CREAT. MATH. INFORM.
Volume **26** (2017), No. 1,
Pages 79 - 84

Online version at https://creative-mathematics.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2017.01.10

Generalized rough relations via ideals

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ABSTRACT. The aim of this paper is to construct a new rough relation structure for a given ideal and to study many of their properties. Further, definitions of lower and upper approximations via ideal have been introduced. These new definitions are compared with Pawlak's definitions. It's therefore shown that the current definitions are more generally.

1. INTRODUCTION

A classic paper of Z. Pawlak [17] on rough sets is considered to mark the birth of the rough set theory. Several mathematicians, logicians, and researchers of computers have become interested in the theory and have done a lot of research work of rough set theory [4, 12] and its applications. Its applications are shown in wide fields such as machine learning [3], data mining [2], decision- making support and analysis [14, 19, 20] and expert system [22].

A great effort of many researchers has been done to design newer, faster, and more efficient methods for solving the concept approximation problem. Rough set theory has been introduced by Pawlak [17] as a tool for concept approximation under uncertainty. The idea is to approximate the concept by two descriptive sets called lower and upper approximations. The lower and upper approximations must be extracted from available training data. Kandil et al. [8, 9] introduced a generalization of rough sets and rough membership functions via ideal. Also, Zakaria et al. [23] introduced the generalization of rough multiset via multiset ideals. Other important research about rough set theory and its generalizations can be found in [1, 5, 7, 10, 13].

Basic idea for the notion of rough relation is connected with the fact that in some cases we might be unable to decide for sure whether some objects, states processes, etc., are in a certain relationship or not. This may be caused by our limited accuracy of observation, measurement or description of some phenomena, processes, states, etc.

The notion of ideal topological space was first studied by Kuratowski [11] and Vaidyanathaswamy [21]. Compatibility of the topology with an ideal $\mathcal{I}$ was first defined by Njastad [16]. In 1990, Jankovic and Hamlett [6] investigated further properties of ideal topological spaces.

2. PRELIMINARIES AND BASIC DEFINITIONS

Let $A_1 = (U_1, R_1), ..., A_n = (U_n, R_n)$ be a family of approximation spaces, where R_i is an equivalence relation on U_i for $i = 1, 2, ..., n$, and let

$$A^n = (U^n, R),$$

Received: 16.09.2015. In revised form: 03.11.2016. Accepted: 17.11.2016
2010 *Mathematics Subject Classification.* 54A05, 54H99.
Key words and phrases. *Rough relation, ideal, lower approximation, upper approximation.*
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where $U^n = U_1 \times U_2 \times ... \times U_n,\ R = R_1 \times R_2 \times ... \times R_n$ defined as

$$((x_1, x_2, ..., x_n),\ (y_1, y_2, ..., y_n)) \in R \Leftrightarrow (x_j, y_j) \in R_j, for\ each\ j = 1, 2, ..., n.$$

Obviously, R is also an equivalence relation and A^n is an approximation space, called the product of A_i. The equivalence classes of the relation R are called R-elementary relations in A^n and a finite union of R-elementary relations is called an R-definable relation in A^n.

Definition 2.1. [18] Let $A^n = (U^n, R)$ be a product of approximation spaces. For any relation $Q \subseteq U^n$, define two relations $\underline{R}(Q)$ and $\overline{R}(Q)$ called the lower and upper approximations of Q in A^n, respectively, and defined as:

$$\underline{R}(Q) = \{(x_1, x_2, ..., x_n) \in U^n : [(x_1, x_2, ..., x_n)]_R \subseteq Q\}, \tag{2.1}$$

$$\overline{R}(Q) = \{(x_1, x_2, ..., x_n) \in U^n : [(x_1, x_2, ..., x_n)]_R \cap Q \neq \emptyset\}, \tag{2.2}$$

where $[(x_1, x_2, ..., x_n)]_R$ denotes the equivalence class of the relation R containing $(x_1, x_2, ..., x_n)$.

