CREAT. MATH. INFORM.
Volume **26** (2017), No. 1,
Pages 89 - 94

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2017.01.12

On a new operator on filter generalized topological spaces

SHYAMAPADA MODAK and AHMAD AL-OMARI

ABSTRACT. The purpose of this paper is to introduce the notion of $\psi_{\mathcal{F}}$ operator induced by a given filter $\mathcal{F}$ and a generalized topology μ, and to investigate some properties of this operator. We shall further discuss some characterizations of this operator with the help of $\mathcal{F}$-codeness and $\mathcal{F}$-compatibility.

1. INTRODUCTION

Let X be a nonempty set and let $\wp(X)$ be the power set of X. Then $\mu \subseteq \wp(X)$ is called a *generalized topology* (briefly GT) [2] on X iff $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $G = \cup_{i\in I} G_i \in \mu$. We call the pair (X, μ) a *generalized topological space* (briefly GTS) on X. A GT μ is said to be a *quasi-topology* [4] on X if $M,\ N \in \mu$ implies $M \cap N \in \mu$.
A *filter* $\mathcal{F}$ (not containing the empty set) on X is a nonempty family $\mathcal{F} \subseteq \wp(X)$ satisfying the following conditions:

(1) $A \subset B,\ A \in \mathcal{F}$ implies $B \in \mathcal{F}$.
(2) $A,\ B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.

Let (X, μ) be a GTS and $\mathcal{F}$ be a filter on X, then $(X, \mu, \mathcal{F})$ is called a *filter generalized topological space* (briefly FGTS).
In [1], Al-Omari and Modak introduced an operator $\Omega : \wp(X) \to \wp(X)$ by using a GT μ with a filter $\mathcal{F}$. They also defined an operator $c^{\Omega} : \wp(X) \to \wp(X)$ by using the operator Ω (i. e., for $A \subset X,\ c^{\Omega}(A) = A \cup \Omega(A)$), which is monotone, enlarging and idempotent. They showed that the operator c^{Ω} induces another generalized topology μ^{Ω} satisfying $\mu \subset \mu^{\Omega}$. Some properties of operators Ω and c^{Ω} were investigated in [1].

The purpose of this paper is to introduce another operator $\psi_{\mathcal{F}}$ and investigate some of its properties.

2. PRELIMINARIES

Let $(X, \mu, \mathcal{F})$ be a FGTS. A mapping $\Omega\ : \wp(X) \to \wp(X)$ is defined as follows: $\Omega(A) \subseteq X$ by $x \in \Omega(A)$ if and only if $x \in M \in \mu$ imply $A \cap U \in \mathcal{F}$. If $\mathcal{M}_{\mu} = \cup\{M : M \in \mu\}$ and $x \notin \mathcal{M}_{\mu}$ then by definition $x \in \Omega(A)$.
The mapping is called the local function associated with the filter $\mathcal{F}$ and generalized topology μ.

Proposition 2.1. [1] *Let μ be a GT on a set X, $\mathcal{F}$, $\mathcal{J}$ filters on X and A, B be subsets of X. The following properties hold:*

(1) If $A \subseteq B$, then $\Omega(A) \subseteq \Omega(B)$,
(2) If $\mathcal{J} \subseteq \mathcal{F}$, then $\Omega(A)(\mathcal{J}) \subseteq \Omega(A)(\mathcal{F})$,
(3) $\Omega(A) = c_{\mu}(\Omega(A)) \subseteq c_{\mu}(A)$ (where c_{μ} denotes the closure operator of (X, μ)),
(4) $\Omega(A) \cup \Omega(B) \subseteq \Omega(A \cup B)$,

Received: 15.06.2016. In revised form: 15.11.2016. Accepted: 29.11.2016
2010 *Mathematics Subject Classification.* 54A05, 54C10.
Key words and phrases. *GTS, FGTS, $\mathcal{F}$-codense, $\mathcal{F}$-compatible.*
Corresponding author: Shyamapada Modak; spmodak2000@yahoo.co.in

(5) $\Omega(\Omega(A)) \subseteq \Omega(A)$.
(6) $\Omega(A)$ *is a* μ*-closed set.*

Proposition 2.2. [1] *Let* $(X, \mu, \mathcal{F})$ *be a FGTS. If* $M \in \mu$, $M \cap A \notin \mathcal{F}$ *imply* $M \cap \Omega(A) = \emptyset$. *Hence* $\Omega(A) = X \setminus \mathcal{M}_\mu$ *if* $A \notin \mathcal{F}$.

