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Fixed point theorems for uniformly generalized Kannan type semigroup of self-mappings

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ABSTRACT. In this paper, we consider a new uniformly generalized Kannan type semigroup of self-mappings defined on a closed convex subset of a real Banach space equipped with uniform normal structure and employ the same to show that such semigroup of self-mappings admits a common fixed point provided the underlying semigroup of self-mappings has a bounded orbit.

1. INTRODUCTION

The origin of metric fixed point theory can be traced back to classical Banach contraction theorem which was originated in the Ph.D. thesis of Banach in 1920. This useful and applicable theorem was later published in the form of a research paper in 1922 which has already earned around 2000 google citations. Indeed, this theorem is one of the most useful theorems ever proved in classical functional analysis. In the long course of last several decade, this natural theorem has been generalized and extended by improving the involved contraction condition or lightening the requirement of completeness and by now there exists an extensive literature which are also available in the form of books and survey articles. To mention a few, we recall the books due to Takahashi [22], Goebel and Kirk [8](Book), Rhoades [20](Survey article) and even many more.

In this paper, we prove our results on the common fixed point of semigroup of operators defined on suitable subsets of a Banach space. Technically speaking, some of the fixed point results proved for uniformly Lipschitzian mappings were extended to uniformly Lipschitzian semigroup of self-mappings and even more generally to Lipschitzian semigroup of self-mappings (e.g. [21, 23–32]). Recall that such mappings were first studied by Goebel and Kirk [9] wherein authors proved the existence of a fixed point of a k-uniformly Lipschitzian mapping T defined on a bounded closed convex subset of a uniformly convex Banach space B provided $k < \gamma$ and $\gamma > 1$ is the unique solution of the equation

$$(1 - \delta_B(1/\gamma))\gamma = 1, \tag{1.1}$$

where δ_B denotes the modulus of convexity of B.

In 1973, Goebel and Kirk [9] posed the following question:

Question. Whether (or not) the constant $\gamma > 1$ satisfying equation (1.1) is the greatest real number for which any k-uniformly Lipschitzian mapping T (with $k < \gamma$) has a fixed point?

In 1993, Tan and Xu [21] answered the question of Goebel and Kirk [9] in the negative by proving the existence of a fixed point of a k-uniformly Lipschitzian one parameter

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semigroup of self-mappings defined on K provided $k < \gamma < \alpha,$ and $\alpha > 1$ is the unique solution of the following equation:

$$\frac{\alpha^2}{N(B)}\delta_B^{-1}(1-\frac{1}{\alpha}) = 1,$$

where $N(B) > 1$ denotes the normal structure coefficient of B.

Now, it remains an interesting question to find another constant α^* strictly greater than α for which every k-uniformly Lipschitzian mapping T (with $k < \alpha^*$) has a fixed point.

In 2010, Ceng et al. [6] answered this question by proving the following theorem:

Theorem 1.1. [6] *Let K be a nonempty closed convex subset of a real Banach space B with normal structure coefficient $N(B) > \max\{1, \varepsilon_0\}$. Let $\tau = \{T_s : s \in S = [0, \infty)\}$ be a k-uniformly Lipschitzian semigroup of self-mappings defined on K wherein $k < \alpha^*$ with*

$$\alpha^* = \sup\Big\{\alpha : \alpha^2\delta_B^{-1}(1-\frac{1}{\alpha})N(B)^{-1} \leq 1 \textit{ and } 1-\frac{1}{\alpha} \in (0, 1-\frac{\varepsilon_0}{2})\Big\},$$

while ε_0 denotes the characteristic of convexity of B. If $\{T_s u_0 : s \in S\}$ is bounded for some $u_0 \in K$, then the semigroup τ admits a common fixed point (i.e., there exists $u^* \in K$ such that $T_s u^* = u^*$, for all $s \in S$.)

He also proved yet another fixed point result for Lipschitzian semigroup of mappings as follows:

Theorem 1.2. [6] *Let K be a nonempty bounded subset of a uniformly convex Banach space B, and $\tau = \{T_s : s \in S\}$ a k-uniformly Lipschitzian semigroup of self-mappings mappings defined on K such that:*

$$k < \sqrt{\gamma_0 N(B)},$$

where $\gamma_0 = \inf\{\gamma \geq 1 : \gamma(1-\delta_B(1/\gamma)) \geq 1/2\}$. If there exists a nonempty bounded closed convex subset E of K enjoying the following property:

$$(\Re)\ u \in E \textit{ implies } w_w(u) \subset E,$$

where $w_w(u)$ stands for set of weak w-limit of τ at u, then semigroup τ admits a common fixed point (i.e., there exists $u^* \in E$ such that $T_s u^* = u^*$ for all $s \in S$).