Proposition 2.1. [15] *Let $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ be two approximation spaces. The product of A_1 by A_2 is the approximation space denoted by $A = (U, R)$, where $U = U_1 \times U_2$. Then*

$$[(x, y)]_R = [x]_{R_1} \times [y]_{R_2}.$$

Example 2.1. Let $A_1 = (U_1, R_1)$ and $A_2 = (U_2, R_2)$ be two approximation spaces, where $U_1 = \{x_1, x_2, x_3, x_4\}$, $R_1 = \Delta \cup \{(x_1, x_2), (x_2, x_1), (x_3, x_4), (x_4, x_3)\}$, $U_2 = \{a, b, c\}$, $R_2 = \Delta \cup \{(a, b), (b, a)\}$, $A = (U, R) = (U_1 \times U_2, R)$, where $U = \{(x_1, a), (x_1, b), (x_1, c), (x_2, a), (x_2, b), (x_2, c), (x_3, a), (x_3, b), (x_3, c), (x_4, a), (x_4, b), (x_4, c)\}$ and R is defined by $((x, y), (z, t)) \in R \Leftrightarrow (x, z) \in R_1$ and $(y, t) \in R_2$. Hence,

$$[(x_1, a)]_R = \{(x_1, a), (x_1, b), (x_2, a), (x_2, b)\},$$
$$[(x_1, c)]_R = \{(x_1, c), (x_2, c)\},$$
$$[(x_3, c)]_R = \{(x_3, c), (x_4, c)\},$$
$$[(x_4, a)]_R = \{(x_3, a), (x_3, b), (x_4, a), (x_4, b)\}.$$

Now, let $X = \{(x_1, a), (x_1, b)\},\ Y = \{(x_1, c), (x_2, c), (x_3, c), (x_4, c)\}$ and $Z = \{(x_1, a), (x_1, c), (x_3, a), (x_3, c), (x_4, c)\}$, then

$$\underline{R}(X) = \emptyset \qquad \overline{R}(X) = \{(x_1, a), (x_1, b), (x_2, a), (x_2, b)\},$$
$$\underline{R}(Y) = Y \qquad \overline{R}(Y) = Y,$$
$$\underline{R}(Z) = \{(x_3, c), (x_4, c)\} \qquad \overline{R}(Z) = U.$$

3. Generalized rough relations via ideals

Definition 3.2. Let $U_1, U_2, ..., U_n$ be a nonempty sets and $\mathcal{I}_{\Re} \subseteq P(U_1 \times U_2 \times ... \times U_n)$. Then $\mathcal{I}_{\Re}$ is called an ideal on U^n if:

(1) $A,\ B \in \mathcal{I}_{\Re} \Rightarrow A \cup B \in \mathcal{I}_{\Re}$,
(2) $A \in \mathcal{I}_{\Re},\ B \subseteq A \Rightarrow B \in \mathcal{I}_{\Re}$.

Example 3.2. Let $U_1, U_2, ..., U_n$ be a nonempty sets, $(x_1, x_2, ..., x_n) \in U^n$ and $B \subseteq U^n$. Then the following are ideals on U^n:

(1) $\mathcal{I}_{\Re} = P(U_1 \times U_2 \times ... \times U_n)$,
(2) $\mathcal{I}_{\Re} = \{\emptyset\}$,
(3) $\mathcal{I}_{\Re(x_1, x_2, ..., x_n)} = \{A \subseteq U^n : (x_1, x_2, ..., x_n) \notin A\}$ called principle ideal of $(x_1, x_2, ..., x_n)$,
(4) $\mathcal{I}_{\Re B} = \{A \subseteq U^n : A \subseteq B\}$ called principle ideal of B,

(5) $\mathcal{I}_{\Re f} = \{A \subseteq U^n : \ A \ is \ finite\}$ called ideal of a finite subsets of U^n,
(6) $\mathcal{I}_{\Re c} = \{A \subseteq U^n : \ A \ is \ countable\}$ called ideal of a countable subsets of U^n.

