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Determining Lucas identities by using Hosoya index

HACÈNE BELBACHIR[1], HAKIM HARIK[1,2] and S. PIRZADA[3]

ABSTRACT. We introduce a new identity of Lucas number by using the Hosoya index. As a consequence we give some properties of Lucas numbers and the extension of the work of Hillard and Windfeldt.

1. INTRODUCTION

We denote by $G = (V(G),\ E(G))$ a simple graph, where $V(G)$ is the set of its vertices and $E(G)$ is the set of its edges. The order of G is $|V(G)|$ and the size of G is $|E(G)|$. For a vertex v of G, $N(v)$ is the set of vertices adjacent to v, $\deg(v) := |N(v)|$ is the degree of v; $Link(v)$ is the set of edges incident to v. In G, an edge between the vertices u and v is denoted by uv. A path P_n, from a vertex v_1 to a vertex v_n, $n \geq 2$, is a sequence of vertices $v_1, \ldots, v_n$ and edges v_iv_{i+1}, for $i = 1, \ldots, n-1$; for simplicity we denote it by $v_1 \ldots v_n$. For $n = 1$, we assume that $P_0P_n = P_nP_0 = P_n$ and for $n = 0$, P_1 is a single vertex v. A cycle is a path with $v_1 = v_n$. A cycle is elementary if all its vertices are different. We denote an elementary cycle on n vertices by C_n.

The graph $G - v$ is obtained from G by deleting the vertex v and removing all the edges which are incident to v. For an edge e of G, we denote by $G - e$ the graph obtained from G by removing e. The contraction of a graph G, associated to an edge e, is the graph G/e obtained by removing e and identifying the end vertices u and v of e and replacing them by a single vertex v' where the edges incident to u or v are now incident to v'. Then we say that in G the adjacent vertices u and v have been contracted into the vertex v'. For further graph theoretical definitions, we refer to [15].

For $n \geq 2$, the well-known Fibonacci $\{F_n\}$ and Lucas $\{L_n\}$ sequences are defined by $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$, where $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively. Moreover, the Fibonacci numbers are connected to the element of Pascal's triangle using the following well known identity

$$F_{n+1} = \sum_k \binom{n-k}{k}.$$

It is well-known that the relation between Lucas and Fibonacci numbers is given by the identity

$$L_n = F_{n+1} + F_{n-1}$$

For some results and properties related to Fibonacci and Lucas numbers, one can see [3]. This sequence finds applications in many areas, particularly in physics and chemistry [13].

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Corresponding author: S. Pirzada; pirzadasd@kashmiruniversity.ac.in

A matching M of a graph G is a subset of $E(G)$ such that no two edges in M share a common vertex. A matching of G is also called an independent edge set of G. A k-matching of a graph G is of cardinality k, that is, an independent edge set of G of cardinality k. We denote by $m(G,k)$ the number of k-matching of G with the convention that $m(G,0)=1$. Note that $m(G,1)=|E(G)|$ and when $k>\frac{n}{2}$, $m(G,k)=0$.

The Hosoya index of a graph G, denoted by $Z(G)$, is an index introduced by Hosoya [12], as follows :

$$Z(G)=\sum_{k=0}^{\lfloor n/2\rfloor} m(G,k),$$

where $n=|V(G)|$, $\lfloor n/2\rfloor$ stands for the integer part of $n/2$. This index has several applications in molecular chemistry such as boiling point, entropy or heat of vaporization. There are several papers on Hosoya index in the literature [1, 2, 4, 5, 6, 8].

2. Preliminary results

First we list the following results. From the definition of the Hosoya index, it is not difficult to deduce the following lemma.

Lemma 2.1. [10] *Let G be a graph, we have the following.*
(1) *If* $uv\in E(G)$, *then* $Z(G)=Z(G-uv)+Z(G-\{u,v\})$.
(2) *If* $v\in V(G)$, *then* $Z(G)=Z(G-v)+\sum_{w\in N_G(v)} Z(G-\{w,v\})$.
(3) *If* $G_1,G_2,\dots,G_t$ *are the components of* G, *then* $Z(G)=\prod_{k=1}^{t} Z(G_k)$.

