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On the continuity of almostlocal contractions

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ABSTRACT. This paper introduces a new class of contraction: the almost local contractions. Then, we prove the existence and uniqueness of a fixed-point for local almost contractions in two cases: with constant and variable coefficients of contraction.

1. INTRODUCTION

Definition 1.1. Let (X,d) be a metric space and $T : X \to X$ is called almost contraction or (δ, L)- contraction if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \cdot d(y, Tx), \forall x, y \in X \tag{1.1}$$

Remark 1.1. The term of almost contraction is equivalent to weak contraction, and it was first introduced by V. Berinde in [3].

Remark 1.2. Because of the simmetry of the distance, the almost contraction condition (1.1) includes the following dual one:

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \cdot d(x, Ty), \forall x, y \in X \tag{1.2}$$

obtained from (1.1) by replacing $d(Tx, Ty)$ by $d(Ty, Tx)$ and $d(x, y)$ by $d(y, x)$, and after that step, changing x with y, and viceversa. Obviously, to prove the almost contractiveness of T, it is necessary to check both (1.1) and (1.2).
The next Theorem show that an almost contraction is continuous at any fixed point of it, according to [1].

Theorem 1.1. *Let (X,d) be a complete metric space and $T : X \to X$ be an almost contraction. Then T is continuous at p, for any $p \in Fix(T)$.*

Example 1.1. Let $T : [0, 1] \to [0, 1]$ a mapping given by $Tx = \frac{2}{3}$ for $x \in [0, 1)$, and $T1 = 0$. Then T has the following properties:
1) T satisfies (2.5) with $h \in [\frac{2}{3}, 1)$, i.e. T is quasi-contraction;
2) T satisfies (1.1), with $\delta \geq \frac{2}{3}$ and $L \geq 0$, i.e. T is also weak contraction;
3) T has a unique fixed point, $x^* = \frac{2}{3}$.
4) T is not continuous.

The concept of local contraction was first introduced by Martins da Rocha and Filipe Vailakis in [5] (2010), here they studied the existence and uniqueness of fixed points for the local contractions.

Definition 1.2. Let F be a set and let $\mathcal{D} = (d_j)_{j \in J}$ a family of semidistances defined on F. We let σ be the weak topology on F defined by the family $\mathcal{D}$.
Let r be a function from J to J. An operator $T : F \to F$ is a *local contraction* with respect

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$(\mathcal{D}, \text{r})$ if, for every j, there exists $\beta_j \in [0,1)$ such that

$$\forall f, g \in F, \quad d_j(Tf, Tg) \le \beta_j d_{r(j)}(f,g)$$

2. Almost local contractions

We try to combine these two different type of contractive mappings: the almost and local contractions, to study their fixed points. This new type of mappings was first introduced in [12]

Definition 2.3. The mapping $d(x,y) : X \times X \to \mathbb{R}_+$ is said to be a pseudometric if:

1. $d(x,y) = d(y,x)$
2. $d(x,y) \le d(x,z) + d(z,y)$
3. $x = y$ implies $d(x,y) = 0$
   (instead of $x = y \Leftrightarrow d(x,y) = 0$ in the metric case)

Definition 2.4. Let X be a set and let $\mathcal{D} = (d_j)_{j \in J}$ be a family of pseudometrics defined on X. We let σ be the weak topology on X defined by the family $\mathcal{D}$.
A sequence $(x_n)_{n \in \mathbb{N}^*}$ is said to be $\sigma - Cauchy$ if it is d_j-Cauchy, $\forall j \in J$.
The subset A of X is said to be sequencially σ-complete if every σ-Cauchy sequence in X converges in X for the σ-topology.
The subset $A \subset X$ is said to be σ-bounded if $diam_j(A) \equiv sup\{d_j(x,y) : x, y \in A\}$ is finite for every $j \in J$.

