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Approximation of fixed point of accretive operators based on a Halpern-Type iterative method

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ABSTRACT. In this study, we introduce a new iterative processes to approximate common fixed points of an infinite family of quasi-nonexpansive mappings and obtain a strongly convergent iterative sequence to the common fixed points of these mappings in a uniformly convex Banach space. Also we prove that this process to approximate zeros of an infinite family of accretive operators and we obtain a strong convergence result for these operators. Our results improve and generalize many known results in the current literature.

1. INTRODUCTION AND PRELIMINARIES

Throughout this study, the set of all non-negative integers and the set of reel numbers will be denote by $\mathbb{N}$ and $\mathbb{R}$, respectively.

A quick look into the vast literature of fixed point theory reveals that geometric properties of Banach spaces play a crucial role in the study of iterative approximations of fixed points. Our exposition begins by recalling some geometric properties of a Banach space.

In 1936, Clarkson [6] achieved a remarkable study on uniform convexity. It signalled the beginning of extensive research efforts on the geometry of Banach spaces and its applications. Most of the results indicated in this work were developed in 1991 or later.

Let C be a nonempty, closed and convex subset of a Banach space B , and B^* be the dual space of B.

The convexity modulus of B is defined as follows:

$$\delta_B(\epsilon) = \inf\left\{1 - \frac{\|a+b\|}{2} : a, b \in \overline{B(0,1)}, \|a-b\| \geq \epsilon\right\}.$$

The modulus of convexity is a real valued function defined from $[0,2]$ to $[0,1]$ which is continuous on $[0,2)$. A Banach space B is uniformly convex if and only if $\delta_B(\epsilon) > 0$ for all $\epsilon > 0$.Let B be a normed space and $S_B = \{a \in B : \|a\| = 1\}$ the unit sphere of B. Then the norm of B is Gâteaux differentiable at a point $a \in S_B$ if for $a \in S_B$, the limit

$$\frac{d}{dt}\left(\|a+tb\|\right)|_{t=0} = \lim_{t\to 0}\frac{\|a+tb\| - \|a\|}{t}$$

exists. The norm of B is said to be Gâteaux differentiable if it is Gâteaux differentiable at each point of S_B. In this case, B is called smooth. The norm of B is said to be uniformly Gâteaux differentiable if for each $b \in S_B$, the limit is approached uniformly for $a \in S_B$. Similarly, if the norm of B is uniformly Gâteaux differentiable, then B is called uniformly smooth. A normed space B is called stricly convex if for all $a, b \in B$, $a \neq b$, $\|a\| = \|b\| = 1$, we have

$$\|\lambda a + (1-\lambda)\, b\| < 1, \text{ for all } \lambda \in (0,1).$$

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Theorem 1.1. [3] *Let B be a Banach space.*

1) *B is uniformly convex if and only if B^* is uniformly smooth.*

2) *B is uniformly smooth if and only if B^* is uniformly smooth.*

Theorem 1.2. [3] *Every uniformly smooth space is reflexive.*

A self mapping ϕ on $[0,\infty)$ is said to be a gauge map if it is countinuos and strictly increasing such that $\phi(0)=0$. Let ϕ be a gauge function, and let B be any normed space. A mapping $J_\phi : B \to 2^{B^*}$ defined by

$$J_\phi a = \{f \in B^* : \langle a, f\rangle = \|a\| \|f\| ; \|f\| = \phi(\|a\|)\}$$

for all $a \in B$, is called the dulaity map with gauge function ϕ.If $\phi(t) = t$, then $J_\phi = J$ duality mapping is called the normalized duality map.

Let

$$\psi(t) = \int_0^t \phi(\varsigma)\, d\varsigma, \quad t \geq 0,$$

then $\psi(\delta t) \leq \delta\phi(t)$ and for each $\delta \in (0,1)$. A mapping $\rho : [0,\infty) \to [0,\infty)$ defined by

$$\rho(t) = \sup\left\{\frac{\|a+b\| + \|a-b\|}{2} - 1 : a, b \in B, \|a\| = 1 \; and \; \|b\| = t\right\}$$

is called the modulus of smoothness of B. Also, $\lim_{t\to 0}\frac{\rho(t)}{t} = 0$ if and only if B is uniformly smooth.