Lemma 2.1 allows us to compute $Z(G)$ recursively for any graph. The following theorem gives the relation between the Hosoya index and the Fibonacci number (see [9, 10]).

Theorem 2.1. *Let* P_n *be a path on* n *vertices, then* $Z(P_n)=F_{n+1}$.

The next theorem gives the relation between the Hosoya index and the Lucas number (see [9, 10]).

Theorem 2.2. *Let* C_n *be a path on* n *vertices, then* $Z(C_n)=L_n$.

3. Main results

In this section, we introduce a new identity of Lucas numbers which generalizes identities of Lucas numbers given in [11] and answers a question of Melham [14].

Theorem 3.3. *For all positive integers* r_i $(1\le i\le s)$ *and each integer* $s\ge 2$, *we have*

$$L_{r_1+r_2+\ldots+r_s}=\sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s)\in\Omega_s}\prod_{i=1}^{s} F_{r_i+\varepsilon_i}, \tag{3.1}$$

where Ω_s *be the set of* $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_s)$ *such that* $\varepsilon_i\in\{-1,0,1\}$ *and in cycle of* s *cases between each pair of zeros there is nothing or only* $-1's$ *and between two consecutive pairs of zeros there is nothing or* $1's$.

Proof. Let $C_{r_1+r_2+\cdots+r_s}$ be a cycle with $r_1+r_2+\cdots+r_s$ vertices. We subdivide $C_{r_1+r_2+\cdots+r_s}$ in consecutive blocs of paths P_{r_i} with r_i $(1\le i\le s)$ vertices as shown in Figure 1.

On one hand, by Theorem 2.2, we have $Z(C_{r_1+\cdots+r_s})=L_{r_1+\cdots+r_s}$ while on the other hand, $Z(P_{r_1+r_2+\cdots+r_s})$ is the number of independent edge subsets in

$$C_{r_1+r_2+\cdots+r_s}.Z(C_{r_1+r_2+\cdots+r_s})=\sum_{k=0}^{s-1}|M_k|,$$

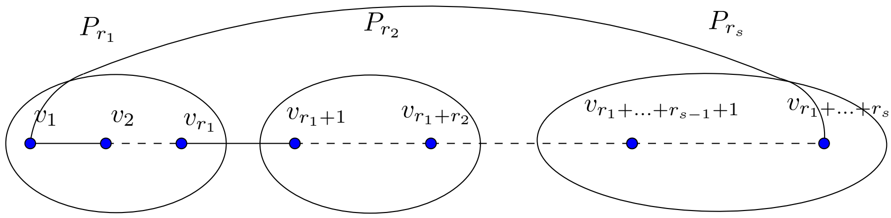

FIGURE 1. A cycle $P_{r_1+r_2+\ldots+r_s}$ subdivide into consecutive blocs of path P_{r_i} with r_i $(1 \leq i \leq s)$ vertices.

where M_k is a set of independent edge subsets in $C_{r_1+r_2+\cdots+r_s}$ such that for every independent edge subset of M_k, there exists k edges between the blocks of paths P_{r_i} $(1 \leq i \leq s)$ which belong to it.

Thus M_0 is a set of independent edge subsets in $C_{r_1+r_2+\cdots+r_s}$ such that for every independent edge subset of M_0, it does not contain any edge between blocs of paths P_{r_i} $(1 \leq i \leq s)$, so all independent edges are in blocks P_{r_i} $(1 \leq i \leq s)$ and using Theorem 2.1 we have $|M_0| = \prod_{i=1}^{s} F_{r_i+1}$.

