CREAT. MATH. INFORM.
Volume **26** (2017), No. 2,
Pages 115 - 124

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2017.02.01

A subclass of meromorphic functions defined by a certain integral operator on Hilbert space

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ABSTRACT. In the present paper, we introduce and investigate a new class of meromorphic functions associated with an integral operator, by using Hilbert space operator. For this class, we obtain coefficient inequality, extreme points, radius of close-to-convex, starlikeness and convexity, Hadamard product and integral means inequality.

1. INTRODUCTION

Let Σ denote the class of meromorphic functions in the punctured unit disc

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \backslash \{0\},$$

of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in $\mathbb{U}^*$.

Denote by $f * g$ the Hadamard product(or convolution) of the functions f and g; that is, if f is given by (1.1) and g is defined by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n, \text{ then } (f * g)(z) := \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n.$$

Various subclasses of Σ were introduced and studied by many authors. In recent years, some subclass of meromorphic functions associated with several families of integral operators and derivate operators were introduced and investigated (see for example [1], [2], [9], [11], [19] and see also [4] and [20]). Lashin [19] defined an integral operator $P_\mu^\gamma : \Sigma \to \Sigma$

$$P_\mu^\gamma = P_\mu^\gamma f(z) = \frac{\mu^\gamma}{\Gamma(\gamma)} \frac{1}{z^{\mu+1}} \int_0^z t^\mu \left(\log \frac{z}{t}\right)^{\gamma-1} f(t)dt, \mu > 0, \gamma > 0; z \in \mathbb{U}^*$$

where Γ is the familiar Gamma function. Using the integral representation of the Gamma and Beta functions, it is easy to see that

$$P_\mu^\gamma f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{\mu}{n+\mu+1}\right)^\gamma a_n z^n = \frac{1}{z} + \sum_{n=1}^{\infty} L(n,\mu,\gamma) a_n z^n \tag{1.2}$$

where

$$L(n,\mu,\gamma) = \left(\frac{\mu}{n+\mu+1}\right)^\gamma.$$

Received: 01.09.2016. In revised form: 14.12.2016. Accepted: 21.12.2016
2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. *Meromorphic functions, coefficient estimates, Hadamard product, Hilbert space operator, integral operator, meromorphically close-to convexity, starlikeness and convexity.*

Remark 1.1. The integral operator P_μ^γ was studied by Atshan and Mohammed [6] for analytic functions. For analytic function$f$ if we take $\mu = 1$ in the equality (1.2), then we obtain the Libera integral operator given by

$$P_1^\gamma f(z) = z + \sum_{n=1}^{\infty} \frac{2}{n+1} a_n z^n.$$

Libera integral operator is generalized as Bernardi integral operator given by Bernardi [7]. Gupta and Sharma [15] introduced certain differential inequalities for the integral operator P_μ^γ. In [20] Piejko and Sokol considered a multipleer transformation and some subclasses of the class of meromorphic functions which was defined by means of the Hadamard product and by using the operator P_μ^γ, introduced by N. E. Cho, O. S.: Khown and H. M. Srivastava [10].

Let H be a Hilbert space on the complex field and $L(H)$ denote the algebra of all bounded linear operators on H. For a complex-valued function f analytic in a domain E of the complex plain containing the spectrum $\sigma(A)$ of the bounded linear operator A, let $f(A)$ denote the operator on H defined by the Riesz-Dunford integral [11]

$$f(A) = \frac{1}{2\pi i} \int_C (zI - A)^{-1} f(z) dz,$$

where I is the identity operator on H and C is a positively oriented simple closed rectifiable closed contour containing the spectrum $\sigma(A)$ in the interior domain [11]. The operator $f(A)$ can also be defined by the following series

$$f(A) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} A^n$$

which converges in the norm topology. The class of all functions $f \in \Sigma$ with $a_n \geq 0$ is denoted by Σ_p. Analytically a function $f \in \Sigma$ given by (1.1) is said to be meromorphically starlike of order α if it satisfies the following

$$R\left(-\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbb{U})$$

for some $\alpha (0 \leq \alpha < 1)$. We say that f is in the class $\sum_p^*(\alpha)$ of such functions.