Definition 3.3. Let $A^n = (U^n, R)$ be an approximation space and $\mathcal{I}_{\Re}$ be an ideal on U^n. For $Q \subseteq U^n$, a pair of lower and upper approximations, denoted by $\underline{apr}(Q)$ and $\overline{apr}(Q)$, are defined respectively as:

$$\underline{apr}(Q) = \{(x_1, x_2, ..., x_n) \in Q : [(x_1, x_2, ..., x_n)]_R \cap Q^c \in \mathcal{I}_{\Re}\}, \tag{3.3}$$

$$\overline{apr}(Q) = Q \cup \{(x_1, x_2, ..., x_n) \in U^n : \ [(x_1, x_2, ..., x_n)]_R \cap Q \notin \mathcal{I}_{\Re}\}. \tag{3.4}$$

Example 3.3. Consider Example 2.1 and let Q_1=$\{(x_1, a), (x_1, b)\}$, Q_2=$\{(x_1, c)$,(x_2, c),(x_3, c), $(x_4, c)\}$ and $Q_3 = \{(x_1, a), (x_2, b)\}$ are subsets of $U_1 \times U_2$ and $\mathcal{I}_{\Re} = \{\emptyset, \{(x_1, a)\}, \{(x_2, b)\}, \{(x_1, a), (x_2, b)\}\}$. Then

$$\begin{array}{ll} \underline{apr}(Q_1) = \emptyset & \overline{apr}(Q_1) = Q_1 \cup \{(x_2, a), (x_2, b)\}, \\ \underline{apr}(Q_2) = Q_2 & \overline{apr}(Q_2) = Q_2, \\ \underline{apr}(Q_3) = \emptyset & \overline{apr}(Q_3) = Q_3 \cup \{(x_1, b), (x_2, a)\}. \end{array}$$

Theorem 3.1. *For any approximation space $A = (U, R)$ the following assertions hold:*
(1) If $\mathcal{I}_{\Re} = P(U_1 \times U_2 \times ... \times U_n)$, then $\overline{apr}(Q) = \emptyset$,
(2) If $\mathcal{I}_{\Re} = \{\emptyset\}$, then $\overline{apr}(Q) = \overline{R}(Q)$.

Proof. (1) Let $\mathcal{I}_{\Re} = P(U_1 \times U_2 \times ... \times U_n)$. It follows that $\overline{apr}(Q) = Q \cup \{(x_1, x_2, ..., x_n) \in U^n : \ [(x_1, x_2, ..., x_n)]_R \cap Q \notin P(U_1 \times U_2 \times ... \times U_n)\} = \emptyset$.
(2) Let $\mathcal{I}_{\Re} = \{\emptyset\}$, then $\overline{apr}(Q) = Q \cup \{(x_1, x_2, ..., x_n) \in U^n : \ [(x_1, x_2, ..., x_n)]_R \cap Q \neq \emptyset\} = \overline{R}(Q)$.

□

It should be noted that Theorem 3.1 part (2) shows that the current definitions are more general than Pawlak's definitions [18].

Theorem 3.2. *Let (U, R) be an approximation space and $Q_1, Q_2 \subseteq U_1 \times U_2$. Then the lower and upper approximations, defined in (3.3) and (3.4), satisfy the following assertions:*

(L_1): $\underline{apr}(U_1 \times U_2) = U_1 \times U_2$,
(L_2): $\underline{apr}(\emptyset) = \emptyset$,
(L_3): $\underline{apr}(Q_1) \subseteq Q_1$,
(L_4): $Q_1 \subseteq Q_2 \Rightarrow \underline{apr}(Q_1) \subseteq \underline{apr}(Q_2)$,
(L_5): $\underline{apr}(Q_1 \cap Q_2) = \underline{apr}(Q_1) \cap \underline{apr}(Q_2)$,
(L_6): $\underline{apr}(Q_1) \cup \underline{apr}(Q_2) \subseteq \underline{apr}(Q_1 \cup Q_2)$,
(L_7): $\underline{apr}(\underline{apr}(Q)) = \underline{apr}(Q)$,
(U_1): $\overline{apr}(U_1 \times U_2) = U_1 \times U_2$,
(U_2): $\overline{apr}(\emptyset) = \emptyset$,
(U_3): $Q_1 \subseteq \overline{apr}(Q_1)$,
(U_4): $Q_1 \subseteq Q_2 \Rightarrow \overline{apr}(Q_1) \subseteq \overline{apr}(Q_2)$,
(U_5): $\overline{apr}(Q_1 \cup Q_2) = \overline{apr}(Q_1) \cup \overline{apr}(Q_2)$,
(U_6): $\overline{apr}(Q_1 \cap Q_2) \subseteq \overline{apr}(Q_1) \cap \overline{apr}(Q_2)$,
(U_7): $\overline{apr}(\overline{apr}(Q)) = \overline{apr}(Q)$,
(LU): $\underline{apr}(Q) = [\overline{apr}(Q^c)]^c$.