Now M_1 is a set of independent edge subsets in $P_{r_1+r_2+\cdots+r_s}$ such that for every independent edge subset of M_1, there exists only one edge between blocs of paths P_{r_i} $(1 \leq i \leq s)$ which belong to it. Let H be a subset of M_1 containing the edge

$$v_{r_1+\cdots+r_k} v_{r_1+\cdots+r_k+1} \, (1 \leq k \leq s-1)$$

in all of its independent edge subsets. We contract the adjacent vertices $v_{r_1+\cdots+r_k}$ and $v_{r_1+\cdots+r_k+1}$ in $C_{r_1+\cdots+r_s}$ into one vertex $v^{'}$ and let $P_{r_1+\cdots+r_s-1}$ be a new path after contraction composed of the consecutive blocks of paths

$$P_{r_1}, P_{r_2}, \ldots, P_{r_k-1}, v^{'}, P_{r_{k+1}-1}, \ldots, P_{r_s}.$$

A cycle $C_{r_1+\cdots+r_s-1}$ does not contain any edge between the blocks which belong to the independent edge subsets of H, so

$$|H| = F_{r_1+1} \times F_{r_2+1} \times \ldots \times F_{r_k} \times F_2 \times F_{r_{k+1}} \ldots \times F_{r_s+1}.$$

Thus,

$$|M_1| = \sum_{(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_s) \in \Delta_1} \prod F_{r_i+\varepsilon_i},$$

where Δ_1 is the set of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s)$ such that for $1 \leq i \leq s$, $\varepsilon_i \in \{0, 1\}$ and $\varepsilon_1 \varepsilon_2 \ldots \varepsilon_s$ form a cycle of s cases such that there is only one pair of zeros and between them there is only ϕ.

Further, M_2 is a set of independent edge subsets in $C_{r_1+r_2+\cdots+r_s}$ such that for every independent edge subset of M_2, there exist two edges between the blocks of paths P_{r_i} $(1 \leq i \leq s)$ which belong to it. As for computing of $|M_1|$ and using the contraction method for the two edges between blocks of paths P_{r_i} $(1 \leq i \leq s)$, the cardinality of M_2 can be counted by $\sum_{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s) \in \Delta_2} \prod F_{r_i+\varepsilon_i}$, where Δ_2 is the set of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s)$ such that for $1 \leq i \leq s$, $\varepsilon_i \in \{-1, 0, 1\}$ and $\varepsilon_1 \varepsilon_2 \ldots \varepsilon_s$ form a cycle of s cases such that there is only one pair of zeros separated only -1 or two pairs of zeros between them (the pairs) there is ϕ or 1.

Continuing in this way, we see that M_s is a set of independent edge subsets in $C_{r_1+r_2+\cdots+r_s}$ such that for every independent edge subset of M_s, there exists s edges between blocs of paths P_{r_i} $(1 \leq i \leq s)$ which belong to it. In this case, all paths P_{r_i} $(1 \leq i \leq s)$ lose two vertices after contraction method. Thus, the cardinality of M_s can be counted by $\prod_{i=1}^{s} F_{r_i-1}$. Hence, the identity (3.1) holds. □

Example 3.1. To calculate L_5, we consider a cycle of five vertices C_5. We subdivided the cycle into two paths, the first one with three vertices P_3 and the second one with two vertices P_2, see Figure 2.

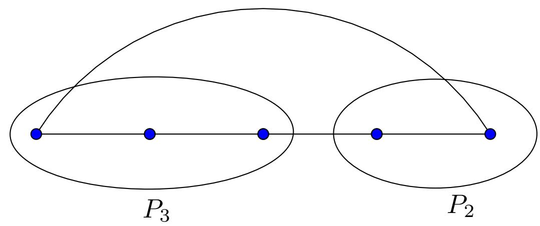

FIGURE 2. A cycle C_5 subdivide into two blocs of paths P_3 and P_2.

Using Theorem 3.3, we have $\Omega_2 = \{(1,1),(0,0),(0,0),(-1,-1)\}$ and $L_5 = F_4F_3 + 2F_3F_2 + F_2F_1 = 11$.

Example 3.2. To calculate $L_6 = L_{3+1+2}$, we consider a C_6, a cycle on six vertices. We subdivided the cycle into three consecutive blocs, the first block contain a path P_3 with three vertices, the second bloc contain one vertex and the third bloc contain a path P_2 with two vertices, see Figure 3.