The object of the present paper is to investigate the following subclass of Σ_p associated with the integral operator $P_\mu^\gamma f(z)$.

Definition 1.1. For $0 \leq \beta < 1$ and $0 \leq \xi < 1$, a function $f \in \Sigma_p$ given by the equation (1.1) is in the class $M_p(\xi, \beta, A)$ if

$$\Re\left(\frac{A(P_\mu^\gamma f(A))'}{(\beta - 1)P_\mu^\gamma f(A) + \beta A(P_\mu^\gamma f(A))'}\right) > \xi \tag{1.3}$$

where P_μ^γ is given by equation (1.2).

Remark 1.2. Kavitha et all [18] considered the generalized subclass of meromorphic functions $M_p(\alpha, \lambda)$: For $0 \leq \alpha < 1$ and $0 \leq \lambda < 1$, a function $f \in \Sigma_p$ given by the equation (1.1) is in the class $M_p(\alpha, \lambda)$ if and only if

$$\Re\left(\frac{zf(z)'}{(\lambda - 1)f(z) + \lambda z(f(z))'}\right) > \alpha. \tag{1.4}$$

Remark 1.3. For a function $f \in \Sigma_p$ given by the equation (1.1) if we take the generalized Dziok Srivastava operator instead of the operator P_μ^γ , then the class is $M_p(\alpha, \lambda)$ considered by Rosy et all [21].

Lemma 1.1. *Let $w = u + iv$. Then $R(w) > \alpha \Leftrightarrow |w-1| < |w+1-2\alpha|$.*

By applying Lemma 1.1 we obtain an equivalent definition of Definition 1.1.

Definition 1.2. For $0 \leq \beta < 1$ and $0 \leq \xi < 1$, a function $f \in \Sigma_p$ given by (1.1) is in the class $M_p(\xi, \beta, A)$ if the following inequality is satisfied

$$\begin{aligned} & \left\| A(P_\mu^\gamma f(A))' - \{(\beta-1)P_\mu^\gamma f(A) + \beta A(P_\mu^\gamma f(A))'\} \right\| \\ < & \left\| A(P_\mu^\gamma f(A))' + (1-2\xi)\{(\beta-1)P_\mu^\gamma f(A) + \beta A(P_\mu^\gamma f(A))'\} \right\| \end{aligned}$$

for all operators A with $\|A\| < 1$ and $A \neq \Theta$ (Θ is the zero operator on H).

In the present paper, we obtain coefficient estimates, radii of starlikeness and convexity for the functions in the class $M_p(\xi, \beta, A)$.

2. COEFFICIENT BOUNDS

Theorem 2.1. *A function $f \in \Sigma_p$ given by (1.1) is in the class $M_p(\xi, \beta, A)$ for all proper contraction T with $A \neq \Theta$ if and only if*

$$\sum_{n=1}^{\infty} [n + \xi - \xi\beta(n+1)] L(n, \mu, \gamma) a_n \leq 1 - \xi. \tag{2.5}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{1-\xi}{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)} z^n \quad (n \geq 1)\,. \tag{2.6}$$

Proof. Assume that (2.5) holds. Then we have

$$\begin{aligned} & \left\| A(P_\mu^\gamma f(A))' - \{(\beta-1)P_\mu^\gamma f(A) + \beta A(P_\mu^\gamma f(A))'\} \right\| \\ & - \left\| A(P_\mu^\gamma f(A))' + (1-2\xi)\{(\beta-1)P_\mu^\gamma f(A) + \beta A(P_\mu^\gamma f(A))'\} \right\| \\ = & \left\| \sum_{n=1}^{\infty} (n+1)(1-\beta) L(n,\mu,\gamma) a_n A^n \right\| \\ & - \left\| 2(1-\xi)A^{-1} - \sum_{n=1}^{\infty} [n + (1-2\xi)(\beta - 1 + \beta n)] L(n,\mu,\gamma) a_n A^n \right\| \\ \leq & \sum_{n=1}^{\infty} (n+1)(1-\beta) L(n,\mu,\gamma) a_n \|A\|^n - 2(1-\xi) \left\| A^{-1} \right\| \\ & + \sum_{n=1}^{\infty} [n + (1-2\xi)(\beta - 1 + \beta n)] L(n,\mu,\gamma) a_n \|A\|^n \\ = & \ 2\sum_{n=1}^{\infty} [n + \xi - \xi\beta(n+1)] L(n,\mu,\gamma) a_n \|A\|^n - 2(1-\xi) \left\| A^{-1} \right\| \\ \leq & \ 2(1-\xi) - 2(1-\xi) = 0. \qquad \text{(by using (2.5))} \end{aligned}$$