Definition 2.5. Let r be a function from J to J. An operator $T : X \to X$ is called an *almost local contraction* with respect $(\mathcal{D}, \text{r})$ if, for every j, there exist the constants $\theta \in (0,1)$ and $L \ge 0$ such that

$$d_j(Tx, Ty) \le \theta \cdot d_j(x,y) + L \cdot d_{r(j)}(y, Tx), \forall x, y \in X \tag{2.3}$$

Remark 2.3. The almost contractions represent a particular case of almost local contractions, by taking (X, d) metric space instead of the pseudometrics d_j and $d_{r(j)}$ defined on X. Also, to obtain the almost contractions, we take in (2.3) for r the identity function, so we have $r(j) = j$.

Definition 2.6. The space X is σ- Hausdorff if the following condition is valid: for each pair $x, y \in X, x \neq y$, there exists $j \in J$ such that $d_j(x,y) > 0$.
If A is a nonempty subset of X, then for each z in X, we let
$d_j(z, A) \equiv inf\{d_j(z,y) : y \in A\}$.

Theorem 2.2 is an existence fixed point theorem for almost local contractions, as they appear in [12].

Theorem 2.2. *Consider a function $r : J \to J$ and let $T : X \to X$ be an almost local contraction with respect to $(\mathcal{D}, r)$. Consider a nonempty, σ- bounded, sequentially σ- complete, and T- invariant subset $A \subset X$. If the condition*

$$\forall j \in J, \quad \lim_{n \to \infty} \theta^{n+1} diam_{r^{n+1}(j)}(A) = 0 \tag{2.4}$$

is satisfied, then the operator T admits a fixed point x^* in A.

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}$$

Take $x := x_{n-1}, y := x_n$ in (2.5) to obtain

$$d_j(Tx_{n-1}, Tx_n) \leq \theta \cdot d_{r(j)}(x_{n-1}, x_n)$$

which yields

$$d_j(x_n, x_{n+1}) \leq \theta \cdot d_{r(j)}(x_{n-1}, x_n), \forall j \in J \tag{2.5}$$

Using (2.5), we obtain by induction with respect to n:

$$d_j(x_n, x_{n+1}) \leq \theta^n \cdot d_{r(j)}(x_0, x_1), \quad n = 0, 1, 2, \cdots \tag{2.6}$$

According to the triangle rule, by (2.6) we get:

$$d_j(x_n, x_{n+p}) \leq \theta^n(1 + \theta + \cdots + \theta^{p-1})d_{r(j)}(x_0, x_1) = \tag{2.7}$$

$$= \frac{\theta^n}{1-\theta}(1 - \theta^p) \cdot d_{r(j)}(x_0, x_1), \quad n, p \in \mathbb{N}, p \neq 0 \tag{2.8}$$

Conditions (2.7), (2.8) show us that the sequence $(x_n)_{n\in\mathbb{N}}$ is d_j- Cauchy for each $j \in J$. The subset A is assumed to be sequentially σ-complete, there exists f^* in$A$ such that $(T^n x)_{n\in\mathbb{N}}$ is σ- convergent to x^*. Besides, the sequence $(T^n x)_{n\in\mathbb{N}}$ converges for the topology σ to x^*, which implies

$$\forall j \in J, \quad d_j(Tx^*, x^*) = \lim_{n\to\infty} d_j(Tx^*, T^{n+1}x).$$

Recall that the operator T is an almost local contraction with respect to $(\mathcal{D}, r)$. From that, we have

$$\forall j \in J, \quad d_j(Tx^*, x^*) \leq \beta_j \lim_{n\to\infty} d_{r(j)}(x^*, T^n x).$$

The convergence for the σ- topology implies convergence for the pseudometric $d_{r(j)}$, we obtain $d_j(Tx^*, x^*) = 0$ for every $j \in J$.
This way, we prove that $Tf^* = f^*$, since σ is Hausdorff.
So, we prove the existence of the fixed point for almost local contractions. □

Remark 2.4. For T verifies (2.3) with $L = 0$, and $r : J \to J$ the identity function, we find Theorem Vailakis [5] by taking $\theta = \beta_j$.
Further, for the case $d_j = d, \forall j \in J$, with d = metric on X, we obtain the well known Banach contraction, with his unique fixed point.

