{\frac{1}{p}}}{(1-|\lambda|)^{\gamma+\beta+2}}$$

where $D_{\lambda,r} = \{z : |z-\lambda| < r(1-|\lambda|)\}$, $K_\lambda^\gamma = \frac{1}{(1-\bar{\xi}z)^{\eta+2}}$ and $\overline{\partial}^\beta K_\lambda(z) = \partial_{\bar{\zeta}}^\beta K_\zeta(z)|_{\zeta=\lambda}$.

Lemma 1.3. [5] *Let $p > 0$, $p_0 > 1$, $\alpha \in (0,1)$, $\eta > -1$ and suppose $p_0 > p$. Assume that $\dfrac{\omega}{(1-|z|^2)^\eta} \in B_{p_0}(\eta)$ and $\gamma \ge (\eta+2)\dfrac{p_0}{p} - 2$. Then*

$$\|K_\lambda^\gamma\|_{A^p(\omega)} \approx \frac{(\int_{D_{\lambda,\alpha}} \omega(z)dA(z))^{\frac{1}{p}}}{(1-|\lambda|)^{\gamma+2}}.$$

Definition 1.7. For each weight σ, we put

$$\omega_\sigma(z) = \int_{|z|}^1 (t-|z|)\sigma(t)dt \qquad (z \in D).$$

Then we see that ω_σ is non-increasing convex and $\omega_\sigma(z) \to 0$ as $|z| \to 1$.

Lemma 1.4. [12] *If σ is an admissible Bekolle weight function, then it holds that*

$$\omega_\sigma(r) = (1-r^2)^2\sigma(r)$$

for every $r \in [0,1)$.

Lemma 1.5. [14] *There is a constant C such that, for every function f analytic on D, $q > 2$ and all $\lambda \in D$,*

$$\int_{D_{(\lambda,\frac{1}{4})}} |f|^{q-2}|f'|^2 dA(z) \preceq \frac{C}{(1-|\lambda|^2)^2} \int_{D_{(\lambda,\frac{1}{2})}} |f|^q dA(z).$$

Lemma 1.6. *Let $p > 1$ and σ be an admissible Bekolle weight function. Then for each $f \in B_p(\sigma)$,*

(i) $|f'(z)| \preceq \dfrac{\|f\|_{B_p(\sigma)}}{\sigma(z)^{\frac{1}{p}}(1-|z|^2)^{\frac{2}{p}}}$,

(ii) $|f''(z)| \preceq \dfrac{\|f\|_{B_p(\sigma)}}{\sigma(z)^{\frac{1}{p}}(1-|z|^2)^{1+\frac{2}{p}}}$.

Proof. Since $f \in B_p(\sigma)$, so $f' \in A^p(\sigma)$. By using Lemma 1.1 the Lemma is proved. □

Definition 1.8. *For $f \in H(D)$, Hardy - Littlewood maximal function f is defined by*

$$M[f](z) = sup_{\delta>0} \frac{1}{A(B(z,\delta))} \int_{B(z,\delta)} |f| dA,$$

where $B(z,\delta) = \{w \in D : |w - z| < \delta\}$.

Note. Since we can find a positive constant c such that $E(z, \frac{1}{2}) \subset B(z, c(1-|z|^2))$ for $z \in D$, it holds that

$$\frac{1}{(1-|z|^2)^2} \int_{E(z,\frac{1}{2})} |f| dA \preceq M[f](z) \ \ z \in D. \tag{1.1}$$

Lemma 1.7. *Let $p > 1$ and σ be a weight function. Then it holds that*

$$\|f\|^p_{B_p(\sigma)} \approx |f'(0)|^p + \int_D |f'(z)|^{p-2}|f''(z)|^2 \{\int_{|z|}^1 (log\frac{r}{|z|})\sigma(r) r dr\} dA(z),$$

for $f \in H(D)$.

