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Some Hermite-Hadamard type inequalities for convex functions via conformable fractional integrals and related inequalities

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ABSTRACT. In this present study, firstly we give some necessary definitions and some results related to Riemann-Liouville fractional and conformable fractional integrals. Secondly, using the given definitions , we established a new identity and Hermite-Hadamard type inequalities via conformable fractional integrals. Relevant connections of the results presented here with those earlier ones are also pointed out.

1. INTRODUCTION

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The following inequality is the classical Hermite Hadamard inequality for convex functions: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

It is a known fact that theory of convex functions is closely related to theory of inequalities. Recently, many researches have studied and investigated this theory. Especially, after 1983, Hermite-Hadamard's inequality has been considered as one of the most useful inequality in mathematical analysis. A number of the papers have been written on this inequality and lots of new results were established on some kinds of convex functions. For more recent results which generalize, improve and extend this inequality, see and references therein ([2], [4], [8], [9], [10]).

Now, for using in this study we need some definitions and mathematical preliminaries of fractional calculus theory. Using classical Hermite Hadamard's inequality some researchers generalized and obtained the results on fractional integrals. For more details, see the papers given in the references ([5], [6], [7], [11], [12], [14]-[20]).

Definition 1.1. ([16]) Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \qquad x < b$$

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respectively. Here $\Gamma(t)$ is the Gamma function and its definition is $\Gamma(t)=\int_0^\infty e^{-x}x^{t-1}dx$.

It is to be noted that $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$ in the case of $\alpha=1$, the fractional integral reduces to the classical integral.

The Beta function defined as follows:

$$B\left(a,b\right)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 t^{a-1}\left(1-t\right)^{b-1}dt,\qquad a,b>0,$$

where $\Gamma\left(\alpha\right)$ is Gamma function. The incomplete beta function is defined by

$$B_x(a,b)=\int_0^x t^{a-1}(1-t)^{b-1}dt.$$

For $x=1$, the incomplete beta function coincides with the complete beta function.

It is remarkable that Sarıkaya et al. gave an integral inequality of Hermite-Hadamard type involving Riemann-Liouville fractional integrals as follows:

Theorem 1.1. ([17]) *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\le a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequality for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\le\frac{f\left(a\right)+f\left(b\right)}{2}\tag{1.2}$$

where $\Gamma\left(\alpha\right)$ is Gamma function.

It is obviously seen that, if we take $\alpha=1$ in Theorem 1.1, then the inequality (1.2) reduces to well known Hermite-Hadamard's inequality as (1.1).

Sarıkaya and Yıldırım established new identity and inequalities via Riemann-Liouville fractional integrals as follows:

Lemma 1.1. ([18]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}f(b)+J_{\left(\frac{a+b}{2}\right)^-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)\\&=\frac{(b-a)}{4}\left\{\int_0^1 t^\alpha f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt-\int_0^1 t^\alpha f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right\}\end{aligned}\tag{1.3}$$

with $\alpha>0$.

Theorem 1.2. ([18]) *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is a convex function on $[a,b]$ for $q\ge1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}f(b)+J_{\left(\frac{a+b}{2}\right)^-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\le\frac{(b-a)}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\left\{\left[(\alpha+1)|f'(a)|^q+(\alpha+3)|f'(b)|^q\right]^{\frac{1}{q}}\right.\\&\left.+\left[(\alpha+3)|f'(a)|^q+(\alpha+1)|f'(b)|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}\tag{1.4}$$

where $\Gamma\left(\alpha\right)$ is the Gamma function.

Theorem 1.3. ([18]) *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|^q$ is a convex function on $[a,b]$ for $q > 1$, then the following inequality for fractional integrals holds:*

$$\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] - f\left(\frac{a+b}{2}\right) \right| \tag{1.5}$$
$$\leq \frac{(b-a)}{4} \left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \left\{ \left[\frac{|f'(a)| + 3|f'(b)|}{4}\right]^{\frac{1}{q}} + \left[\frac{3|f'(a)| + |f'(b)|}{4}\right]^{\frac{1}{q}} \right\}.$$
$$\leq \frac{b-a}{4} \left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}} \left[|f'(a)| + |f'(b)|\right]$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\Gamma(\alpha)$ is the Gamma function.