Recall that a Banach space B is said to be with normal structure if every weakly compact convex subset K of B with more than one point contains a non-diametral point, $i.e.,$ $u_0 \in K$ for which

$$\sup\{||u_0 - v|| : v \in K\} < diam(K).$$

Every Banach space having normal structure also has weak normal structure, but the converse is not true in general. In reflexive Banach spaces these properties are equivalent.

As every Lipschitzian mapping is uniformly continuous, it is natural to ask if there exist contractive mappings which are discontinuous. This question was answered in affirmative by Kannan [15] which has greatly influenced the recent research of this domain. In fact, Lipschitzian mappings are always continuous and Kannan type mappings are not necessarily continuous. In what follows, a semigroup of self-mappings is called uniformly Kannan if the following condition is satisfied:

$$d(T(t)u, T(t)v) \leq \beta[d(u, T(t)u) + d(v, T(t)v)], \quad t \in S,$$

for all $u, v \in B$ and $0 < \beta < \frac{1}{2}$.

There already exists extensive literature around the Kannan type of mappings. For the work of this kind one can be referred [2,3,14].

The aim of this paper is to introduce a relatively larger class of generalized uniformly Kannan type semigroup of self-mappings and utilize the same to prove our results by replacing the k-uniformly Lipschitzian semigroup of self-mappings with generalized uniformly Kannan type semigroup of self-mappings.

2. Preliminaries

Let K be a nonempty closed convex subset of a real Banach space B and S an unbounded subset of $[0, \infty)$ such that $t + h \in S$ and $t - h \in S$ (for all $t, h \in S$ with $t > h$). A collection $\tau = \{T_s : s \in S\}$ of self-mappings on K is said to be generalized uniformly Kannan type semigroup of self-mappings defined on K if $T_{s+t}u = T_sT_tu$ (for all $s, t \in S$ and $u \in K$), for each $x \in K$, $s \mapsto T_sx$ is continuous and there exists a constant $0 \leq \beta < 1$ (for all $u, v \in K$) such that

$$||T_su - T_sv|| \leq \beta\{||u - T_su|| + ||v - T_sv||\}, \text{for each } s \in S. \quad (2.2)$$

Example 2.1. Let $X = \mathbb{R}, K = [0, 1]$ and $S = [0, \infty)$. Define $T_t : K \to K$ by

$$T_s(x) = \left(\frac{1}{2}\right)^s x, \text{ for all } x \in K \text{ and each } s \in S.$$

Here for each $s \in S$, T_s satisfies the requirement of uniformly Lipschitzian mapping but fails to satisfy the condition (2.2) (e.g., $x = 1, y = 0, s = 1$). Observe that this example exhibit that this two concepts are genuinely difference.

Geometrically speaking, B is strictly convex if its unit spheres do not contain any line segments. Technically speaking, B is strictly convex if and only if the following implication holds:

$$u, v \in B, \ \ ||u|| = ||v|| = 1 \text{ and } ||(u + v)/2|| = 1 \ \ \Rightarrow u = v.$$

The Modulus of convexity of a Banach space B, is a function $\delta_B : [0, 2] \to [0, 1]$ by

$$\delta_B(\varepsilon) = \inf\{1 - ||(u + v)/2|| : ||u|| \leq 1, ||v|| \leq 1 \text{ and } ||u - v|| \geq \varepsilon\}.$$

Also, the characteristic of convexity of B is a number $\varepsilon_0(B) = \sup\{\varepsilon \in [0, 2] : \delta_B(\varepsilon) = 0\}$. It is easy to see (cf. [12]) that B is uniformly convex if and only if $\varepsilon_0(B) = 0$; uniformly nonsquare if and only if $\varepsilon_0(B) < 2$; and strictly convex if and only if $\delta(2) = 1$. Moreover, if $\varepsilon_0(B) < 1$; then B has a normal structure, $i.e.$, each bounded convex subset H of B which contains more than one non-diametral point $i.e.$, a point u_0 such that

$$\sup\{||u_0 - u|| : u \in H\} < diam(H).$$

The modulus of convexity δ_B of a Banach space B has well-known following properties (see [13]):