Lemma 2.1. [1] *Let* $(X, \mu, \mathcal{F})$ *be a FGTS.* $\Omega(X) = X$ *if and only if* $\mu \setminus \{\emptyset\} \subseteq \mathcal{F}$.

Corollary 2.1. *Let* (X, μ) *be a quasi-topological space with a filter* $\mathcal{F}$. *Then* $\mu \setminus \{\emptyset\} \subseteq \mathcal{F}$ *if and only if* $U \subseteq \Omega(U)$, *for* $U \in \mu$.

Proof. Suppose $\mu \setminus \{\emptyset\} \subseteq \mathcal{F}$. Then for $U \in \mu$ and $x \in U$, any $U_x \in \mu(x)$, $U \cap U_x \in \mu$. This implies that $U \cap U_x \in \mathcal{F}$, and so $x \in \Omega(U)$.
Conversely suppose that $U \subseteq \Omega(U)$. Then for $x \in U \subseteq \Omega(U)$, $U_x \cap U \in \mathcal{F}$, where $U_x \in \mu(x)$. Therefore, $U \in \mathcal{F}$. □

3. $\psi_{\mathcal{F}}$ operator

Let $\mathcal{F}$ be a filter on a space (X, μ), an operator $\psi_{\mathcal{F}} : \wp(X) \to \wp(X)$ is defined as follows: for every $A \in \wp(X)$, $\psi_{\mathcal{F}}(A) = \{x \in X :$ there exits $M \in \mu$ such that $M \setminus A \notin \mathcal{F}\}$.

Before discussing the properties of $\psi_{\mathcal{F}}$ operator, we shall give an Example to illustrate the difference between the two operators:

Example 3.1. Let $X = \{a, b, c\}$, a GT $\mu = \{\emptyset, \{a, c\}\}$ and $\mathcal{F} = \{\{a, b\}, X\}$. Then $\Omega(\{b, c\}) = \emptyset$, but $\psi_{\mathcal{F}}(\{b, c\}) = X \setminus \Omega(X \setminus \{b, c\}) = X \setminus \Omega(\{a\}) = X \setminus \emptyset = X$.

The following theorem gives a characterization of the function $\psi_{\mathcal{F}}$.

Theorem 3.1. *Let* $(X, \mu, \mathcal{F})$ *be a FGTS. Then* $\psi_{\mathcal{F}}(A) = X \setminus \Omega(X \setminus A)$.

Proof. Suppose $x \in X \setminus \Omega(X \setminus A)$. Then $x \notin \Omega(X \setminus A)$ and so there exists $M \in \mu$ containing x such that $M \cap (X \setminus A) \notin \mathcal{F}$ which implies that $M \setminus A \notin \mathcal{F}$. Therefore, $X \setminus \Omega(X \setminus A) \subset \{x \in X :$ there exists $M \in \mu(x)$ such that $M \setminus A \notin \mathcal{F}\}$.
Conversely, assume that $y \in \psi_{\mathcal{F}}(A)$. Then there exists $M \in \mu$ containing y such that $M \setminus A \notin \mathcal{F}$. Since $M \setminus A \notin \mathcal{F}$, $M \cap (X \setminus A) \notin \mathcal{F}$ which implies that $y \notin \Omega(X \setminus A)$. Therefore $y \in X \setminus \Omega(X \setminus A)$. Thus $\psi_{\mathcal{F}}(A) = X \setminus \Omega(X \setminus A)$. □

Theorem 3.2. *Let* (X, μ) *be a space with a filter* $\mathcal{F}$ *and* $A, B \subset X$. *Then the following hold:*