Proof. Straightforward. □

4. PROPERTIES OF ROUGH RELATIONS

In this section, an approximation space $A = (U, S)$ and $B = (U^2, R)$ as the approximation product space are considered, where $S \subseteq U \times U$ and $R \subseteq (U \times U)^2$.

Proposition 4.2. *Let Q be a reflexive relation on U. Then $\overline{apr}(Q)$ is a reflexive relation on U.*

Proof. Since $Q \subseteq \overline{apr}(Q)$ and Q is reflexive on U, hence $\overline{apr}(Q)$ is reflexive on U. □

The following example shows that Proposition 4.2 is invalid for $\underline{apr}(Q)$, in general.

Example 4.4. Let (U, S) be an approximation space such that $U = \{a, b, c, d\}$, $S = \Delta \cup \{(a, b), (b, a), (b, d), (d, b), (a, d), (d, a)\}$ and $\mathcal{I}_{\Re} = \{\emptyset, \{(a, c)\}, \{(b, d)\}, \{(a, b)\}, \{(a, b), (a, c)\}, \{(a, c), (b, d)\}, \{(b, d), (a, b)\}, \{(a, c), (b, d), (a, b)\}\}$. Hence

$$[(a, b)]_R = \{(a, a), (b, b), (d, d), (a, b), (b, a), (a, d), (d, a), (b, d), (d, b)\},$$
$$[(a, c)]_R = \{(a, c), (b, c), (d, c)\},$$
$$[(c, a)]_R = \{(c, a), (c, b), (c, d)\},$$
$$[(c, c)]_R = \{(c, c)\}.$$

Let $Q = \Delta$ be a reflexive relation on U. Then we get $\underline{apr}(Q) = \{(c, c)\}$; that is, $\underline{apr}(Q)$ is non-reflexive relation on U.

Remark 4.1. Let Q be a symmetric relation on U. Then

(1) If $\overline{apr}(Q) \neq Q$, then $\overline{apr}(Q)$ is not necessary to be symmetric,
(2) If $\underline{apr}(Q) \neq Q$, then $\underline{apr}(Q)$ is not necessary to be symmetric.

The following example clarify Remark 4.1.

Example 4.5. Consider Example 4.4.

(1) Let $Q = \{(a, c), (c, a)\}$, then $\overline{apr}(Q) = Q \cup \{(c, b), (c, d)\}$, which is not symmetric relation on U
(2) Let $\mathcal{I}_{\Re} = \{\emptyset, \{(a, c)\}, \{(d, c)\}, \{(a, c), (d, c)\}\}$ and $Q = \{(c, b), (b, c)\}$, then $\underline{apr}(Q) = \{(b, c)\}$, which is not symmetric relation on U.

Proposition 4.3. *Let Q be an antisymmetric relation on U. Then $\underline{apr}(Q)$ is antisymmetric relation on U.*

Proof. If $\underline{apr}(Q) \neq \emptyset$, then there exists $(x, y) \in \underline{apr}(Q)$. But if $(x, y), (y, x) \in \underline{apr}(Q)$, since $\underline{apr}(Q) \subseteq Q$ and Q is antisymmetric, we have $x = y$. □

The following example shows that Proposition 4.3 is incorrect for $\overline{apr}(Q)$, in general.

Example 4.6. Consider Example 4.4, let $Q = \{(a, d), (b, c)\}$ is an antisymmetric relation. Then $\overline{apr}(Q) = Q \cup \{(a, a), (b, b), (d, d), (a, b), (b, a), (b, d), (d, b), (d, a), (d, c), (a, c)\}$ which is not antisymmetric relation on U.

Remark 4.2. Let Q be a non-symmetric relation on U.

(1) If $\underline{apr}(Q) \neq Q$, then $\underline{apr}(Q)$ is not necessary to be non-symmetric,
(2) If $\overline{apr}(Q) \neq Q$, $\overline{apr}(Q)$ is not necessary to be non-symmetric.

the following example illustrates Remark 4.2.