Remark 2.5. In Theorem 2.2, the coefficient of contraction $\theta \in (0, 1)$ is constant, but local contractions have a coefficient of contraction $\theta_j \in [0, 1)$ whitch depends on $j \in$ J. Our first goal is to extend the local almost contractions to the most general case of $\theta_j \in (0, 1)$.
The next Theorem represent an existence and uniqueness theorem for the almost local contractions with constant coefficient of contraction.

Theorem 2.3. *If to the conditions of Theorem 2.2, we add:*
(U) for every fixed $j \in J$ there exists:

$$\lim_{n\to\infty} (\theta + L)^n diam_{r^n(j)}(z, A) = 0, \forall x, y \in X \tag{2.9}$$

then the fixed point x^* of T is unique.

Proof. Suppose, by contradiction, there are two different fixed points x^* and y^* of T. Then for every fixed j∈J we have:

$$\begin{aligned} 0 < d_j(x^*, y^*) &= d_j(Tx^*, Ty^*) \leq \theta d_{r(j)}(x^*, y^*) + Ld_{r(j)}(y^*, Tx^*) = \\ &= (\theta{+}L) \cdot d_{r(j)}(x^*, y^*) \leq \cdots \leq (\theta{+}L)^n d_{r^n(j)}(x^*, y^*) \leq \\ &\leq (\theta{+}L)^n diam_{r^n(j)}(z, A) \end{aligned}$$

Now, letting n$\to\infty$, we obtained a contradiction with condition (2.9), i.e. the fixed point is unique. □

3. Main results

This paper can be regarded as an extension of V. Berinde and M. Păcurar (2015, [1]) analysis about the continuity of almost contractions in their fixed points. The main results of this paper are given by Theorem 3.4, which give us the answer about the continuity of local almost contractions in their fixed points.

Theorem 3.4. *Let X be a set and $\mathcal{D} = (d_j)_{j\in J}$ be a family of pseudometrics defined on X; let $T : X \to X$ be an almost local contraction satisfying condition (2.3), so T admits a fixed point. Then T is continuous at f, for any $f \in Fix(T)$.*

Proof. The mapping T is an almost local contraction, i.e. there exist the constants $\theta \in (0, 1)$ and some $L \geq 0$

$$d_j(Tx, Ty) \leq \theta \cdot d_j(x, y) + L \cdot d_{r(j)}(y, Tx), \forall x, y \in X \tag{3.10}$$

For any sequence $\{y_n\}_{n=0}^{\infty}$ in X converging to f, we take $y := y_n, x := f$ in (3.10), and we get

$$d_j(Tf, Ty_n) \leq \theta \cdot d_j(f, y_n) + L \cdot d_{r(j)}(y_n, Tf), n = 0, 1, 2, ... \tag{3.11}$$

Using $Tf = f$, since f is a fixed point of T, we obtain:

$$d_j(Ty_n, Tf) \leq \theta \cdot d_j(f, y_n) + L \cdot d_{r(j)}(y_n, f), n = 0, 1, 2, ... \tag{3.12}$$

Now by letting $n \to \infty$ in (3.12) we get $Ty_n \to Tf$, which shows that T is continuous at f. The fixed point has been chosen arbitrarily, so the proof is complete. □

According to Definition 2.4, the almost local contractions are defined in a subset $A \subset X$. In the case $A = X$, then an almost local contraction is actually an usual almost contraction.

Example 3.2. Let $X = [1, n] \times [1, n] \subset \mathbb{R}^2$, $\quad T : X \to X$,

$$T(x, y) = \begin{cases} (x/2, y/2) & \text{if (x,y)} \neq (1, 0) \\ (0, 0) & \text{if (x,y)} = (1, 0) \end{cases} \tag{3.13}$$

The diameter of the subset $X = [1, n] \times [1, n] \subset \mathbb{R}^2$ is given by the diagonal line of the square with $(n - 1)$ side.
We shall use the pseudometric:

$$d_j((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| \cdot e^j, \forall j \in \mathcal{Q}. \tag{3.14}$$

This is a pseudometric, but not a metric, take for example:
$d_j((1, 4), (1, 3)) = |1 - 1| \cdot e^j = 0$, however $(1, 4) \neq (1, 3)$
In this case, the mapping T is contraction, which implies that is an almost local contraction, with the unique fixed point $x = 0, y = 0$.
According to Theorem 3.4, T is continuous in $(0, 0) \in Fix(T)$, but is not continuous in $(1, 0) \in X$.