In the following, we give some definitions and properties of conformable fractional integrals which helps to obtain main identity and results in section 2. In [13], Khalil et al. defined a new well-behaved simple fractional derivative called conformable fractional derivative depending just on the basic limit definition of the derivative. They also defined the fractional integral of order $0 < \alpha \leq 1$ only, for interesting results see also [3].

In [1], Abdeljawad gave the definition of left and right conformable fractional integrals of any order $\alpha > 0$ as follows:

Definition 1.2. Let $\alpha \in (n, n+1]$ and set $\beta = \alpha - n$ then the left conformable fractional integral starting at a of order α is defined by

$$(I^a_\alpha f)(t) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx.$$

Analogously, the right conformable fractional integral is defined by

$$({}^b I_\alpha f)(t) = \frac{1}{n!} \int_t^b (x-t)^n (b-x)^{\beta-1} f(x)dx.$$

Notice that if $\alpha = n+1$ then $\beta = \alpha - n = n+1-n = 1$ where $n = 0, 1, 2, ...$ and hence $(I^a_\alpha f)(t) = (J^a_{n+1} f)(t)$.

Recently, a few studies can be found on conformable fractional integral. In ([21]-[25]) Set et al. obtained some new results and generalizations for convex functions via conformable fractional integrals. In [21], they gave a new generalization of Hermite-Hadamard inequality via conformable fractional integral as follows:

Theorem 1.4. *Let $f : [a,b] \to \mathbb{R}$ be a function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for conformable fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha \Gamma(\alpha-n)} \left[(I^a_\alpha f)(b) + ({}^b I_\alpha f)(a)\right] \leq \frac{f(a) + f(b)}{2} \tag{1.6}$$

with $\alpha \in (n, n+1]$, where Γ is Euler Gamma function.

The aim of this present work is to establish new Hermite-Hadamard type inequalities for conformable fractional integral related to fractional integral.

2. Main results

Now, using the definitions and properties of conformable fractional integral, we will establish some new inequalities connected with the left-side of Hermite Hadamard inequality via conformable fractional integrals. Before establish the inequalities , we need the following useful lemma.

Lemma 2.2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equality for conformable fractional integrals holds;*

$$\begin{aligned} & \frac{2^{\alpha-1}n!}{(b-a)^{\alpha}}\left[(I^{\alpha}_{(\frac{a+b}{2})^{+}}f)(b)+(I^{\alpha}_{(\frac{a+b}{2})^{-}}f)(a)\right]-B(n+1,\alpha-n)f\left(\frac{a+b}{2}\right) \qquad (2.7)\\ = & \frac{b-a}{4}\left\{\int_0^1 B_t(n+1,\alpha-n)f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\right.\\ & \left.-\int_0^1 B_t(n+1,\alpha-n)f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right\}\end{aligned}$$

with $\alpha \in (n, n+1]$, $n = 0,1,2,3...$ where $B_t(a,b)$ is incompleted beta function and Γ is Euler's Gamma function.

Proof. Integrating by parts and changing variables with $x = \frac{t}{2}a + \frac{2-t}{2}b$, we get the following results via conformable fractional integrals;

$$\begin{aligned} I_1 &= \int_0^1 B_t(n+1,\alpha-n)f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt \qquad (2.8)\\ &= B_t(n+1,\alpha-n)f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\frac{2}{a-b}\Big|_0^1\\ &\quad -\frac{2}{a-b}\int_0^1 t^n(1-t)^{\alpha-n-1}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\\ &= \frac{2}{a-b}B(n+1,\alpha-n)f\left(\frac{a+b}{2}\right)\\ &\quad -\left(\frac{2}{a-b}\right)^2\int_b^{\frac{a+b}{2}}\left(2.\frac{x-b}{a-b}\right)^n\left(2.\frac{\frac{a+b}{2}-x}{a-b}\right)^{\alpha-n-1}f(x)dx\\ &= \frac{2}{a-b}B(n+1,\alpha-n)f\left(\frac{a+b}{2}\right)\\ &\quad -\left(\frac{2}{a-b}\right)^2\int_b^{\frac{a+b}{2}}2^n\left(\frac{b-x}{b-a}\right)^n 2^{\alpha-n-1}\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{\alpha-n-1}f(x)dx\\ &= \frac{2}{a-b}B(n+1,\alpha-n)f\left(\frac{a+b}{2}\right)+\frac{4.2^{\alpha-1}n!}{(b-a)^{\alpha+1}}(I^{\alpha}_{(\frac{a+b}{2})^{+}}f)(b),\end{aligned}$$