(i) δ_B is increasing on [0,2], and moreover strictly increasing on $[\varepsilon_0, 2]$;
(ii) δ_B is continuous on [0,2) (but not necessarily at $\varepsilon = 2$);
(iii) B is strictly convex if and only if $\delta_B(2) = 1$;
(iv) $\delta_B(0) = 0$ and $\lim_{\varepsilon \to 2^-} \delta_B(\varepsilon) = 1 - \varepsilon_0/2$;
(v) $[||a - u|| \leq r, ||a - v|| \leq r$ and $||u - v|| \geq \varepsilon] \ \Rightarrow \ ||a - (u + v)/2|| \leq r(1 - \delta_B(\varepsilon/r))$.

Recall that the normal structure coefficient $N(B)$ of B is the number (see [5])

$$\inf\left\{\frac{diamK}{r_K(K)}\right\},$$

where the infimum is taken over all bounded closed convex subsets K of B with more than one members, and $r_K(K)$ and $diam(K)$ respectively, stand for Chebyshev radii of K relative to itself and the diameter of K, $i.e.$, $r_K(K) = \inf_{u \in K} \sup_{v \in K} ||u - v||$ and $diamK =$

$\sup_{u,v\in K} ||u-v||$. A Banach space B is said to have uniform normal structure if $N(B) > 1$. It is known that a Banach space with uniform normal structure is reflexive and that all uniformly smooth or uniformly convex Banach spaces have uniform normal structure (see [32]). It is also well known that $N(H) = \sqrt{2}$ for a Hilbert space H. Here it can be pointed out that computations of the normal structure coefficient $N(B)$ for general Banach spaces is quite complicated. No exact values of $N(B)$ are known except for some special cases (e.g., all Hilbert as well as L^p spaces). In general, we have the following lower bounded for $N(B)$ (see [1,5,18]):

$$N(B) \geq \frac{1}{1-\delta_B(1)}.$$

Other lower bounds for $N(B)$ in terms of some Banach space parameters or constants can be found in [16,19].

Tan and Xu [21] have also shown that if B is uniformly convex and $\gamma > 1$ is the unique solution of (1.1), then $N(B) > \gamma$. Recall that for a Hilbert space H, we have $N(H) = \sqrt{2}$ and $\gamma = \sqrt{5}/2$.

If B is uniformly convex Banach space, then it can be easily seen that

$$\alpha^2\delta_B^{-1}(1-\frac{1}{\alpha})\widetilde{N}(B) = 1 \tag{2.3}$$

has a unique solution $\alpha > 1$, where $\widetilde{N}(B) = 1/N(B)$. Tan and Xu [21] proved that if $\gamma > 1$ and $\alpha > 1$ are the solution of (1.1) and (2.3), respectively, then $\gamma < \alpha$. Notice that $\gamma = \sqrt{5}/2$, and $\alpha = \frac{1}{\sqrt{\sqrt{3}-1}} > \gamma$.

We also use the notation of asymptotic center essentially due to Edelstein [7]. Let K be a nonempty closed convex subset of a Banach space B and $\{u_t : t \in S\}$ a bounded net of elements of B. Then the asymptotic radius and asymptotic center of $\{u_t\}_{t\in S}$ with respect to K are the number

$$r_K\{u_t\} = \inf_{v\in K} \limsup_t ||u_t - v||,$$

and respectively, the (possibly empty) set

$$A_K(\{u_t\}) = \{v \in K : \limsup_t ||u_t - v|| = r_K(\{u_t\})\}.$$

The following lemma is required.

Lemma 2.1. [21, Lemma 2.1] *If K is a nonempty closed convex subset of a reflexive Banach space B, then for every bounded net $\{u_t\}_{t\in S}$ of elements of B, $A_K(\{u_t\})$ is a nonempty bounded closed convex subset of K. In particular, if B is a uniformly convex Banach space, then $A_K(\{u_t\})$ consists of a single point.*

The following lemma can be proved on the lines of Lim [17], hence proof is omitted.