Example 3.3. With the presumptions of Example 3.2 and the pseudometric defined by (3.14), we get another example for almost local contractions.
Considering $T : X \to X$,

$$T(x, y) = \begin{cases} (x, -y) & \text{if (x,y)} \neq (1, 1) \\ (0, 0) & \text{if (x,y)} = (1, 1) \end{cases}$$

T is not a contraction because the contractive condition:

$$d_j(Tx, Ty) \leq \theta \cdot d_j(x, y), \tag{3.15}$$

is not valid $\forall x, y \in X$, and for any $\theta \in (0, 1)$. Indeed, (3.15) is equivalent with:

$$| x_1 - x_2| \cdot e^j \leq \theta \cdot |x_1 - x_2| \cdot e^j$$

The last inequality leads us to $1 \leq \theta$, which is obviously false, considering $\theta \in (0, 1)$.
However, T becomes an almost local contraction if:

$$| x_1 - x_2| \cdot e^j \leq \theta \cdot |x_1 - x_2| \cdot e^j + L \cdot |x_2 - x_1| \cdot e^{\frac{j}{2}}$$

which is equivalent to : $e^{j/2} \leq \theta \cdot e^{j/2} + L$

$$(1 - \theta) \cdot e^{j/2} \leq L \tag{3.16}$$

For $\theta = 1/3 \in (0, 1)$, $L = 1 \geq 0$ and $j < 0$, the (3.16) inequality becomes true, i.e. T is an almost local contraction with many fixed points:
$FixT = \{(x, 0) : x \in \mathbb{R}\}$
In this case, we have:

$$\forall j \in J, \quad \lim_{n\to\infty} \theta^{n+1} diam_{r^{n+1}(j)}(A) = \lim_{n\to\infty} (1/3)^{n+1} \cdot (n - 1)^2 = 0$$

This way, the existence of the fixed point is assured, according to condition (2.4) from Theorem 2.2.
Theorem 3.4 is again valid, because the continuity of T in $(0, 0) \in Fix(T)$, but discontinuous in $(1, 1)$, which is not a fixed point of T.

Example 3.4. Let X the set of positive functions: $X = \{f | f : [0, \infty) \to [0, \infty)\}$ and $d_j(f, g) = |f(0) - g(0)| \cdot e^j, \forall f, g \in X$.
d_j is indeed a pseudometric, but not a metric, take for example $d_j(x, x^2) = 0$, but $x \neq x^2$
Considering the mapping $Tf = |f|, \forall f \in X$, and using condition (2.3) for almost local contractions:

$$|f(0) - g(0)| \cdot e^j \leq \theta \cdot |f(0) - g(0)| \cdot e^j + L \cdot |g(0) - f(0)| \cdot e^{\frac{j}{2}}$$

which is equivalent to: $e^{j/2} \leq \theta \cdot e^{j/2} + L$
This inequality becames true if $j < 0, \quad \theta = \frac{1}{3} \in (0, 1), L = 3 > 0$
However, T is also not a contraction, because the contractive condition (3.14) leads us again to the false presumption: $1 \leq \theta$. The mapping T has infinite number of fixed points: $FixT = \{f \in X\}$, by taking:

$$|f(x)| = f(x), \forall f \in X, x \in [0, \infty)$$

4. Conclusions

This paper analyse the continuity of contractions in their fixed points (if they exists) initiated by V. Berinde and M. Păcurar in [1]. The main extension of Theorem 1.1 in [1] to the more general class of almost local contractions is given by Theorem 3.4, with constant coefficient of contraction. The case of variable coefficient of contraction is an interesting open problem.

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