again in a same way, changing variables with $x = \frac{2-t}{2}a + \frac{t}{2}b$ and integrating on $[0,1]$, we get;

$$\begin{aligned} I_2 &= \int_0^1 B_t(n+1,\alpha-n)f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt \qquad (2.9)\\ &= \frac{2}{b-a}B_t(n+1,\alpha-n)f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\Big|_0^1\\ &\quad -\left(\frac{2}{b-a}\right)\int_0^1 t^n(1-t)^{\alpha-n-1}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\\ &= \frac{2}{b-a}B(n+1,\alpha-n)f\left(\frac{a+b}{2}\right)\end{aligned}$$

$$
\begin{aligned}
& -\left(\frac{2}{b-a}\right)^{2}\int_{a}^{\frac{a+b}{2}}\left(2.\frac{x-a}{b-a}\right)^{n}\left(2.\frac{\frac{a+b}{2}-x}{b-a}\right)^{\alpha-n-1} f(x)dx \\
= \quad & \frac{2}{b-a}B(n+1,\alpha-n)f\left(\frac{a+b}{2}\right)-\frac{4.2^{\alpha-1}n!}{(b-a)^{\alpha+1}}(I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f)(a).
\end{aligned}
$$

Adding the equations I_1, $-I_2$ and multtiplaying with $\frac{b-a}{4}$, which proof is completed. □

Remark 2.1. Noticed that, taking $\alpha = n+1$ in conformable fractional integral inequalities, they reduce to Riemann-Liouville fractional integral inequalities. If we choose $\alpha = n+1$ in equality (2.7), our main identity reduces to the equality (1.3).

Theorem 2.5. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|^q$ is convex on $[a,b]$, then the following inequality for conformable fractional integrals holds;*

$$
\begin{aligned}
&\left|\frac{2^{\alpha-1}n!}{(b-a)^{\alpha}}\left[(I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f)(b)+(I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f)(a)\right]-B(n+1,\alpha-n)f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4}\Psi^{1-\frac{1}{q}}\left\{\left(\Psi_1|f'(a)|^q+\Psi_2|f'(b)|^q\right)^{\frac{1}{q}}+\left(\Psi_2|f'(a)|^q+\Psi_1|f'(b)|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}
\tag{2.10}
$$

where

$$
\begin{aligned}
\Psi &= \big[B(n+1,\alpha-n)-B(n+2,\alpha-n)\big], \\
\Psi_1 &= \Big[\frac{B(n+1,\alpha-n)-B(n+3,\alpha-n)}{4}\Big], \\
\Psi_2 &= \Big[\frac{3B(n+1,\alpha-n)-4B(n+2,\alpha-n)+B(n+3,\alpha-n)}{4}\Big]
\end{aligned}
\tag{2.11}
$$

with $q \geq 1$.

Proof. Taking modulus in Lemma 2.2 and using the well-known Power mean inequality with convexity of $|f'|^q$, we get;