Thus $f \in \Sigma_p$ is in the class $M_p(\xi, \beta, A)$. Conversely, suppose that $f \in M_p(\xi, \beta, A)$ that is,

$$\begin{aligned} & \left\| A(P_\mu^\gamma f(A))' - \{(\beta-1)P_\mu^\gamma f(A) + \beta A(P_\mu^\gamma f(A))'\} \right\| \\ < & \left\| A(P_\mu^\gamma f(A))' + (1-2\xi)\{(\beta-1)P_\mu^\gamma f(A) + \beta A(P_\mu^\gamma f(A))'\} \right\|. \end{aligned}$$

and from last inequality, it is obtained that

$$\left\|\sum_{n=1}^{\infty}(n+1)(1-\beta)L(n,\mu,\gamma)a_nA^{n+1}\right\|$$
$$< \left\|2(1-\xi)-\sum_{n=1}^{\infty}[n+(1-2\xi)(\beta-1+\beta n)]L(n,\mu,\gamma)a_nA^{n+1}\right\|.$$

Selecting $A=rI\ \ (0<r<1)$ in above inequality, we have

$$\frac{\sum\limits_{n=1}^{\infty}(n+1)(1-\beta)L(n,\mu,\gamma)a_nr^{n+1}}{2(1-\xi)-\sum\limits_{n=1}^{\infty}[n+(1-2\xi)(\beta-1+\beta n)]L(n,\mu,\gamma)a_nr^{n+1}}<1.$$

As $r\to 1^-$, (2.5) is obtained. ☐

Remark 2.4. For $\beta=0$, we get

$$\Re\left(\frac{-A(P_\mu^\gamma f(A))'}{P_\mu^\gamma f(A)}\right)>\xi \tag{2.7}$$

and hence $P_\mu^\gamma f(A)$ is in the class $\sum_p^*(\xi)$ if and only if

$$\sum_{n=1}^{\infty}(n+\xi)L(n,\mu,\gamma)a_n\leq 1-\xi. \tag{2.8}$$

Remark 2.5. If we take the function $f\in\Sigma_p$ given by (1.1) for $z\in\mathbb{U}^*$ and for $\beta=0$, then we obtain the generalized result given by Kavitha el all [18].

Corollary 2.1. If a function $f\in\Sigma_p$ given by (1.1) is in the class $M_p(\xi,\beta,A)$, then

$$a_n\leq\frac{1-\xi}{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)}\quad(n\geq 1).$$

The result is sharp for the function f of the form (2.6).

Remark 2.6. If $P_\mu^\gamma f(A)\in\sum_p^*(\gamma)$, then

$$a_n\leq\frac{1-\xi}{(n+\xi)L(n,\mu,\gamma)}\quad(n\geq 1).$$

Remark 2.7. If we take the function $f\in\Sigma_p$ given by (1.1) for $z\in\mathbb{U}^*$ and for $\beta=0$ in (2.1), then we obtain the generalized corollary given by Kavitha el all [18].