(1) $\psi_{\mathcal{F}}(A)$ *is* μ*-open.*
(2) $\Omega(A) = X \setminus \psi_{\mathcal{F}}(X \setminus A)$.
(3) *If* $A \subset B$, *then* $\psi_{\mathcal{F}}(A) \subset \psi_{\mathcal{F}}(B)$.
(4) $\psi_{\mathcal{F}}(A \cap B) \subset \psi_{\mathcal{F}}(A) \cap \psi_{\mathcal{F}}(B)$.
(5) *If* $U \in \mu^{\Omega}$, *then* $U \subset \psi_{\mathcal{F}}(U)$.
(6) $\psi_{\mathcal{F}}(A) \subset \psi_{\mathcal{F}}(\psi_{\mathcal{F}}(A))$.
(7) $\psi_{\mathcal{F}}(A) = \psi_{\mathcal{F}}(\psi_{\mathcal{F}}(A))$ *if and only if* $\Omega(X \setminus A) = \Omega(\Omega(X \setminus A))$.
(8) $A \cap \psi_{\mathcal{F}}(A) = i^{\Omega}_{\mu}(A)$ *(where* i^{Ω}_{μ} *denotes the interior operator of* (X, μ^{Ω})*).*
(9) $\psi_{\mathcal{F}}(X) = X$ *or* $\mathcal{M}_\mu$.
(10) *For* $X \setminus K \notin \mathcal{F}$, $\psi_{\mathcal{F}}(K) = \mathcal{M}_\mu$.
(11) $\psi_{\mathcal{F}}(\emptyset) = \mathcal{M}_\mu \setminus \Omega(X)$.

Proof. (1) Proof is obvious from Proposition 2.1.

(2) Obvious from definition of $\psi_{\mathcal{F}}$.

(3) Proof is obvious from Proposition 2.1.
(4) Obvious from (3).

(5) If $U \in \mu^{\Omega}$, then $X \setminus U$ is μ^{Ω}-closed. Therefore $\Omega(X \setminus U) \subset X \setminus U$ which implies that $X \setminus (X \setminus U) \subset X \setminus \Omega(X \setminus U)$ and so $U \subset \psi_{\mathcal{F}}(U)$.

(6) Obvious from the fact that $\psi_{\mathcal{F}}(A) \in \mu^{\Omega}$.

(7) Suppose $\Omega(X \setminus A) = \Omega(\Omega(X \setminus A))$. Then $\psi_{\mathcal{F}}(A) = X \setminus \Omega(X \setminus A)$ implies that $\psi_{\mathcal{F}}(\psi_{\mathcal{F}}(A)) = X \setminus \Omega(X \setminus \psi_{\mathcal{F}}(A)) = X \setminus \Omega(\Omega(X \setminus A)) = X \setminus \Omega(X \setminus A) = \psi_{\mathcal{F}}(A)$.

Conversely, $\psi_{\mathcal{F}}(A) = \psi_{\mathcal{F}}(\psi_{\mathcal{F}}(A))$ implies that $X \setminus \Omega(X \setminus A) = X \setminus \Omega(X \setminus \psi_{\mathcal{F}}(A)) = X \setminus \Omega(\Omega(X \setminus A))$. Therefore, $\Omega(X \setminus A) = \Omega(\Omega(X \setminus A))$.

(8) Let $x \in A \cap \psi_{\mathcal{F}}(A)$. Then $x \in A$ and $x \in \psi_{\mathcal{F}}(A)$. Since $x \in \psi_{\mathcal{F}}(A)$, there exists $M_x \in \mu$ containing x such that $M_x \setminus A \notin \mathcal{F}$. Therefore, $x \in M_x \setminus (M_x \setminus A) \subset A$. Since $\beta = \{V \setminus F : V$ is a μ-open set of $(X, \mu), F \notin \mathcal{F}\}$ is a basis for μ^{Ω} (see [1]) and $M_x \setminus (M_x \setminus A) \in \beta,\ x \in i_{\mu}^{\Omega}(A)$, where $i_{\mu}^{\Omega}(A)$ is the interior operator in (X, μ^{Ω}). Conversely, assume that $x \in i_{\mu}^{\Omega}(A)$. Then there exists a μ-open set M_x containing x and $F \in \mathcal{F}$ such that $x \in M_x \setminus F \subset A$. Now $M_x \setminus F \subset A$ implies that $M_x \setminus A \subset F$ which turn implies that $M_x \setminus A \notin \mathcal{F}$ and so $x \in \psi_{\mathcal{F}}(A)$. Therefore $x \in A \cap \psi_{\mathcal{F}}(A)$. Hence $A \cap \psi_{\mathcal{F}}(A) = i_{\mu}^{\Omega}(A)$.