Proof. By using $\|f\|^p_{A^p(\sigma)} \approx |f(0)|^p + \int_D |f(z)|^{p-2}|f'(z)|^2\{\int_{|z|}^1 (log\frac{r}{|z|})\sigma(r)rdr\}dA(z)$, (see [12]), the Lemma is proved. □

Lemma 1.8. *Let $p > 1$ and σ is an admissible Bekolle weight function.Then we have*

$$\int_D \frac{1}{|1-\overline{z}w|^{p(\alpha+2)}} \sigma(z) dA(z) \approx \frac{\sigma(z)}{(1-|z|^2)^{p(\alpha+2)-2}}.$$

Proof. The proof follows by using Lemmas 1.2 and 1.3 . □

2. Boundedness

In this section, we study the boundedness of the composition operator on weighted Besov spaces. As an easy cosequence of Littlewood's subordination principle (see [6] or [11]) we see that C_ϕ is bounded on H_σ for each I-admissible weight σ. For the case of II-admissible weight σ and $\phi \in H_\sigma$, C_ϕ is bounded on H_σ if and only if

$$Sup_{|z|<1} \frac{N_{\phi,\sigma}(z)}{\sigma(z)} < \infty. \ \ (see[9]) \tag{2.2}$$

One can say that our main result (Theorem 2.2) is a generalization of the relation (2.2) for $B_p(\sigma)$.

Theorem 2.1. *Let $p > 1$, σ is an admissible Bekolle weight function, p and p' are conjugate exponents. Then $C_{\alpha_\lambda} : B_p(\sigma) \to B_{\frac{p}{p'}}(\sigma)$ is bounded.*

Proof.

$$\begin{aligned}
\|C_{\alpha_\lambda} f\|^{\frac{p}{p'}}_{B_{\frac{p}{p'}}(\sigma)} &= \int_D |\alpha'_\lambda(z) f'(\alpha_\lambda(z))|^{\frac{p}{p'}} \sigma(z) dA(z) \\
&\le \{\int_D \{|\alpha'_\lambda(z)|^{\frac{p}{p'}}\}^p \sigma(z) dA(z)\}^{\frac{1}{p}} \\
&\quad \{\int_D |f'(\alpha_\lambda(z))|^p \sigma(z) dA(z)\}^{\frac{1}{p'}} < \infty.
\end{aligned}$$

The last inequality follows from Hölder inequality and

$$\int_D |\alpha_\lambda'(z)|^p\sigma(z)dA(z) = \int_D \frac{(1-|\lambda|^2)^p}{|1-\overline{\lambda}z|^{2p}}\sigma(z)dA(z) \approx \frac{\sigma(z)}{(1-|z|^2)^{2p-2}} < \infty,$$

(from Lemma 1.8). □

Lemma 2.9. *Let $p > 1$, ϕ be an analytic self map of D and σ is an admissible Bekolle weight function.If*

$$sup_{z\in D}\frac{|\phi'(z)|^p}{(1-|\phi(z)|)^2\sigma(\phi(z))} < \infty, \tag{2.3}$$

then The composition operator C_ϕ is bounded on $B_p(\sigma)$.

Proof. By using Lemma 1.6 we have

$$\|f\circ\phi\|^p_{B_p(\sigma)} = \int_D |f'(\phi(z))|^p|\phi'(z)|^p\sigma(z)dA(z) \le sup_{z\in D}\frac{|\phi'(z)|^p}{(1-|\phi(z)|)^2\sigma(\phi(z))}\|f\|^p_{B_p(\sigma)}.$$

So the claim follows. □

Lemma 2.10. *Let $p > 1$, ϕ be an analytic self map of D, σ is a weight function and the relation (2.3) holds. Then*

$$\|f\circ\phi\|^p_{B_p(\sigma)} \approx |\phi'(0)f'(\phi(0))|^p + \int_D |(f\circ\phi)'(z)|^{p-2}|(f\circ\phi)''(z)|^2\{\int_0^1 N_I(r,z)\sigma(r)rdr\}dA(z)$$

for $f \in H(D)$. Here $N_I(r,w)$ denote the partial counting function for I defined by

$$N_I(r,w) = \Sigma_{|w|\le r}log\frac{r}{|w|}\ \ (w \in D - \{0\}, r \in (0,1)).$$