Example 4.7. Consider Example 4.4 and the ideal $\mathcal{I}_{\Re} = \{\emptyset, \{(d, c)\}, \{(a, c)\}, \{(c, d)\}, \{(c, a)\}, \{(d, c), (a, c)\}, \{(d, c), (c, d)\}, \{(d, c), (c, a)\}, \{(a, c), (c, d)\}, \{(a, c), (c, a)\}, \{(c, d), (c, a)\}, \{(a, c), (d, c), (c, d)\}, \{(a, c), (d, c), (c, a)\}, \{(d, c), (c, d), (c, a)\}, \{(a, c), (c, d), (c, a)\}, \{(a, c), (c, d), (c, a)\}, \{(a, c)(d, c), (c, d), (c, a)\}\}$.

Let $Q = \{(b,c),(c,b),(a,b)\}$ be a non-symmetric relation, then $\underline{apr}(Q) = \{(b,c),(c,b)\}$ which is symmetric relation on U.
Let $Q = \{(a,a),(b,b),(a,b)\}$ be a non-symmetric relation, then $\overline{apr}(Q) = Q \cup \{(d,d), (b,a),(a,d),(d,a),(b,d),(d,b)\}$ which is symmetric relation on U.

Remark 4.3. Let Q be a transitive relation on U.

(1) If $\underline{apr}(Q) \neq Q$, then $\underline{apr}(Q)$ is not necessary to be transitive,
(2) If $\overline{apr}(Q) \neq Q$, then$\overline{apr}(Q)$ is not necessary to be transitive.

The following example explain Remark 4.3.

Example 4.8. Consider Example 4.4 and the ideal mentioned in Example 4.7.

(1) Let $Q = \{(b,a),(b,c),(c,a)\}$ be a transitive relation on U, then $\underline{apr}(Q) = \{(b,c),(c,a)\}$ which is not transitive relation.
(2) Let $Q = \{(b,a),(b,c),(c,a),(c,c)\}$ be a transitive relation on U, then $\overline{apr}(Q) = (U_1 \times U_2)\backslash\{(c,b),(c,d)\}$ which is not transitive relation because $(c,a),(a,d) \in \overline{apr}(Q)$, but $(c,d) \notin \overline{apr}(Q)$.

Proposition 4.4. *Let Q be a binary relation on U and $\mathcal{I}_{\Re}$ be an ideal on $U \times U$ satisfies condition $Q \in \mathcal{I}_{\Re} \Rightarrow Q^{-1} \in \mathcal{I}_{\Re}$. Then*

(1) $\underline{apr}(Q^{-1}) = (\underline{apr}(Q))^{-1}$,
(2) $\overline{apr}(Q^{-1}) = (\overline{apr}(Q))^{-1}$.

Proof.

$$
\begin{aligned}
(1) \qquad Let\ (x,y) \in \underline{apr}(Q^{-1}) &\Leftrightarrow [(x,y)]_R \cap (Q^{-1})^c \in \mathcal{I}_{\Re} \\
&\Leftrightarrow [(x,y)]_R \cap (Q^c)^{-1} \in \mathcal{I}_{\Re} \\
&\Leftrightarrow ([(x,y)]_R \cap (Q^c)^{-1})^{-1} \in \mathcal{I}_{\Re} \\
&\Leftrightarrow [(y,x)]_R \cap Q^c \in \mathcal{I}_{\Re} \\
&\Leftrightarrow (y,x) \in \underline{apr}(Q) \\
&\Leftrightarrow (x,y) \in (\underline{apr}(Q))^{-1}
\end{aligned}
$$

$$
\begin{aligned}
(2) \qquad Let\ (x,y) \in \overline{apr}(Q^{-1}) &\Leftrightarrow [(x,y)]_R \cap Q^{-1} \notin \mathcal{I}_{\Re} \\
&\Leftrightarrow ([(x,y)]_R \cap Q^{-1})^{-1} \notin \mathcal{I}_{\Re} \\
&\Leftrightarrow [(y,x)]_R \cap Q \notin \mathcal{I}_{\Re} \\
&\Leftrightarrow (y,x) \in \overline{apr}(Q) \\
&\Leftrightarrow (x,y) \in (\overline{apr}(Q))^{-1}
\end{aligned}
$$

□

Acknowledgement. The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

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