(9) Since $\emptyset \notin \mathcal{F}$ by Proposition 2.2 we have $\Omega(\emptyset) = X \setminus \mathcal{M}_{\mu}$. If μ is strong, then $\mathcal{M}_{\mu} = X$, and $\psi_{\mathcal{F}}(X) = X \setminus \Omega(\emptyset) = X \setminus (X \setminus \mathcal{M}_{\mu}) = X$. Otherwise $\psi_{\mathcal{F}}(X) = X \setminus \Omega(\emptyset) = X \setminus (X \setminus \mathcal{M}_{\mu}) = \mathcal{M}_{\mu}$.

(10) For $X \setminus K \notin \mathcal{F}$, by Proposition 2.2 $\psi_{\mathcal{F}}(K) = X \setminus \Omega(X \setminus K) = X \setminus (X \setminus \mathcal{M}_{\mu}) = \mathcal{M}_{\mu}$.

(11) By Theorem 3.1 $\psi_{\mathcal{F}}(\emptyset) = X \setminus \Omega(X) = (\mathcal{M}_{\mu} \cup (X \setminus \mathcal{M}_{\mu})) \setminus \Omega(X) = (\mathcal{M}_{\mu} \setminus \Omega(X)) \cup ((X \setminus \mathcal{M}_{\mu}) \setminus \Omega(X)) = \mathcal{M}_{\mu} \setminus \Omega(X)$, since $\Omega(X)$ is μ-closed by Proposition 2.1 and $X \setminus \mathcal{M}_{\mu}$ is the smallest μ-closed set contained in every μ-closed set. □

Theorem 3.3. *Let (X, μ) be a quasi-topological space and $\mathcal{F}$ be a filter on X. If $A,\ B \subset X$, then $\psi_{\mathcal{F}}(A \cap B) = \psi_{\mathcal{F}}(A) \cap \psi_{\mathcal{F}}(B)$.*

Proof. Let$x \in \psi_{\mathcal{F}}(A) \cap \psi_{\mathcal{F}}(B)$. Then there exist μ-open sets U and V containing x such that $U \setminus A \notin \mathcal{F}$ and $U \setminus B \notin \mathcal{F}$. If $G = U \cap V$, then G is a μ-open set containing x such that $G \setminus A \notin \mathcal{F}$ and $G \setminus B \notin \mathcal{F}$. Now $G \setminus (A \cap B) = (G \setminus A) \cup (G \setminus B) \notin \mathcal{F}$ and so $x \in \psi_{\mathcal{F}}(A \cap B)$. □

Theorem 3.4. *Let $(X, \mu, \mathcal{F})$ be a FGTS. If $\sigma = \{A \subset X :\ A \subset \psi_{\mathcal{F}}(A)\}$, then σ is called a generalized topology on X and $\sigma = \mu^{\Omega}$.*

Proof. Let $A \in \sigma$. Then $A \subset \psi_{\mathcal{F}}(A) = X \setminus \Omega(X \setminus A)$ which implies that $\Omega(X \setminus A) \subset (X \setminus A)$. Therefore, $X \setminus A$ is μ^{Ω}-closed and so A is μ^{Ω}-open. Therefore, $\sigma \subset \mu^{\Omega}$.