Proof. Puting $\Phi(z) = |\phi'(z)f'(\phi(z))|^p$ and $F = I$ in Stanton's formula ([7]) we obtain

$$(2\pi)^{-1}\int_0^{2\pi} |\phi'(re^{i\theta})f'(\phi(re^{i\theta}))|^p d\theta = |\phi'(0)f'(\phi(0))|^p + \int_{rD} N_I(r,w)d\mu(w) \tag{2.4}$$

where $d\mu$ is the Riesz measure of $\Phi = |(f\circ\phi)'|$. Since Riesz measure of $|(f\circ\phi)'|^p$ (see [15]) is given by

$$d\mu(w) = p^2|(f\circ\phi)'(w)|^{p-2}|(f\circ\phi)''(w)|^2dA(w),$$

multiplying both sides by $2r\sigma(r)$ and integrating with respect to r from 0 to 1, in relation (2.4), we obtain the desired formula. □

Theorem 2.2. *Let σ_1 be an admissible Bekolle weight function, σ_2 a weight function, $1 < p \le q < \infty$, $3 < q$ and ϕ is analytic self-map of D with $sup_{z\in D}\frac{|\phi'(z)|^p}{(1-|\phi(z)|)^2\sigma_1(\phi(z))} < \infty$. Then $C_\phi : B_p(\sigma_1) \to B_q(\sigma_2)$ is bounded if and only if*

$$\aleph_{I,\omega_{\sigma_2}}(z) = O(\omega_{\sigma_1}(z)^{\frac{q}{p}})\ \ (|z| \to 1). \tag{2.5}$$

Proof. Let that relation (2.5) holds . Hence we can choose a constant $K > 0$ and $r_0 \in [\frac{1}{2}, 1)$ such that

$$\aleph_{I,\omega_{\sigma_2}}(z) \le K\omega_{\sigma_1}(z)^{\frac{q}{p}}\ \ z \in D\backslash r_0\overline{D}.$$

For fixed $f \in B_p(\sigma_1)$, by Lemma 2.10, we have

$$\|C_\phi f\|^q_{B_q(\sigma_2)} \preceq |\phi'(0)f'(\phi(0))|^q + \int_D |(f\circ\phi)'(z)|^{q-2}|(f\circ\phi)''(z)|^2\{\int_0^1 N_I(r,z)\sigma_2(r)rdr\}dA(z)$$

Put

$$I_1(C_\phi f) = |\phi'(0)f'(\phi(0))|^q + \int_{r_0\overline{D}} |(f\circ\phi)'(z)|^{q-2}|(f\circ\phi)''(z)|^2\{\int_0^1 N_I(r,z)\sigma(r)rdr\}dA(z)$$

$$I_2(C_\phi f) = |\phi'(0)f'(\phi(0))|^q + \int_{D\backslash r_0\overline{D}} |(f\circ\phi)'(z)|^{q-2}|(f\circ\phi)''(z)|^2\{\int_0^1 N_I(r,z)\sigma(r)rdr\}dA(z).$$

Since $1 < q-2$, by using Lemmas 1.4 and 1.6 we have

$$|(f\circ\phi)'(z)|^{q-2} \preceq \frac{\|(f\circ\phi)\|^{q-2}_{B_p(\sigma_1)}}{\{\sigma_1(z)(1-|z|^2)^2\}^{\frac{q-2}{p}}} \approx \frac{\|f\circ\phi\|^{q-2}_{B_p(\sigma_1)}}{\omega_{\sigma_1}(z)^{\frac{q-2}{p}}},$$

and

$$|(f\circ\phi)''(z))|^2 \preceq \frac{\|(f\circ\phi)\|^2_{B_p(\sigma_1)}}{\{\sigma_1(z)(1-|z|^2)^2\}^{\frac{2}{p}}(1-|z|^2)^2} \approx \frac{\|f\circ\phi\|^2_{B_p(\sigma_1)}}{\omega_{\sigma_1}(z)^{\frac{2}{p}}(1-|z|^2)^2}.$$