$$
\begin{aligned}
&\left|\frac{2^{\alpha-1}n!}{(b-a)^{\alpha}}\left[(I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f)(b)+(I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f)(a)\right]-B(n+1,\alpha-n)f\left(\frac{a+b}{2}\right)\right| \\
\leq \quad & \frac{b-a}{4}\left\{\left|\int_0^1 B_t(n+1,\alpha-n)f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\right|\right. \\
&\left.+\left|\int_0^1 B_t(n+1,\alpha-n)f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right|\right\} \\
\leq \quad & \frac{b-a}{4}\left\{\left[\int_0^1 B_t(n+1,\alpha-n)dt\right]^{1-\frac{1}{q}}\left[\int_0^1 B_t(n+1,\alpha-n)\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\int_0^1 B_t(n+1,\alpha-n)dt\right]^{1-\frac{1}{q}}\left[\int_0^1 B_t(n+1,\alpha-n)\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right\}
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
\leq \quad & \frac{b-a}{4}\left[\int_0^1 B_t(n+1,\alpha-n)dt\right]^{1-\frac{1}{q}} \\
& \times\left\{\left[|f'(a)|^q\int_0^1 B_t(n+1,\alpha-n)\frac{t}{2}dt+|f'(b)|^q\int_0^1 B_t(n+1,\alpha-n)\left(\frac{2-t}{2}\right)dt\right]^{\frac{1}{q}}\right. \\
& \left.+\left[|f'(a)|^q\int_0^1 B_t(n+1,\alpha-n)\left(\frac{2-t}{2}\right)dt+|f'(b)|^q\int_0^1 B_t(n+1,\alpha-n)\frac{t}{2}dt\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Integrating by parts we can write the following calculations;

$$
\begin{aligned}
\Psi_1 \quad &= \int_0^1 B_t(n+1,\alpha-n)\frac{t}{2}dt \qquad (2.13)\\
&= B_t(n+1,\alpha-n)\frac{t^2}{4}\Big|_0^1-\frac{1}{4}\int_0^1 t^{n+2}(1-t)^{\alpha-n-1}dt \\
&= \left[\frac{B(n+1,\alpha-n)-B(n+3,\alpha-n)}{4}\right], \\
\Psi_2 \quad &= \int_0^1 B_t(n+1,\alpha-n)\left(1-\frac{t}{2}\right)dt \qquad (2.14)\\
&= B_t(n+1,\alpha-n)\left(t-\frac{t^2}{4}\right)\Big|_0^1-\int_0^1 t^n(1-t)^{\alpha-n-1}\left(t-\frac{t^2}{4}\right)dt \\
&= \left[\frac{3B(n+1,\alpha-n)-4B(n+2,\alpha-n)+B(n+3,\alpha-n)}{4}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi \quad &= \int_0^1 B_t(n+1,\alpha-n)dt \qquad (2.15)\\
&= B_t(n+1,\alpha-n)t\Big|_0^1-\int_0^1 t^{n+1}(1-t)^{\alpha-n-1}dt \\
&= \big[B(n+1,\alpha-n)-B(n+2,\alpha-n)\big].
\end{aligned}
$$

Thus, combining (2.13)-(2.15) in (2.12), the proof is completed. □

Corollary 2.1. *If we choose $q=1$ in Theorem 2.5, we get:*

$$
\begin{aligned}
&\left|\frac{2^{\alpha-1}n!}{(b-a)^\alpha}\left[\left(I_{\left(\frac{a+b}{2}\right)^+}f\right)(b)+\left(I_{\left(\frac{a+b}{2}\right)^-}f\right)(a)\right]-B(n+1,\alpha-n)f\left(\frac{a+b}{2}\right)\right| \qquad (2.16)\\
&\leq \frac{b-a}{4}\left[B(n+1,\alpha-n)-B(n+2,\alpha-n)\right]\left(\left|f'(a)\right|+\left|f'(b)\right|\right)
\end{aligned}
$$

Remark 2.2. If we choose $\alpha=n+1$ in Theorem 2.5, we get the inequality (1.4) as in the Theorem 1.2.

Theorem 2.6. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|^q$ is convex on $[a,b]$, then the following inequality for conformable fractional integrals holds;*

$$\left| \frac{2^{\alpha-1}n!}{(b-a)^\alpha}\left[\left(I_{\left(\frac{a+b}{2}\right)^+}f\right)(b) + \left(I_{\left(\frac{a+b}{2}\right)^-}f\right)(a)\right] - B(n+1,\alpha-n)f\left(\frac{a+b}{2}\right)\right| \tag{2.17}$$

$$\leq \frac{b-a}{4}[\Omega]^{\frac{1}{p}}\left[\left(\frac{\left|f'(a)\right|^q + 3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\left|f'(a)\right|^q + \left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{4}[4\Omega]^{\frac{1}{p}}\left[|f'(a)| + |f'(b)|\right]$$

where $\Omega = \displaystyle\int_0^1 \left(B_t(n+1,\alpha-n)\right)^p dt$, $\frac{1}{p}+\frac{1}{q}=1,\ q>1.$