Conversely, $A \in \mu^{\Omega}$ and $x \in A$. Then there exists $M \in \mu$ and $F \notin \mathcal{F}$ such that $x \in M \setminus F \subset A$. Now $M \setminus F \subset A$ implies that $M \setminus A \subset F$ which in turn implies $M \setminus A \notin \mathcal{F}$ and so $x \in \psi_{\mathcal{F}}(A)$. Therefore, $\mu^{\Omega} \subset \sigma$. Hence $\sigma = \mu^{\Omega}$. Since μ^{Ω} is generalized topology [1], it follows that σ is a generalized topology. □

Theorem 3.5. *Let $(X, \mu, \mathcal{F})$ be a GFTS and $A \subset X$. Then the following statement hold.*

(1) $\psi_{\mathcal{F}}(A) = \cup\{U \in \mu : U \setminus A \notin \mathcal{F}\}$.
(2) $\psi_{\mathcal{F}}(A) = \cup\{U \in \mu : (U \setminus A) \cup (A \setminus U) \notin \mathcal{F}\}$, *if* A *is* μ*-open.*

Proof. (1). Follows immediately from the definition of $\psi_{\mathcal{F}}$.

(2). Denote $\cup\{U \in \mu : (U\setminus A)\cup(A\setminus U) \notin \mathcal{F}\}$ by A_1. Then $A_1 \subset \psi_{\mathcal{F}}(A)$ for every $A \subset X$. Assume $A \in \mu$ and $x \in \psi_{\mathcal{F}}(A)$. Then there exists $M \in \mu$ such that $x \in M$ and $M \setminus A \notin \mathcal{F}$. If $M \cup A = U$, then $U \in \mu$ and $x \in U$. Now $(U \setminus A) \cup (A \setminus U) = (M \setminus A) \cup \emptyset = M \setminus A$ implies $(U \setminus A) \cup (A \setminus U) \notin \mathcal{F}$ and so $x \in A_1$. Hence $\psi_{\mathcal{F}}(A) = A_1$. □

Theorem 3.6. *Let* (X, μ) *be a quasi-topological space with a filter* $\mathcal{F}$. *Then the following statements are equivalent:*
(1) $\mu \setminus \{\emptyset\} \subseteq \mathcal{F}$.
(2) $\psi_{\mathcal{F}}(\emptyset) = \emptyset$.
(3) If $A \subseteq X$ *is* μ*-closed, then* $\psi_{\mathcal{F}}(A) \setminus A = \emptyset$.
(4) If $A \subseteq X$, *then* $i_\mu(c_\mu(A)) = \psi_{\mathcal{F}}(i_\mu(c_\mu(A)))$ *(where* i_μ *denotes the interior operator of* (X, μ) *).*
(5) If $A = i_\mu(c_\mu(A))$, *then* $A = \psi_{\mathcal{F}}(A)$.
(6) If $U \in \mu$, *then* $\psi_{\mathcal{F}}(U) \subseteq i_\mu(c_\mu(U)) \subseteq \Omega(U)$.

Proof. (1)⇒ (2). $\psi_{\mathcal{F}}(\emptyset) = \cup\{U \in \mu : U \setminus \emptyset = U \notin \mathcal{F}\} = \emptyset$, since $\mu \setminus \{\emptyset\} \subset \mathcal{F}$.
(2) ⇒ (3). Suppose $A \subseteq X$ is μ-closed. If $x \in \psi_{\mathcal{F}}(A) \setminus A$, there exists a $U_x \in \mu$ containing x such that $U_x \setminus A \notin \mathcal{F}$. But $U_x \setminus A \notin \mu$ implies that $U_x \setminus A \in \{U : U \notin \mathcal{F}\}$ and so $\psi_{\mathcal{F}}(\phi) \neq \emptyset$, a contradiction. Therefore, $\psi_{\mathcal{F}}(A) \setminus A = \emptyset$.

(3) ⇒ (4). Since $i_\mu(c_\mu(A)) \in \mu$ for every subset A of X, by Theorem 3.2(5), $i_\mu(c_\mu(A)) \subseteq \psi_{\mathcal{F}}(i_\mu(c_\mu(A)))$. By (3) $\psi_{\mathcal{F}}(c_\mu(A)) \subseteq c_\mu(A)$ and so $\psi_{\mathcal{F}}(c_\mu(A)) = i_\mu(\psi_{\mathcal{F}}(c_\mu(A))) \subseteq i_\mu(c_\mu(A))$. By Theorem 3.1, $\psi_{\mathcal{F}}(i_\mu(c_\mu(A)) \subseteq \psi_{\mathcal{F}}(c_\mu(A)) \subseteq i_\mu(c_\mu(A))$ and so $i_\mu(c_\mu(A)) = \psi_{\mathcal{F}}(i_\mu(c_\mu(A)))$.