Theorem 2.2. *The class $M_p(\xi,\beta,A)$ is closed under convex combinations.*

Proof. Let the functions

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n\qquad\text{and}\qquad g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}b_nz^n$$

be in the class $M_p(\xi,\beta,A)$. Then, by Theorem 2.1, we have

$$\sum_{n=1}^{\infty}[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)a_n\quad\leq\quad 1-\xi$$

and

$$\sum_{n=1}^{\infty}[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)b_n \quad \leq \quad 1-\xi.$$

For $0 \leq \tau \leq 1$, we define the function h as

$$h(z) = \tau f(z) + (1-\tau)g(z) \text{ and we get } h(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left[\tau a_n + (1-\tau)b_n\right]z^n.$$

Now, we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty}[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)\left[\tau a_n + (1-\tau)b_n\right] \\ = \quad & \tau\sum_{n=1}^{\infty}[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)a_n + (1-\tau)\sum_{n=1}^{\infty}[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)b_n \\ \leq \quad & \tau(1-\xi) + (1-\tau)(1-\xi) = (1-\xi). \text{ So, } h \in M_p(\xi,\beta,T). \end{aligned}$$

□

3. Extreme points

Theorem 3.3. *If*

$$f_0(z) = \frac{1}{z}$$

and

$$f_n(z) = \frac{1}{z} + \frac{1-\xi}{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)}z^n \quad (n = 1, 2, \dots), \tag{3.9}$$

then $f \in M_p(\xi,\beta,A)$ if and only if it can be represented in the form

$$f(z) = \sum_{n=0}^{\infty}\delta_n f_n(z) \quad \left(\delta_n \geq 0, \ \sum_{n=0}^{\infty}\delta_n = 1\right).$$

Proof. Assume that $f(z) = \sum_{n=0}^{\infty}\delta_n f_n(z)$, $(\delta_n \geq 0,\ n = 0, 1, 2, \dots;\ \sum_{n=0}^{\infty}\delta_n = 1)$. Then, we have

$$f(z) = \sum_{n=0}^{\infty}\delta_n f_n(z) = \delta_0 f_0(z) + \sum_{n=1}^{\infty}\delta_n f_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\delta_n\frac{1-\xi}{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)}z^n.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty}[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)\delta_n\frac{1-\xi}{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)} &= (1-\xi)\sum_{n=1}^{\infty}\delta_n \\ = (1-\xi)(1-\delta_0) \leq (1-\xi). \end{aligned}$$

Hence, by Theorem 2.1, $f \in M_p(\xi,\beta,A)$. Conversely, suppose that $f \in M_p(\xi,\beta,A)$. Since, by Corollary 2.2,

$$a_n \leq \frac{1-\xi}{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)} \quad (n \geq 1),$$

setting

$$\delta_n = \frac{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)}{1-\xi}a_n \qquad (n \geq 1)$$

and $\delta_0 = 1 - \sum_{n=1}^{\infty}\delta_n$, we obtain$f(z) = \delta_0 f_0(z) + \sum_{n=1}^{\infty}\delta_n f_n(z)$. □

4. RADII OF STARLIKENESS AND CONVEXITY

We now find the radii of meromorphically close-to-convexity, starlikeness and convexity for functions f in the class $M_p(\xi, \beta, A)$.

Theorem 4.4. *Let $f \in M_p(\xi, \beta, A)$. Then f is meromorphically close-to-convex of order δ $(0 \leq \delta < 1)$ in the disk $|z| < r_1$, where*

$$r_1 = \inf_n \left[\frac{(1-\delta)\left[n + \xi - \xi\beta(n+1)\right] L(n, \mu, \gamma)}{n(1-\xi)} \right]^{\frac{1}{n+1}} \qquad (n \geq 1).$$

The result is sharp for the extremal function f given by equation (1.1)*.*

Proof. It is sufficient to show that

$$\left\| \frac{f'(A)}{A^{-2}} + 1 \right\| < 1 - \delta. \tag{4.10}$$

By Theorem 2.1, we have $\sum\limits_{n=1}^{\infty} \frac{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)}{1-\xi} a_n \leq 1$. So, the inequality

$$\left\| \frac{f'(A)}{A^{-2}} + 1 \right\| = \left\| \sum_{n=1}^{\infty} n a_n A^{n+1} \right\| \leq \sum_{n=1}^{\infty} n a_n \left\| A \right\|^{n+1} < 1 - \delta$$

holds true if

$$\frac{n \left\| A \right\|^{n+1}}{1-\delta} \leq \frac{[n + \xi - \xi\beta(n+1)] L(n, \mu, \gamma)}{1-\xi}.$$