Lemma 2.2. *If B is a Banach space equipped with uniformly normal structure, then for every bounded net $\{u_t\}_{t\in S}$ of elements of B, there exists $v \in \overline{co}(\{u_t : t \in S\})$ such that*

$$\limsup_t ||u_t - v|| \leq \widetilde{N}(B)D(\{u_t\}),$$

where $\widetilde{N}(B) = 1/N(B)$, $\overline{co}(E)$ is the closure of the convex hull of a set $E \subset B$ and $D(\{u_t\}) = \lim_t(\sup\{||u_i - u_j|| : t \leq i, j \in S\})$ the asymptotic diameter of $\{u_t\}$.

3. MAIN RESULTS

Our main result runs as follows.

Theorem 3.3. *Let B be a real Banach space with $N(B) > \max\{1, \varepsilon_0\}$ and K a nonempty closed convex subset of B wherein ε_0 stands for the characteristic of convexity of B. Let $\tau = \{T_s : s \in S\}$ be a generalized uniformly Kannan type semigroup of self-mappings defined on K with*

$$3\xi\beta\widetilde{N}(B)\delta_B^{-1}(1 - \frac{1}{\xi}) < 1, \tag{3.4}$$

where $\xi = \frac{3\beta}{1-\beta}$, $0 < \beta < 1$ and $\widetilde{N}(B) = N(B)^{-1}$. If $\{T_s u_0 : s \in S\}$ is bounded for some $u_0 \in K$, then the semigroup τ admits a common fixed point ($\exists u^* \in K$ such that $T_s u^* = u^*$ for all $s \in S$).

Proof. Since B is equipped with uniform normal structure, B is reflexive. Owing to the boundedness of $\{T_s u_0 : s \in S\}$ and Lemma 2.1, $A_K(\{T_t u_0\}_{t\in S})$ is nonempty bounded, closed and convex subset of K. Therefore, we can choose $u_1 \in A_K(\{T_t u_0\}_{t\in G})$ such that

$$\limsup_t ||T_t u_0 - u_1|| = \inf_{v\in K} \limsup_t ||T_t u_0 - v||.$$

Recursively, we can choose $u_2 \in A_K(\{T_t u_1\}_{t\in S})$ such that

$$\limsup_t ||T_t u_1 - u_2|| = \inf_{v\in K} \limsup_t ||T_t u_1 - v||.$$

Continuing this process indefinitely, we furnish a sequence $\{u_n\}_{n=0}^{\infty}$ in K with the following properties (for each $n \geq 0$):

(i) $\{T_t u_n\}_{t\in S}$ is bounded;
(ii) $u_{n+1} \in A_K(\{T_t u_n\}_{t\in S})$; *i.e.*, u_{n+1} is a point in K such that

$$\lim_t ||T_t u_n - u_{n+1}|| = \inf_{v\in K} \lim_t ||T_t u_n - v||.$$

Write $r_n = r_K(\{T_t u_n\}_{t\in S})$. Then by Lemma 2.2, we have

$$\begin{aligned} r_n &= \limsup_t ||T_t u_n - u_{n+1}|| \\ &\leq \widetilde{N}(B)D(\{T_t u_n\}_{t\in S}) \\ &= \widetilde{N}(B)\lim_t(\sup\{||T_i u_n - T_j u_n|| : t \leq i, j \in S\}) \\ &= \widetilde{N}(B)\lim_t(\sup\{||T_i u_n - T_i T_{j-i} u_n|| : t \leq i, j \in S\}) \\ &\leq \widetilde{N}(B)\beta \limsup_t\{||u_n - T_i u_n|| + ||T_j u_n - T_{j-i} u_n||\} \\ &\leq \widetilde{N}(B).\beta.3d(u_n), \end{aligned}$$

so that

$$r_n \leq 3\beta\widetilde{N}(B)d(u_n), \tag{3.5}$$

where

$$d(u_n) = \sup\{||u_n - T_t u_n|| : t \in S\}.$$

Without loss of generality one may assume that $d(u_n) > 0$ for all $n \geq 0$ (otherwise u_n is a common fixed point of the semigroup τ and hence proof is completed). If $n \geq 0$ is fixed and $\varepsilon > 0$ is small enough, then we can choose $j \in S$ such that