Therefore

$$I_1(C_\phi f) \preceq max_{|z|\leq r_0}\frac{1}{\omega_{\sigma_1}(z)^{\frac{q}{p}}(1-|z|^2)^2}\|(f\circ\phi)\|^q_{B_p(\sigma_1)}\int_{r_0\overline{D}}\{\int_0^1 N_I(r,z)\sigma_2(r)rdr\}dA(z). \tag{2.6}$$

Put $f(z) = \frac{z^2}{2} + z$,

$$\|C_\phi f\|^q_{B_q(\sigma_2)} = \int_D |(\phi+1)(z)|^q|\phi'(z)|^q\sigma_2(z)dA(z).$$

Puting $\phi = I$ in lemma 2.10 and using above equality we have

$$\int_{r_0\overline{D}}\{\int_0^1 N_I(r,z)\sigma_2(r)rdr\}dA(z) \preceq 2^q. \tag{2.7}$$

From relation (2.6), we get

$$I_1(C_\phi f) \preceq max_{|z|\leq r_0}\frac{2^q\|f\circ\phi\|^q_{B_p(\sigma_1)}}{\omega_{\sigma_1}(z)^{\frac{q}{p}}(1-|z|^2)^2} < \infty. \tag{2.8}$$

Put $c = inf_{v\in\overline{E(z,t)}}|v|$ where $\overline{E(z,t)} = \{w\in D : |\phi_z(w)| \leq t\}$. Since $I(0) = 0$ Schwartz' s lemma shows that each $u\in D$ satisfies $c \leq |u|$. Thus we have the following inequalities

$$log\frac{r}{|u|} \leq \frac{1}{|u|}(r-|u|) \leq \frac{1}{c}(r-|u|)$$

for $|u| < r < 1$. These give that for fix $z\in D\backslash r_0\overline{D}$.

$$\int_0^1 N_I(r,z)\sigma_2(r)dr \leq \frac{1}{r_0}\aleph_{I,\omega_{\sigma_2}}(z). \tag{2.9}$$

Hence

$$I_2(C_\phi f) \preceq K\int_{D\backslash r_0\overline{D}} |(f\circ\phi)'(z)|^{q-2}||(f\circ\phi)''(z)|^2\omega_{\sigma_1}(z)^{\frac{q}{p}}dA(z).$$

By Lemmas 1.4 and 1.6, we have

$$|(f\circ\phi)'|^{q-2} \preceq \frac{\|f\circ\phi\|^{q-p}_{B_p(\sigma_1)}}{\omega_{\sigma_1}(z)^{\frac{q-p}{p}}}|(f\circ\phi)'|^{p-2}.$$

Therefore,

$$I_2(C_\phi f) \preceq K\|C_\phi f\|^{q-p}_{B_p(\sigma_1)}\int_D |(f\circ\phi)'|^{p-2}|(f\circ\phi)''|^2\omega_{\sigma_1}(z)dA(z).$$

Since we have

$$\omega_{\sigma_1}(z) \leq \int_{|z|}^1 (log\frac{r}{|z|})\sigma_1(r)rdr, \tag{2.10}$$

by lemma 1.6 we obtain

$$\int_D |(f\circ\phi(z))'|^{p-2}|(f\circ\phi(z))''|^2\omega_{\sigma_1}(z)dA(z) \preceq \|f\circ\phi\|_{B_p(\sigma_1)}.$$

So, $I_2(C_\phi f) \preceq \|f\circ\phi\|_{B_q(\sigma_1)}$. Thus, we conclude that $C_\phi(B_p(\sigma_1)) \subseteq B_q(\sigma_2)$. By the closed graph theorem,

$$C_\phi : B_p(\sigma_1) \to B_q(\sigma_2)$$

is bounded.