Then, inequality (4.10) holds true if

$$\left\| A \right\|^{n+1} \leq \frac{(1-\delta)\left[n + \xi - \xi\beta(n+1)\right] L(n, \mu, \gamma)}{n(1-\xi)} \qquad (n \geq 1),$$

which yields the close-to-convexity of the family and completes the proof. □

Theorem 4.5. *Let $f \in M_p(\xi, \beta, A)$. Then f is meromorphically starlike of order δ $(0 \leq \delta < 1)$ in the disk $|z| < r_2$, where*

$$r_2 = \inf_n \left[\left(\frac{1-\delta}{n+2-\delta} \right) \frac{[n + \xi - \xi\beta(n+1)] L(n, \mu, \gamma)}{1-\xi} \right]^{\frac{1}{n+1}} \qquad (n \geq 1).$$

The result is sharp for the extremal function f given by (2.6)*.*

Proof. By using the technique employed in the proof of Theorem 4.4, we can show that

$$\left\| \frac{A f'(A)}{f(A)} + 1 \right\| < 1 - \delta, \text{ for } |z| < r_2.$$

□

Theorem 4.6. *Let $f \in M_p(\xi, \beta, A)$. Then f is meromorphically convex of order δ $(0 \leq \delta < 1)$ in the disk $|z| < r_3$, where*

$$r_3 = \inf_n \left[\left(\frac{1-\delta}{n+2-\delta} \right) \frac{[n + \xi - \xi\beta(n+1)] L(n, \mu, \gamma)}{n(1-\xi)} \right]^{\frac{1}{n+1}} \qquad (n \geq 1).$$

The result is sharp for the extremal function f given by

$$f_n(z) = \frac{1}{z} + \frac{n(1-\xi)}{[n + \xi - \xi\beta(n+1)] L(n, \mu, \gamma)} z^n \qquad (n \geq 1).$$

Proof. By using the technique employed in the proof of Theorem (4.4), we can show that

$$\left\| \frac{Af''(A)}{f'(A)} + 2 \right\| < 1 - \delta,$$

for $|z| < r_3$ and prove that the assertion of the theorem is true. □

5. HADAMARD PRODUCT

Theorem 5.7. *For functions $f, g \in \Sigma_p$ defined by equation (1.1) let $f, g \in M_p(\xi, \beta, A)$. Then the Hadamard product $f * g \in M_p(\rho, \beta, A)$, where*

$$\rho \leq 1 - \frac{(1-\xi)^2(n+1)(1-\beta)}{(1-\xi)^2(1-\beta(n+1)) + [n+\xi-\xi\beta(n+1)]^2 L(n,\mu,\gamma)}$$

Proof. Under the hypothesis, it follows from Theorem(2.1) we have

$$\sum_{n=1}^{\infty} \frac{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)}{1-\xi} a_n \leq 1 \tag{5.11}$$

and

$$\sum_{n=1}^{\infty} \frac{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)}{1-\xi} b_n \leq 1. \tag{5.12}$$

We need to find the largest ρ such that

$$\sum_{n=1}^{\infty} \frac{[n+\rho-\rho\beta(n+1)]L(n,\mu,\gamma)}{1-\rho} a_n b_n \leq 1.$$

From inequalities(5.11) and (5.12) we find, by means of the Cauchy-Schwarz inequality, that

$$\sum_{n=1}^{\infty} \frac{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)}{1-\xi} \sqrt{a_n b_n} \leq 1. \tag{5.13}$$

Thus it is enough to show that

$$\frac{[n+\rho-\rho\beta(n+1)]L(n,\mu,\gamma)}{1-\rho} a_n b_n \leq \frac{[n+\xi-\xi\delta(n+1)]L(n,\mu,\gamma)}{1-\xi} \sqrt{a_n b_n}.$$