Proof. Taking modulus in Lemma 2.2 and using the well-known Hölder inequality with convexity of $|f'|^q$, we get;

$$\left| \frac{2^{\alpha-1}n!}{(b-a)^\alpha}\left[(I_{\left(\frac{a+b}{2}\right)^+}f)(b) + (I_{\left(\frac{a+b}{2}\right)^-}f)(a)\right] - f\left(\frac{a+b}{2}\right)\right| \tag{2.18}$$

$$\begin{aligned}
\leq\ & \frac{b-a}{4}\left\{\left|\int_0^1 B_t(n+1,\alpha-n)f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\right|\right. \\
&\left.+\left|\int_0^1 B_t(n+1,\alpha-n)f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right|\right\} \\
\leq\ & \frac{b-a}{4}\left\{\left[\int_0^1 \left(B_t(n+1,\alpha-n)\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\int_0^1 \left(B_t(n+1,\alpha-n)\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right\} \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 \left(B_t(n+1,\alpha-n)\right)^p dt\right]^{\frac{1}{p}} \\
&\times\left\{\left[\int_0^1 \frac{t}{2}|f'(a)|^q dt + \int_0^1 \frac{2-t}{2}|f'(b)|^q dt\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\int_0^1 \frac{2-t}{2}|f'(a)|^q dt + \int_0^1 \frac{t}{2}|f'(b)|^q dt\right]^{\frac{1}{q}}\right\} \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 \left(B_t(n+1,\alpha-n)\right)^p dt\right]^{\frac{1}{p}} \\
&\times\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Let $a_1 = 3|f'(a)|^q,\ a_2 = |f'(a)|^q,\ b_1 = 3|f'(b)|^q,\ b_2 = |f'(b)|^q$. Here, $0 < \frac{1}{q} < 1$ for $q > 1$. Using the fact that,

$$\sum_{k=1}^n (a_k+b_k)^s \leq \sum_{k=1}^n a_k^s + \sum_{k=1}^n b_k^s$$

For $(0 \leq s < 1)$. $, a_1, a_2..., a_n \geq 0, \;\; b_1, b_2..., b_n \geq 0,$ we obtain

$$\begin{aligned} & \left| \frac{2^{\alpha-1} n!}{(b-a)^{\alpha}} \left[\left(I_{\left(\frac{a+b}{2}\right)^{+}} f\right)(b) + \left(I_{\left(\frac{a+b}{2}\right)^{-}} f\right)(a) \right] - f\left(\frac{a+b}{2}\right) \right| \qquad (2.19) \\ \leq \;& \frac{b-a}{16} \left[4 \int_0^1 \left(B_t(n+1, \alpha-n)\right)^p dt \right]^{\frac{1}{p}} \\ & \times \left[\left(|f^{'}(a)|^q + 3|f^{'}(b)|^q \right)^{\frac{1}{q}} + \left(3|f^{'}(a)|^q + |f^{'}(b)|^q \right) \right]^{\frac{1}{q}} \\ \leq \;& \frac{b-a}{16} \left[4 \int_0^1 \left(B_t(n+1, \alpha-n)\right)^p dt \right]^{\frac{1}{p}} (3^{\frac{1}{q}} + 1) \left[|f^{'}(a)| + |f^{'}(b)| \right] \\ \leq \;& \frac{b-a}{16} \left[4 \int_0^1 \left(B_t(n+1, \alpha-n)\right)^p dt \right]^{\frac{1}{p}} 4 \left[|f^{'}(a)| + |f^{'}(b)| \right] \end{aligned}$$

which completes the proof. □

Remark 2.3. For $\alpha = n+1$, by considering the fact that

$$B_t(n+1, \alpha-n) = \int_0^t x^n (1-x)^{\alpha-n-1} dx = \int_0^t x^{\alpha-1} dx = \frac{t^{\alpha}}{\alpha}$$

inequalitiy (2.17) in Theorem (2.6) becomes inequality (15) as in Theorem 1.3.

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