(4)⇒ (5). Let $A = i_\mu(c_\mu(A))$. Then $A = i_\mu(c_\mu(A))$ and so $\psi_{\mathcal{F}}(A) = \psi_{\mathcal{F}}(i_\mu(c_\mu(A))) = i_\mu(c_\mu(A)) = A$.

(5)⇒(6). Let $U \in \mu$. Then $\psi_{\mathcal{F}}(i_\mu(c_\mu(i_\mu(c_\mu(U))))) = \psi_{\mathcal{F}}(i_\mu(c_\mu(U))) = i_\mu(c_\mu(U))$. Implies that $\psi_{\mathcal{F}}(U) \subseteq i_\mu(c_\mu(U))$, since $\psi_{\mathcal{F}}(U) \subseteq \psi_{\mathcal{F}}(i_\mu(c_\mu(U)))$.

Again $i_\mu(c_\mu(U)) \subseteq c_\mu(U) \subseteq c_\mu(\Omega(U)) = \Omega(U)$.
(6)⇒ (1). Proof is obvious from $U \subseteq \psi_{\mathcal{F}}(U)$ and the Corollary 2.1.

□

Theorem 3.7. *Let* $(X, \mu, \mathcal{F})$ *be a GFTS. Then for* $A \subseteq X$, $i_\mu(A) \subseteq \psi_{\mathcal{F}}(A)$

Proof. Let $x \in i_\mu(A)$, then there exists $M \in \mu$ containing x such that $M \subseteq A$. This implies that $M \setminus A = \emptyset \notin \mathcal{F}$ and hence by definition of $\psi_{\mathcal{F}}(A)$, $x \in \psi_{\mathcal{F}}(A)$. □

The revers inclusion of the above theorem may be not hold as shown in the next example:

Example 3.2. Let $X = \{a, b, c\}$, a GT $\mu = \{\emptyset, \{a, c\}\}$ and $\mathcal{F} = \{\{a, b\}, X\}$. Then $\psi_{\mathcal{F}}(\{a\}) = X \setminus \Omega(X \setminus \{a\}) = X \setminus \Omega(\{b, c\}) = X \setminus \emptyset = X$ and $i_\mu(\{a\}) = \emptyset$. Therefore, $i_\mu(A) \neq \psi_{\mathcal{F}}(A)$.

Definition 3.1. Let $(X, \mu, \mathcal{F})$ be a GFTS. We say the μ is $\mathcal{F}$-compatible with a filter $\mathcal{F}$, denoted $\mu \sim \mathcal{F}$, if the following holds for every $A \subseteq X$, if for every $x \in A$ there exists $M \in \mu(x)$ such that $M \cap A \notin \mathcal{F}$, then $A \notin \mathcal{F}$.

Theorem 3.8. *Let* $(X, \mu, \mathcal{F})$ *be a GFTS. Then the implications (1) ⇒ (2) ⇒ (3) ⇒ (4) hold.*

(1) $\mu \sim \mathcal{F}$;
(2) *If a subset A of X has a cover of μ-open sets each of whose intersection with A is not in $\mathcal{F}$, then $A \notin \mathcal{F}$;*
(3) *For every $A \subseteq X$, $A \cap \Omega(A) = \emptyset$ implies that $A \notin \mathcal{F}$;*
(4) *For every $A \subseteq X$, $A \setminus \Omega(A) \notin \mathcal{F}$.*

Proof. (1) $\Rightarrow$ (2): The proof is obvious.

(2) $\Rightarrow$ (3): Let $A \subseteq X$ and $x \in A$. Then $x \notin \Omega(A)$ and there exists $V_x \in \mu(x)$ such that $V_x \cap A \notin \mathcal{F}$. Therefore, we have $A \subseteq \cup\{V_x : x \in A\}$ and $V_x \in \mu(x)$ and by (2) $A \notin \mathcal{F}$.