That is,

$$\sqrt{a_n b_n} \leq \frac{(1-\rho)\,[n+\xi-\xi\beta(n+1)]}{(1-\xi)\,[n+\rho-\rho\beta(n+1)]}. \tag{5.14}$$

On the other hand, from equation(5.3) we have

$$\sqrt{a_n b_n} \leq \frac{1-\xi}{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)}. \tag{5.15}$$

Therefore in view of equations (5.14) and (5.15) it is enough to show that

$$\frac{1-\xi}{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)} \leq \frac{(1-\rho)\,[n+\xi-\xi\beta(n+1)]}{(1-\xi)\,[n+\rho-\rho\beta(n+1)]}$$

which simplifies to

$$\rho \leq \frac{[n+\xi-\xi\beta(n+1)]^2 L(n,\mu,\gamma) - n(1-\xi)^2}{[n+\xi-\xi\beta(n+1)]^2 L(n,\mu,\gamma) + (1-\xi)^2\,[1-\beta(n+1)]} = \phi(n).$$

A simple computation shows that $\phi(n+1) - \phi(n) > 0$ for all n. This means that $F(n)$ is increasing and $\phi(n) \geq \phi(1)$. Using this, the result follows □

Theorem 5.8. *For functions $f, g \in \Sigma_p$ defined by (1.1) let $f, g \in M_p(\xi, \beta, A)$. Then the function $k(z) = \frac{1}{z} + \sum\limits_{n=1}^{\infty} (a_n^2 + b_n^2) z^n$ is in the class $M_p(\xi, \beta, A)$ and*

$$\rho \leq 1 - \frac{2(1-\xi)^2 L(n,\mu,\gamma)\left[1-\beta(n+1)+n\right]}{\left\{\left[n+\xi-\xi\beta(n+1)\right] L(n,\mu,\gamma)\right\}^2 + 2(1-\xi)^2 L(n,\mu,\gamma)\left[1-\beta(n+1)\right]}$$

Proof. Since $f, g \in M_p(\xi, \beta, A)$ we have

$$\sum_{n=1}^{\infty} \left\{ \frac{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)a_n}{1-\xi} \right\}^2 \leq 1 \tag{5.16}$$

and

$$\sum_{n=1}^{\infty} \left\{ \frac{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)b_n}{1-\xi} \right\}^2 \leq 1 \tag{5.17}$$

combining the inequalities (5.16) and (5.17),we get

$$\sum_{n=1}^{\infty} \frac{1}{2} \left\{ \frac{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)}{1-\xi} \right\}^2 (a_n^2 + b_n^2) \leq 1,$$

But, we need to find the largest ρ such that

$$\sum_{n=1}^{\infty} \frac{[n+\rho-\rho\beta(n+1)]L(n,\mu,\gamma)}{1-\rho} (a_n^2 + b_n^2) \leq 1. \tag{5.18}$$

The inequality (5.18) would hold if

$$\frac{[n+\rho-\rho\beta(n+1)]L(n,\mu,\gamma)}{1-\rho} \leq \frac{1}{2} \left\{ \frac{[n+\rho-\rho\beta(n+1)]L(n,\mu,\gamma)}{1-\rho} \right\}^2.$$

Then we have

$$\rho \leq \frac{\left\{\left[n+\xi-\xi\beta(n+1)\right] L(n,\mu,\gamma)\right\}^2 - 2n(1-\xi)^2 L(n,\mu,\gamma)}{\left\{\left[n+\xi-\xi\beta(n+1)\right] L(n,\mu,\gamma)\right\}^2 + 2(1-\xi)^2 L(n,\mu,\gamma)\left[1-\beta(n+1)\right]}$$

$$= 1 - \frac{2(1-\xi)^2 L(n,\mu,\gamma)\left[1-\beta(n+1)+n\right]}{\left\{\left[n+\xi-\xi\beta(n+1)\right] L(n,\mu,\gamma)\right\}^2 + 2(1-\xi)^2 L(n,\mu,\gamma)\left[1-\beta(n+1)\right]} = \phi(n)$$