$$||T_j u_{n+1} - u_{n+1}|| > d(u_{n+1}) - \varepsilon$$

and thereafter choose $s_0 \in S$ large enough so that

$$||T_s u_n - u_{n+1}|| < (r_n + \varepsilon), \ \ \forall\, s \geq s_0.$$

Now, for $s \geq s_0 + j$,

$$\begin{aligned} ||T_s u_n - T_j u_{n+1}|| &\leq \beta[||T_s u_n - T_{s-j} u_n|| + ||u_{n+1} - T_j u_{n+1}||] \\ &< \beta[3(r_n + \varepsilon) + ||T_s u_n - T_j u_{n+1}||] \end{aligned}$$

so that

$$||T_s u_n - T_j u_{n+1}|| < \xi(r_n + \varepsilon).$$

Now, it follows from property (v) that

$$||T_s u_n - \frac{1}{2}(u_{n+1} + T_j u_{n+1})|| \leq \xi(r_n + \varepsilon)\Big(1 - \delta_B\Big(\frac{d(u_{n+1}) - \varepsilon}{\xi(r_n + \varepsilon)}\Big)\Big)$$

for $s \geq s_0 + j$ and hence

$$r_n \leq \limsup_s ||T_s u_n - \frac{1}{2}(u_{n+1} + T_j u_{n+1})|| \leq \xi(r_n + \varepsilon)\Big(1 - \delta_B\Big(\frac{d(u_{n+1}) - \varepsilon}{\xi(r_n + \varepsilon)}\Big)\Big).$$

On taking limit as $\varepsilon \to 0$, we obtain

$$r_n \leq \xi r_n\Big(1 - \delta_B\Big(\frac{d(u_{n+1})}{\xi r_n}\Big)\Big),$$

which implies that

$$\delta_B\Big(\frac{d(u_{n+1})}{\xi r_n}\Big) \leq 1 - \frac{1}{\xi}$$

or

$$d(u_{n+1}) \leq \xi r_n \delta_B^{-1}(1 - \frac{1}{\xi}). \tag{3.6}$$

Therefore, utilizing (3.5) and (3.6), we obtain

$$d(u_{n+1}) \leq 3\beta\xi\widetilde{N}(B)\delta_B^{-1}(1 - \frac{1}{\xi})d(u_n). \tag{3.7}$$

Write $A = 3\beta\xi\widetilde{N}(B)\delta_B^{-1}(1 - \frac{1}{\xi})$. Then $A < 1$. Hence, it follows from (3.7) that

$$d(u_n) \leq Ad(u_{n-1}) \leq \cdots \leq A^n d(u_0). \tag{3.8}$$

Since

$$\begin{aligned} ||u_{n+1} - u_n|| &\leq \limsup_t ||T_t u_n - u_{n+1}|| + \limsup_t ||T_t u_n - u_n|| \\ &\leq r_n + d(u_n) \\ &\leq (3\beta\widetilde{N}(B) + 1)d(u_n) \to 0 \text{ as } n \to \infty, \end{aligned}$$

In view of (3.8), $\sum_{n=1}^{\infty} ||u_{n+1} - u_n|| < \infty$, and hence the sequence $\{u_n\}$ is a Cauchy. Let $u^* = \lim_{n\to\infty} u_n$. Finally, we have (for each $s \in S$,)

$$\begin{aligned} ||u^* - T_s u^*|| &\leq ||u^* - u_n|| + ||T_s u_n - u_n|| + ||T_s u_n - T_s u^*|| \\ &\leq ||u^* - u_n|| + d(u_n) + \beta[d(u_n) + ||u^* - T_s u^*||] \\ &\leq ||u^* - u_n|| + (1 + \beta)d(u_n) + \beta||u^* - T_s u^*|| \end{aligned}$$

or

$$||u^* - T_s u^*|| \leq \frac{1}{1 - \beta}[||u^* - u_n|| + (1 + \beta)d(u_n)] \to 0 \text{ as } n \to \infty.$$

Hence, $T_s u^* = u^*$ for all $s \in S$ and this concludes the proof. □

Our next theorem is proved in uniformly convex Banach spaces.