Suppose $C_\phi : B_p(\sigma_1) \to B_q(\sigma_2)$ is bounded. Fix $|z| > \frac{1}{3}$ and put

$$f_z(w) = \frac{(1-|z|^2)^{\alpha+2-\frac{2}{p}}}{\sigma_1(z)^{\frac{1}{p}}(1-\overline{z}w)^{\alpha+1}(\alpha+2)\overline{z}} \quad w \in D$$

and

$$\|f_z\|^p_{B_p(\sigma_1)} = \int_D |\frac{(1-|z|^2)^{(\alpha+2)p-2}}{\sigma_1(z)(1-\overline{z}w)^{(\alpha+2)p}}|\sigma_1(w)dA(w).$$

By Lemma 1.8, it follows that $f_z \in B_p(\sigma_1)$. Therefore, Lemma 2.10 gives

$$\begin{aligned}\|C_\phi f_z\|^q_{B_q(\sigma_2)} &\succeq \int_D |(f_z\circ\phi)'(z)|^{q-2}|(f_z\circ\phi)''(z)|^2\{\int_0^1 N_I(r,w)\sigma_2(r)rdr\}dA(w)\\ &\geq \int_D |(f_z\circ\phi)'(z)|^{q-2}|(f_z\circ\phi)''(z)|^2\aleph_{I,\omega_{\sigma_2}}dA(w)\\ &\geq \int_{E(z,\frac{1-|z|}{2})} |(f_z\circ\phi)'(z)|^{q-2}|(f_z\circ\phi)''(z)|^2\aleph_{I,\omega_{\sigma_2}}dA(w).\end{aligned} \tag{2.11}$$

Since $|1-\overline{z}w| \approx 1-|z|^2$ for $w \in E(z, \frac{1-|z|}{2})$, Lemma 1.4 gives

$$|(f_z\circ\phi)'(w)|^{q-2}|(f_z\circ\phi)''(w)|^2 \approx \frac{(\alpha+2)^2|z|^2}{\omega_{\sigma_1}(z)^{\frac{q}{p}}(1-|z|^2)^2}.$$

By the above inequality and relation (2.11) we get

$$\frac{1}{\omega_{\sigma_1}(z)^{\frac{q}{p}}}\frac{1}{(1-|z|^2)^2}\int_{E(z,\frac{1-|z|}{2})}\aleph_{I,\omega_{\sigma_2}}(w)dA(w) \preceq 1$$

for $z \in D$ with $|z| > \frac{1}{3}$. Since one has $1-|z| < 2|z|$, for $|z| > \frac{1}{3}$, by using

$$\aleph_{\phi,\omega_{\sigma_2}}(z) \preceq \frac{1}{t^2(1-|z|^2)^2}\int_{E(z,t)}\aleph_{\phi,\omega_\sigma}(w)dA(w) \;\; (t<|z|<1), \;\; (\text{ see [12]}), \tag{2.12}$$

we get

$$\aleph_{I,\omega_{\sigma_2}}(z) \preceq \omega_{\sigma_1}(z)^{\frac{q}{p}} \qquad for \;\; |z| > \frac{1}{3}.$$

□

Theorem 2.3. *Let σ_1 be an admissible Bekolle weight function, σ_2 a weight function, $1 < p < q < \infty$, $q > 3$ and ϕ is analytic self-map on D with*

$$\sup_{z\in D}\frac{|\phi'(z)|^p}{(1-|\phi(z)|)^2\sigma_1(\phi(z))} < \infty.$$

If

$$\int_D |\frac{\aleph_{I,\omega_{\sigma_2}}(z)}{\omega_{\sigma_1}(z)}|^{\frac{p}{p-q}}\sigma_1(z)dA(z) < \infty, \tag{2.13}$$

then $C_\phi : B_p(\sigma_1) \to B_q(\sigma_2)$ is bounded.