(3) $\Rightarrow$ (4): For any $A \subseteq X$, $A \setminus \Omega(A) \subseteq A$ and $(A \setminus \Omega(A)) \cap \Omega(A \setminus \Omega(A)) \subseteq (A \setminus \Omega(A)) \cap \Omega(A) = \emptyset$. By (3), $A \setminus \Omega(A) \notin \mathcal{F}$. □

Theorem 3.9. *Let $(X, \mu, \mathcal{F})$ be a GFTS. If μ is $\mathcal{F}$-compatible with $\mathcal{F}$. If for every $A \subseteq X$, $A \cap \Omega(A) = \emptyset$ implies that $\Omega(A) = X \setminus \mathcal{M}_\mu$, then $\Omega(A \setminus \Omega(A)) = X \setminus \mathcal{M}_\mu$.*

Proof. First, we show that (1) holds if μ is $\mathcal{F}$-compatible with $\mathcal{F}$. Let A be any subset of X and $A \cap \Omega(A) = \emptyset$. By Theorem 3.8, $A \notin \mathcal{F}$ and by Proposition 2.1, $\Omega(A) = X \setminus \mathcal{M}_\mu$.

Assume that for every $A \subseteq X$, $A \cap \Omega(A) = \emptyset$ implies that $\Omega(A) = X \setminus \mathcal{M}_\mu$. Let $B = A \setminus \Omega(A)$, then

$$\begin{aligned} B \cap \Omega(B) =& (A \setminus \Omega(A)) \cap \Omega(A \setminus \Omega(A)) \\ =& (A \cap (X \setminus \Omega(A))) \cap (A \cap \Omega(X \setminus \Omega(A))) \\ \subseteq& [A \cap (X \setminus \Omega(A))] \cap [\Omega(A) \cap (\Omega(X \setminus \Omega(A)))] = \emptyset. \end{aligned}$$

By (1), we have $\Omega(B) = X \setminus \mathcal{M}_\mu$. Hence $\Omega(A \setminus \Omega(A)) = X \setminus \mathcal{M}_\mu$. □

Theorem 3.10. *Let $(X, \mu, \mathcal{F})$ be a GFTS. Then $\mu \sim \mathcal{F}$ if and only if $\psi_{\mathcal{F}}(A) \setminus A \notin \mathcal{F}$ for every $A \subseteq X$.*

Proof. Necessity. Assume $\mu \sim \mathcal{F}$ and let $A \subseteq X$. Observe that $x \in \psi_{\mathcal{F}}(A) \setminus A$ if and only if $x \notin A$ and $x \notin \Omega(X \setminus A)$ if and only if $x \notin A$ and there exists $U_x \in \mu(x)$ such that $U_x \setminus A \notin \mathcal{F}$ if and only if there exists $U_x \in \mu(x)$ such that $x \in U_x \setminus A \notin \mathcal{F}$. Now, for each $x \in \psi_{\mathcal{F}}(A) \setminus A$ and $U_x \in \mu(x)$, $U_x \cap (\psi_{\mathcal{F}}(A) \setminus A) \notin \mathcal{F}$ by heredity and hence $\psi_{\mathcal{F}}(A) \setminus A \notin \mathcal{F}$ by assumption that $\mu \sim \mathcal{F}$.

Sufficiency. Let $A \subseteq X$ and assume that for each $x \in A$ there exists $U_x \in \mu(x)$ such that $U_x \cap A \notin \mathcal{F}$. Observe that $\psi_{\mathcal{F}}(X \setminus A) \setminus (X \setminus A) = A \setminus \Omega(A) = \{x \in X : \text{there exists } U_x \in \mu(x) \text{ such that } x \in U_x \cap A \notin \mathcal{F}\}$. Thus we have $A \subseteq \psi_{\mathcal{F}}(X \setminus A) \setminus (X \setminus A) \notin \mathcal{F}$ and hence $A \notin \mathcal{F}$ by heredity of $\mathcal{F}$. □

Theorem 3.11. *Let $(X, \mu, \mathcal{F})$ be a GFTS with $\mu \sim \mathcal{F}$, $A \subseteq X$. If N is a nonempty μ-open subset of $\Omega(A) \cap \psi_{\mathcal{F}}(A)$, then $N \setminus A \notin \mathcal{F}$ and $N \cap A \in \mathcal{F}$.*