Theorem 3.4. *Let K be a nonempty bounded subset of a uniformly convex Banach space B and $\tau = \{T_s : s \in S\}$ a k-uniformly generalized Kannan type semigroup of self-mappings defined on K with*

$$3\beta\xi < \nu_0 N(B), \tag{3.9}$$

where $\nu_0 = \inf\{\nu > 1 : \nu(1 - \delta_B(1/\nu)) \geq 1/2\},\ 0 < \beta < 1$ and $\xi = \frac{3\beta}{1-\beta}$. Also, suppose that there exists a nonempty bounded closed convex subset E of K with the following property

$$(\Re) : u \in E \text{ implies } w_w(u) \subset E.$$

Then there exists $u^* \in E$ such that $T_s u^* = u^*$, for all $s \in S$.

Proof. Choose $u_0 \in E$ and consider for each $t \in S$, $\{T_s u_0 : t \leq s \in S\}$ is a bounded net. Owing to Lemma 2.2, there exists $z_t \in \overline{co}\{T_s u_0 : t \leq s \in S\}$ such that

$$\limsup_s ||T_s u_0 - z_t|| \leq \widetilde{N}(B) D(\{T_s u_0\}_{t \leq s \in S}), \tag{3.10}$$

where $\widetilde{N}(B) = 1/N(B)$ and $D(\{v_t\})$ denotes the asymptotic diameter of a net $\{v_t\}$, *i.e.*, the number

$$D(\{v_t\}) = \lim_t (sup\{||v_i - v_j|| : t \leq i, j \in S\}).$$

Since B is reflexive, $\{z_t\}$ admits a subnet $\{z_{t_l}\}$ converging weakly to some $u_1 \in B$. On using (3.10) and the weakly lower semicontinuity of the functional $\limsup_t ||T_t u_0 - v||$, it follows that

$$\limsup_t ||T_t u_0 - u_1|| \leq \widetilde{N}(B) D(\{T_t u_0\}_{t \in S}).$$

Also, $u_1 \in \bigcap_{t \in S} \overline{co}\{T_s u_0 : t \leq s \in S\}$ and that

$$||z - u_1|| \leq \limsup_t ||z - T_t u_0|| \text{ for all } z \in B.$$

Using Property $(\Re)$ and the fact that $\bigcap_{t \in G} \overline{co}\{T_s u_0 : t \leq s \in S\} = \overline{co}\{w_w(u_0)\}$ (which is easy to prove by using Separation Theorem (see [4])), we conclude that u_1 lies in E. By repeating this process inductively, we obtain a sequence $\{u_n\}_{n=0}^{\infty}$ in E with the properties: (for all nonnegative integers $n \geq 0$)

$$\limsup_t ||T_t u_n - u_{n+1}|| \leq \widetilde{N}(B) D(\{T_t u_n\}_{t \in G}) \tag{3.11}$$

and

$$||z - u_{n+1}|| \leq \limsup_t ||z - T_t u_n|| \text{ for all } z \in B. \tag{3.12}$$

Write $r_n = \limsup_t ||T_t u_n - u_{n+1}||$ and $d(u_n) = \sup\{||T_t u_n - u_n|| : t \in S\}$. Now, in view of (3.11), we have

$$\begin{aligned} r_n &= \limsup_t ||T_t u_n - u_{n+1}|| \\ &\leq \widetilde{N}(B) D(\{T_t u_n\}_{t \in S}) \\ &= \widetilde{N}(B) \lim_t (\sup\{||T_i u_n - T_j u_n|| : t \leq i, j \in G\}) \\ &\leq \widetilde{N}(B) \beta \limsup_t \{||u_n - T_i u_n|| + ||T_j u_n - T_{j-i} u_n||\} \\ &\leq \widetilde{N}(B).\beta.3d(u_n), \end{aligned}$$

that is,

$$r_n < 3\beta \widetilde{N}(B) d(u_n). \tag{3.13}$$

We may assume that $d(u_n) > 0$ for all $n \geq 0$. Let $n \geq 0$ be fixed and $\varepsilon > 0$ small enough. Firstly, choose $j \in G$ such that