A simple computation shows that $\phi(n+1) - \phi(n) > 0$ for all n. This means that $F(n)$ is increasing and $\phi(n) \geq \phi(1)$. Using this, we get

$$\rho = 1 - \frac{2(1-\xi)^2 L(n,\mu,\gamma)\left[1-\beta(n+1)+n\right]}{\left\{\left[n+\xi-\xi\beta(n+1)\right] L(n,\mu,\gamma)\right\}^2 + 2(1-\xi)^2 L(n,\mu,\gamma)\left[1-\beta(n+1)\right]}.$$

□

6. INTEGRAL OPERATORS

In this section, we consider integral transforms of functions in the class $M_p(\xi, \beta, A)$ of the type considered by Goel and Sohi [16].

Theorem 6.9. *Let the function* $f \in \Sigma_p$ *given by* (1.1) *be in the class* $M_p(\xi,\beta,A)$. *Then the integral operator*

$$F(z) = c\int_0^z u^c f(uz)du, \quad 0 < u \leq 1\, 0 < c < \infty$$

is in $M_p(\rho,\beta,A)$ *where*

$$\rho = 1 - \frac{(1-\xi)(1+2\beta)+c}{(1+\xi-2\xi\beta)(c+2)+(1-\xi)(1-2\beta)}.$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(1-\xi)(\mu+2)}{(1+\xi-2\xi\beta)\mu}z.$$

Proof. Let $f \in \Sigma_p$ given by (1.1) be in the class $M_p(\xi,\beta,A)$. Then

$$F(z) = c\int_0^1 u^c f(uz)\,du = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n z^n$$

We have to show that

$$\sum_{n=1}^{\infty} \frac{c\,[n+\rho-\rho\beta(n+1)]\,L(n,\mu,\gamma)}{(1-\rho)(c+n+1)} a_n \leq 1. \tag{6.19}$$

Since $f \in M_p(\xi,\beta,A)$ we have

$$\sum_{n=1}^{\infty} \frac{[n+\xi-\xi\beta(n+1)]L(n,\mu,\gamma)}{1-\xi} a_n \leq 1.$$

The inequality (6.19) satisfied if

$$\frac{c\,[n+\rho-\rho\beta(n+1)]}{(1-\rho)(c+n+1)} \leq \frac{[n+\xi-\xi\beta(n+1)]}{1-\xi}.$$

Then we get

$$\begin{aligned} \rho &\leq \frac{[n+\xi-\xi\beta(n+1)](n+c+1)-(1-\xi)cn}{[n+\xi-\xi\beta(n+1)](n+c+1)+c(1-\xi)(1-\beta(n+1))} \\ &= 1 - \frac{(1-\xi)[1+\beta(n+1)]+cn}{[n+\xi-\xi\beta(n+1)](n+c+1)+(1-\xi)[1-\beta(n+1)]} \end{aligned} \tag{6.20}$$

By a simple computation, we can show that the function

$$\phi(n) = 1 - \frac{(1-\xi)[1+\beta(n+1)]+cn}{[n+\xi-\xi\beta(n+1)](n+c+1)+(1-\xi)[1-\beta(n+1)]}$$

is an increasing function of $n (n \geq 1)$ and $\phi(n) \geq \phi(1)$. Using this, we obtain the desired result. ☐

Remark 6.8. If we let the function $f \in \Sigma_p$ given by (1.1) is in the class $\sum_p^*(\gamma)$, then the integral operator $F(z) = c\int_0^z u^c f(uz)du \quad 0 < u \leq 1\, 0 < c < \infty$ is in $\sum_p^*(\gamma)$, where

$$\rho = \frac{(2\xi)(c+1)+2}{(1+\xi)(c+2)+2}.$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(1-\xi)(\mu+2)}{(1+\xi)\mu}z.$$

Remark 6.9. If we let the function $f \in \Sigma_p$ given by (1.1) is in the class $\sum_p^*(\gamma)$ for $\beta = 0$, then we obtain the result given by Kavitha et all [18].

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