Proof. If $N \subseteq \Omega(A) \cap \psi_{\mathcal{F}}(A)$, then $N \setminus A \subseteq \psi_{\mathcal{F}}(A) \setminus A \notin \mathcal{F}$ by Theorem 3.10 and hence $N \setminus A \notin \mathcal{F}$ by heredity. Since $N \in \mu \setminus \{\emptyset\}$ and $N \subseteq \Omega(A)$, we have $N \cap A \in \mathcal{F}$ by the Definition of $\Omega(A)$. □

We shall say that a filter $\mathcal{F}$ is $\mathcal{F}$-codense if and only if $\mu \setminus \{\emptyset\} \subseteq \mathcal{F}$.

Lemma 3.2. *Let μ be a GT in X and $\mathcal{F}$ a filter on X. $\psi_{\mathcal{F}}(\emptyset) = \emptyset$ if and only if a filter $\mathcal{F}$ is $\mathcal{F}$-codense.*

Proof. Since $\psi_{\mathcal{F}}(\emptyset) = X \setminus \Omega(X)$, $\psi_{\mathcal{F}}(\emptyset) = \emptyset$ if and only if $X = \Omega(X)$ and hence by Lemma 2.1 $\psi_{\mathcal{F}}(\emptyset) = \emptyset$ if and only if a filter $\mathcal{F}$ is $\mathcal{F}$-codense. □

Proposition 3.3. *Let μ be a GT in X and $\mathcal{F}$ a filter on X. Then the following are equivalent.*

(1) $\mathcal{F}$ is $\mathcal{F}$-codense.
(2) $\Omega(\mathcal{M}_\mu) = X$.
(3) $\psi_{\mathcal{F}}(X \setminus \mathcal{M}_\mu) = \emptyset$.

Proof. (1) ⇔ (2). Suppose $x \in X$ and $x \notin \Omega(\mathcal{M}_\mu)$. Then there exists $M \in \mu$ such that $x \in M$ and $M \cap \mathcal{M}_\mu \notin \mathcal{F}$ which implies that $M \notin \mathcal{F}$ and hence $M = \emptyset$ since $\mathcal{F}$ is $\mathcal{F}$-codense which is a contradiction. Therefore, $x \in \Omega(\mathcal{M}_\mu)$. Hence $\Omega(\mathcal{M}_\mu) = X$. Conversely, suppose $M \in \mu \setminus \{\emptyset\}$ and $M \notin \mathcal{F}$, $M \in \mu$. If $M \neq \emptyset$, then there exists $x \in M$ and hence $x \in \Omega(\mathcal{M}_\mu)$ which implies that $M \cap \mathcal{M}_\mu = M \in \mathcal{F}$, a contradiction. Therefore, $\mu \setminus \{\emptyset\} \subseteq \mathcal{F}$.

(2) ⇔ (3). It is obvious from $\psi_{\mathcal{F}}(X \setminus \mathcal{M}_\mu) = X \setminus \Omega(X \setminus (X \setminus \mathcal{M}_\mu)) = X \setminus \Omega(\mathcal{M}_\mu)$. Hence (2) and (3) are equivalent. □

Acknowledgement. Authors are thankful to the referees for their valuable comments.

References

[1] Al-omari, A. and Modak, S., *Filter on generalized topological spaces*, Scientia Magna, **9** (2013), 62–71
[2] Császár, Á., *Generalized open sets*, Acta Math. Hungar, **75** (1997), No. 1-2, 65–87
[3] Császár, Á., *Generalized topology, generalized continuity*, Acta Math. Hungar, **96** (2002), 351–357
[4] Császár, Á., *Remarks on quasi-topologies*, Acta Math. Hungar, **119** (2008), No. 1-2, 197–200

Department of Mathematics
University of Gour Banga
P. O. Mokdumpur, 732 103 Malda, India
Email address: spmodak2000@yahoo.co.in

Department of Mathematics
Al al-Bayt University
P.O. Box 130095, Mafraq 25113, Jordan
Email address: omarimutah1@yahoo.com