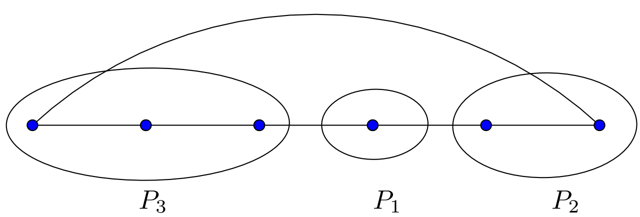

FIGURE 3. A cycle C_6 subdivide into three consecutive blocs of paths P_3 with three vertices, P_1 with one vertex and P_2 with two vertices.

Using Theorem 3.3, we have $\Omega_3 = \{(1,1,1),(1,0,0),(0,1,0),(0,0,1),(-1,0,0),$ $(0,-1,0),(0,0,-1),(-1,-1,-1)\}$ then $L_5 = F_4F_3F_2 + F_4F_1F_2 + F_3F_2F_2 + F_3F_1F_3 + F_2F_1F_2 + F_3F_0F_2 + F_3F_1F_1 + F_2F_1F_0 = 18$.

According to Theorem 3.3, it is easy to see that $|\Omega_s| = 2^s$. The following corollaries are the main results of [11].

Corollary 3.1. *For any non-negative integers r and t, we have*

$$L_{r+t} = F_{r+1}F_{t+1} + 2F_rF_t + F_{r-1}F_{t-1}. \tag{3.2}$$

Proof. From Theorem 3.3 with $s = 2$ and $\Omega_2 = \{(1,1),(0,0),(0,0),(-1,-1)\}$, we obtain the identity. □

Corollary 3.2. *For any non-negative integers u, v and w, we have*

$$L_{u+v+w} = F_{u+1}F_{v+1}F_{w+1} + F_{u+1}F_vF_w + F_uF_{v+1}F_w + F_uF_vF_{w+1} + F_{u-1}F_vF_w + F_uF_{v-1}F_w + F_uF_vF_{w-1} + F_{u-1}F_{v-1}F_{w-1}.$$

Proof. From Theorem 3.3 with $s = 3$ and $\Omega_3 = \{(1,1,1), (1,0,0), (0,1,0), (0,0,1), (-1,0,0), (0,-1,0), (0,0,-1), (-1,-1,-1)\}$, the identity holds. □

- The lines of the following table represents the elements of Ω_4,

1	1	1	1	0	1	0	−1
0	0	1	1	0	1	1	0
1	0	0	1	−1	0	0	−1
1	1	0	0	−1	0	1	0
0	−1	0	1	−1	−1	0	0
1	0	−1	0	0	0	0	0
0	−1	−1	0	0	0	0	0
0	0	−1	−1	−1	−1	−1	−1

- The lines of the following table represents the elements of Ω_5,

1	1	1	1	1	0	0	−1	0	0
0	0	1	1	1	−1	0	1	0	−1
1	0	0	1	1	−1	0	1	1	0
1	1	0	0	1	−1	−1	0	0	−1
1	1	1	0	0	−1	−1	0	1	0
0	−1	0	1	1	−1	−1	−1	0	0
1	0	−1	0	1	0	0	0	0	1
1	1	0	−1	0	0	0	1	0	0
0	−1	−1	0	1	1	0	0	0	0
1	0	−1	−1	0	0	−1	0	0	0
0	−1	−1	−1	0	0	0	0	−1	0
0	0	−1	−1	−1	0	0	0	0	−1
0	1	0	−1	−1	−1	0	0	0	0
0	1	1	0	−1	0	1	0	0	0
0	1	1	1	0	0	0	0	1	0
−1	0	0	−1	−1	−1	−1	−1	−1	−1

Another identity of Lucas number is given in the following result, and this is the equivalent form of Theorem 3.3.

Theorem 3.4. *For any non-negative integer* r_i $(1 \leq i \leq s)$, *we have:*

$$L_{\sum_{i=1}^{s} r_i} = F_{\sum_{i=1}^{s-1} r_i+1} F_{r_s+1} + \sum_{\substack{i+k<s \\ i\neq 0,\, j\neq 0}} \left[\left(\prod_{j=0}^{i-1} F_{r_{s-j}-1} \right) \left(\prod_{j=1}^{k-1} F_{r_j-1} \right) F_{\sum_{j=k+1}^{s-i-1} r_j+1} F_{r_k} F_{r_{s-i}} \right]$$
$$+ \sum_{i=1}^{s-2} \left[\left[\left(\prod_{j=1}^{i} F_{r_{s-j}-1} \right) F_{\sum_{j=1}^{s-i-2} r_j+1} + \left(\prod_{j=1}^{s-i-2} F_{r_j-1} \right) F_{\sum_{j=1}^{i} r_{s-j}+1} \right] F_{r_{s-i-1}} F_{r_s} \right]$$