Proof. By Lemma 2.10, it is enough to prove that

$$\int_D |(f\circ\phi)'(z)|^{q-2}|(f\circ\phi)''(z)|^2\{\int_0^1 N_I(r,z)\sigma(r)rdr\}dA(z)<\infty$$

for any $f\in B_p(\sigma_1)$. To prove this we will decompose the integral over D into two integrals over $\frac{1}{4}\overline{D}$ and $D\backslash\frac{1}{4}\overline{D}$. By relation (2.8) we have

$$\int_{\frac{1}{4}\overline{D}} |(f\circ\phi)'(z)|^{q-2}|(f\circ\phi)''(z)|^2\{\int_0^1 N_I(r,z)\sigma(r)rdr\}dA(z)$$

$$\preceq max_{|z|\leq r_0}\frac{2^q\|f\circ\phi\|^q_{B_p(\sigma_1)}}{\omega_{\sigma_1}(z)^{\frac{q}{p}}(1-|z|^2)^2}<\infty.$$

Hence we may consider the integral over $D\backslash\frac{1}{4}\overline{D}$. From relation (2.9) we have that

$$\int_0^1 N_I(r,z)\sigma_2(r)rdr\preceq\aleph_{I,\omega_{\sigma_2}}(z)$$

for $z\in D\backslash\frac{1}{4}\overline{D}$. From relation (2.12) we obtain

$$\begin{aligned}&\textstyle\int_{D\backslash\frac{1}{4}\overline{D}} |(f\circ\phi)'(z)|^{q-2}|(f\circ\phi)''(z)|^2\{\int_0^1 N_I(r,z)|\sigma(r)rdr\}dA(z)\\ \preceq&\textstyle\int_D |(f\circ\phi)'(z)|^{q-2}|(f\circ\phi)''(z)|^2\frac{1}{(1-|z|^2)^2}\{\int_{E(z,\frac{1}{4})}\aleph_{I,\omega_{\sigma_2}}(w)dA(w)\}dA(z).\end{aligned}$$

By noting that $\chi_{E(z,\frac{1}{4})}(w)=\chi_{E(w,\frac{1}{4})}(z)$ and $1-|z|^2\approx 1-|w|^2$ for $w\in E(z,\frac{1}{4})$, and applying Fubini's theorem, we have

$$\int_D |(f\circ\phi)'(z)|^{q-2}|(f\circ\phi)''(z)|^2\frac{1}{(1-|z|^2)^2}\{\int_{E(z,\frac{1}{4})}\aleph_{I,\omega_{\sigma_2}}(w)dA(w)\}dA(z)$$

$$\approx\int_D\{\int_{E(w,\frac{1}{4})}|(f\circ\phi)'(z)|^{q-2}|(f\circ\phi)''(z)|^2dA(z)\}\frac{\aleph_{I,\omega_{\sigma_2}}(w)}{(1-|w|^2)^2}dA(w).$$

Since Lemma 1.5 gives that

$$\int_{E(w,\frac{1}{4})}|(f\circ\phi)'(z)|^{q-2}|(f\circ\phi)''(z)|^2dA(z)\preceq\frac{1}{(1-|w|^2)^2}\int_{E(w,\frac{1}{2})}|(f\circ\phi)'(z)|^q dA(z),$$

we obtain

$$\begin{aligned}&\textstyle\int_{D\backslash\frac{1}{4}\overline{D}} |(f\circ\phi)'(z)|^{q-2}|(f\circ\phi)''(z)|^2\{\int_0^1 N_I(r,z)\sigma_2(r)rdr\}dA(z)\\ \preceq&\textstyle\int_D\{\int_{E(w,\frac{1}{2})}|(f\circ\phi)'(z)|^q dA(z)\}\frac{\aleph_{I,\omega_{\sigma_2}}(w)}{(1-|w|^2)^4}dA(w).\end{aligned}$$