$$||T_j u_{n+1} - u_{n+1}|| > d(u_{n+1}) - \varepsilon$$

and thereafter, choose $s_0 \in S$ so large that

$$||T_s u_n - u_{n+1}|| < r_n + \varepsilon, \text{ for all } s \geq s_0.$$

It turns out, for $s \geq s_0 + j$,

$$\begin{array}{rcl} ||T_s u_n - T_j u_{n+1}|| & \leq & \beta[||T_s u_n - T_{s-j} u_n|| + ||u_{n+1} - T_j u_{n+1}||] \\ & < & \beta[3(r_n + \varepsilon) + ||T_s u_n - T_j u_{n+1}||] \end{array}$$

so that

$$||T_s u_n - T_j u_{n+1}|| < \xi(r_n + \varepsilon).$$

Making use of property (v) ($i.e.$ for $s \geq s_0 + j$), we have

$$||T_s u_n - \frac{1}{2}(u_{n+1} + T_j u_{n+1})|| \leq \xi(r_n + \varepsilon)\Big(1 - \delta_B\Big(\frac{d(u_{n+1}) - \varepsilon}{\xi(r_n + \varepsilon)}\Big)\Big).$$

Substituting $z := (u_{n+1} + T_j u_{n+1})/2$ in (3.12), we obtain

$$\begin{array}{rcl} \frac{1}{2}(d(u_{n+1}) - \varepsilon) & < & ||\frac{1}{2}(T_j u_{n+1} - u_{n+1})|| \\ & \leq & ||\frac{1}{2}(T_j u_{n+1} + u_{n+1}) - u_{n+1}|| \\ & \leq & \limsup_t ||T_t u_n - \frac{1}{2}(u_{n+1} + T_j u_{n+1})|| \\ & \leq & \xi(r_n + \varepsilon)\Big(1 - \delta_B\Big(\frac{d(u_{n+1}) - \varepsilon}{\xi(r_n + \varepsilon)}\Big)\Big). \end{array}$$

On taking the limit as $\varepsilon \to 0$ we have

$$\frac{1}{2}d(u_{n+1}) \leq \xi r_n\Big(1 - \delta_B\Big(\frac{d(u_{n+1})}{\xi r_n}\Big)\Big). \tag{3.14}$$

On the other hand, owing to (3.12), for each $j \in S$, we have

$$||T_j u_{n+1} - u_{n+1})|| \leq \limsup_t ||T_j u_{n+1} - T_t u_n|| < \xi r_n$$

so that

$$d(u_{n+1}) < \xi r_n. \tag{3.15}$$

Now, by using the definition of ν_0 in (3.9) and combining (3.14) and (3.15), we infer that $(\xi r_n)/d(u_{n+1}) \geq \nu_0$. It turns out from (3.13) that

$$d(u_{n+1}) \leq \frac{\xi}{\nu_0} r_n < \frac{3\beta\xi}{\nu_0 N(B)} d(u_n).$$

Consequently, we obtain

$$d(u_n) \leq A d(u_{n-1}) \leq \ldots \leq A^n d(u_0),$$

where $A = 3\beta\xi[\nu_0 N(B)]^{-1} < 1$ by assumption. Since

$$\begin{array}{rl} ||u_{n+1} - u_n|| & \leq \limsup_t ||T_t u_n - u_{n+1}|| + \limsup_t ||T_t u_n - u_n|| \leq r_n + d(u_n) \\ & \leq (1 + 3\beta\widetilde{N}(B))d(u_n) \leq (1 + 3\beta\widetilde{N}(B))A^n d(u_0), \end{array}$$

therefore $\sum_{n=1}^{\infty} ||u_{n+1} - u_n||$ is convergent which amounts to saying that $\{u_n\}$ is strongly convergent. Let $u^* = \lim_n u_n$. Then, for each $s \in S$, we have

$$\begin{aligned} ||u^* - T_s u^*|| &\leq ||u^* - u_n|| + ||T_s u_n - u_n|| + ||T_s u_n - T_s u^*|| \\ &\leq ||u^* - u_n|| + d(u_n) + \beta[d(u_n) + ||u^* - T_s u^*||] \\ &\leq ||u^* - u_n|| + (1+\beta)d(u_n) + \beta||u^* - T_s u^*||. \end{aligned}$$

$$||u^* - T_s u^*|| \leq \frac{1}{1-\beta}[||u^* - u_n|| + (1+\beta)d(u_n)] \to 0 \text{ as } n \to \infty$$

Hence, $T_s u^* = u^*$ for all $s \in S$ and the proof is complete. □

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