Proof. As mentioned in Theorem 3.3,

$$L_{\sum_{i=1}^{i=s} r_i+1} = \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_s)\in\Omega_s} \prod_{i=1}^{s} F_{r_i+\varepsilon_i}$$

where Ω_s is the set of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s)$ such that $\varepsilon_i \in \{-1, 0, 1\}$ and in cycle of s cases between each pair of zeros there is nothing or only $-1's$ and between two consecutive pairs of zeros there is nothing or $1's$.That means to count $L_{\sum_{i=1}^{i=s} r_i}$we have three cases:

Case 1. $\varepsilon_s = 1$. In this case for all $s - uplet\ (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s = 1)$, the expression reduces to the following quantity

$$F_{r_s+1}\left(\sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{s-1})\in\Omega_{s-1}}\prod_{i=1}^{s-1}F_{r_i+\varepsilon_i}\right)$$

where Ω_{s-1} is the set of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{s-1})$ such that $\varepsilon_i \in \{-1, 0, 1\}$ and between each pair of zeros there is nothing or only $-1's$ and between two consecutive pairs of zeros there is nothing or $1's$, so for $\varepsilon_s = 1$ we have $F_{\sum_{i=1}^{i=s-1} r_i+1}F_{r_s+1}$.

Case 2. $\varepsilon_s = -1$, in this case, we search a pair of zeros associate that contains ε_s, Hence, to count $F_{\sum_{i=1}^{i=s} r_i+1}$, for $\varepsilon_s = -1$, we have

$$\sum_{\substack{i+k<s\\ i\neq 0,\, j\neq 0}}\left[\left(\prod_{j=0}^{i-1}F_{r_{s-j}-1}\right)\left(\prod_{j=1}^{k-1}F_{r_j-1}\right)F_{\sum_{j=k+1}^{s-i-1}r_j+1}F_{r_k}F_{r_{s-i}}\right].$$

Case 3. $\varepsilon_s = 0$, in this case, we search a pair of zeros associate to ε_s. Hence, to count $F_{\sum_{i=1}^{i=s} r_i+1}$, for $\varepsilon_s = 0$, we have

$$\sum_{i=1}^{s-2}\left[\left[\left(\prod_{j=1}^{i}F_{r_{s-j}-1}\right)F_{\sum_{j=1}^{s-i-2}r_j+1}+\left(\prod_{j=1}^{s-i-2}F_{r_j-1}\right)F_{\sum_{j=1}^{i}r_{s-j}+1}\right]F_{r_{s-i-1}}F_{r_s}\right].$$

□

The following corollary is a particular case of Theorem 3.4.

Corollary 3.3. *For any non-negative integers s and r, we have:*

$$L_{sr} = F_{r+1}F_{(s-1)r+1} + \sum_{\substack{i+k<s\\ i\neq 0,\, j\neq 0}} F_{r-1}^{i+k-1}F_{(s-i-k-1)r-1}F_r^2$$
$$+\sum_{k=0}^{s-2}\left[F_{r-1}^{k}F_{(s-k-2)r+1}F_{r-1}^{s-k-2}F_{kr+1}\right]F_r^2$$

Proof. This is obtained by using Theorem 3.4 with $r_1 = \cdots = r_s = r$. □

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[1]USTHB, Faculty of Mathematics
RECITS Laboratory,
El Alia, 16111, Algiers, Algeria
Email address: hbelbachir@usthb.dz; hacenebelbachir@gmail.com

[2]CERIST
5 Rue des frères Aissou
Algiers, Algeria
Email address: hhakim@mail.cerist.dz

[3]University of Kashmir
Department of Mathematics
Hazratbal, Srinagar, Kashmir, India
Email address: pirzadasd@kashmiruniversity.ac.in