By applying Fubinis theorem to the last formula once again, we have

$$\begin{aligned}&\textstyle\int_{D\backslash\frac{1}{4}\overline{D}} |(f\circ\phi)'(z)|^{q-2}|(f\circ\phi)''(z)|^2\{\int_0^1 N_I(r,z)\sigma_2(r)rdr\}dA(z)\\ \preceq&\textstyle\int_D |(f\circ\phi)'(z)|^q\{\int_{E(z,\frac{1}{2})}\frac{\aleph_{I,\omega_{\sigma_2}}(w)}{(1-|w|^2)^4}dA(w)\}dA(z).\end{aligned}$$

Hence, Hölder's inequality gives us

$$\begin{aligned}&\textstyle\int_D |(f\circ\phi)'(z)|^q\{\int_{E(z,\frac{1}{2})}\frac{\aleph_{I,\omega_{\sigma_2}}(w)}{(1-|w|^2)^4}dA(w)\}dA(z)\\ \preceq&\textstyle[\int_D |(f\circ\phi)'(z)|^q\sigma_1(z)dA(z)]^{\frac{q}{p}}[\int_D\{\int_{E(z,\frac{1}{2})}\frac{\aleph_{I,\omega_{\sigma_2}}(w)}{(1-|w|^2)^4}dA(w)\}^{\frac{p}{p-q}}\sigma_1(z)^{\frac{-p}{p-q}}dA(z)]^{\frac{p-q}{p}}.\end{aligned}$$

Since $\sigma_1(z) \approx \sigma_1(w)$ and $1-|z|^2 \approx 1-|w|^2$, for $w \in E(z, \frac{1}{2})$, it follows by Lemma 1.4 and relation (1.1) that

$$\begin{aligned}
&\int_{E(z,\frac{1}{2})} \frac{\aleph_{I,\omega_{\sigma_2}}(w)}{(1-|w|^2)^4} dA(w)\} \\
&\preceq \frac{\sigma_1(z)}{(1-|z|^2)^2} \int_{E(z,\frac{1}{2})} \frac{\aleph_{I,\omega_{\sigma_2}}(w)}{\sigma_1(w)(1-|w|^2)^2} dA(w) \\
&\preceq \frac{\sigma_1(z)}{(1-|z|^2)^2} \int_{E(z,\frac{1}{2})} \frac{\aleph_{I,\omega_{\sigma_2}}(w)}{\omega_{\sigma_1}(w)} dA(w) \\
&\preceq \sigma_1(z) M[\frac{\aleph_{I,\omega_{\sigma_2}}}{\omega_{\sigma_1}}](z) \qquad\qquad (z \in D).
\end{aligned}$$

Thus the Hardy-Littlewood maximal theorem shows that

$$\begin{aligned}
&[\int_D \{\int_{E(z,\frac{1}{2})} \frac{\aleph_{I,\omega_{\sigma_2}}(w)}{(1-|w|^2)^4} dA(w)\}^{\frac{p}{p-q}} \sigma_1(z)^{\frac{-q}{p-q}} dA(z)]^{\frac{p-q}{p}} \\
&\preceq \|M[\frac{\aleph_{I,\omega_{\sigma_2}}}{\omega_{\sigma_1}}]\|_{L^{\frac{p}{p-q}}(\sigma_1(w))} \\
&\preceq \|\frac{\aleph_{I,\omega_{\sigma_2}}}{\omega_{\sigma_1}}\|_{L^{\frac{p}{p-q}}(\sigma_1(w))}.
\end{aligned}$$

Finally, the above inequalities gives

$$\|C_\phi(f)\|_{B_p(\sigma_2)} < \infty.$$

□

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TABRIZ
FACULTY OF MATHEMATICAL SCIENCES
TABRIZ, IRAN
Email address: E.Zamani@tabrizu.ac.ir
Email address: hvaezi@tabrizu.ac.ir