Let $q \in (1,2]$ be a real number. If a Banach space B is $q-$uniformly smooth, then the following conditions hold:

$$(i) \text{ there exists a fix } c > 0; \quad (ii) \rho(t) \leq ct^q.$$

For $q > 2$,there is no q-uniformly smooth Banach space. In [5], this assertion was showed by Cioranescu. We say that the mapping J is single-valued and also smooth if the Banach space B having a sequentially continuous duality mapping J from weak topology to weak* topology. The space B is said to have weakly sequentially continuous duality map if duality mapping J is continuous and single-valued, see [5, 19].

Let C be a nonempty subset of Banach space B and $T : C \to B$ be a nonself mapping. Also, let $F(T) = \{a \in C : Ta = a\}$ denote the set of fixed point of T. The map $T : C \to B$ is said to be:

1) Nonexpansive if $\|Ta - Tb\| \leq \|a - b\|$ for all $a, b \in C$;

2) Quasi-nonexpansive if $\|Ta - p\| \leq \|a - p\|$ for all $a \in C$ and $p \in F(T)$.

In 1967, Halpern [9] was the first who introduced the following iteration process under the nonexpansive mapping T. For any initial value $a_0 \in C$ and any fix $u \in C$, $\varphi_n \in [0,1]$ such that $\varphi_n = n^{-b}$,

$$a_{n+1} = \varphi_n u + (1-\varphi_n) Ta_n \qquad \forall n \in \mathbb{N}, \tag{1.1}$$

where $b \in (0,1)$. In 1977, Lions [11] showed that the iteration parocess (1.1) converges strongly to a fixed point of T, where $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfies the following first three conditions:

(C_1) $\lim_{n\to\infty}\varphi_n = 0$; $\quad (C_2)\sum_{n=1}^\infty \varphi_n = \infty$;

(C_3) $\lim_{n\to\infty}\frac{\varphi_{n+1}-\varphi_n}{\varphi_{n+1}^2} = 0$; $\quad (C_4)\sum_{n=1}^\infty |\varphi_{n+1} - \varphi_n| < \infty$

(C_5) $\lim_{n\to\infty}\frac{\varphi_{n+1}-\varphi_n}{\varphi_{n+1}} = 0$; $\quad (C_6)\ |\varphi_{n+1} - \varphi_n| \leq o(\varphi_{n+1}) + \sigma_n$, $\sum_{n=1}^\infty \sigma_n < \infty$.

Afterwards, several authors obtained various results by imposing different conditions on the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ as well as ambient.

(1) In [28], Wittmann showed that the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges strongly to a fixed point of T by the conditions C_1, C_2 and C_4.

(2) In [17, 18], Reich showed that the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges strongly to a fixed point of T in the uniformly smooth Banach spaces by the conditions C_1, C_2 and C_6.

(3) In [22], Shioji and Takahashi showed that the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges strongly to a fixed point of T in the Banach spaces with uniformly Găteaux differentiable norms by the conditions C_1, C_2 and C_4.

(4) In [29], Xu showed that the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges strongly to a fixed point of T by the conditions C_1, C_2 and C_5.

Open question: Are the conditions $C_1$and C_2 enough to guarantee the strong convergence of the iteration process (1.1) for the quasi-nonexpansive mappings, (see [9])?

This question was answered positively in $[13-21]$. But, in [25], it was shown that the answer to open question is not positive for nonexpansive mappings in Hilbert spaces.

The effective domain and range of an operator $A: B \to 2^B$ will be denoted by $dom\,(A) = \{a \in B : Aa \neq \varnothing\}$ and $R\,(A)$, respectively. Let $J : B \to 2^{B^*}$ be a duality mapping. The operator A is said to be accretive if there exists $j \in J\,(a-b)$ such that $\langle a-b, j\rangle \geq 0$ for all $a, b \in B$.An accretive operator A is called m-accretive operator if $R\,(I+rA) = B$, for each $r \geq 0$. For the rest of this manuscript, the operator $A : B \to 2^B$ will be considered as an accretive operator having a zero. A single-valued mapping $J_r = (I+rA)^{-1} : B \to dom\,(A)$ for all $r > 0$ is called resolvent operator of A. It is well known that $A^{-1} = F\,(J_r)$ for all $r > 0$, where $A^{-1} = \{a \in B : 0 \in Aa\}$, (see ,[31, 27]) .

Let B be a reflexive, smooth and strictly convex Banach space and C be a nonempty, closed and convex subset (ccs) of B. Under these conditions, for any $a \in B$, there exists a unique point $z \in C$ such that

$$\|z-a\| \leq \min_{t\in C} \|t-a\|\,;\text{see [27].}$$

Definition 1.1. [27] If $P_C a = z$, then the map $P_C : B \to C$ is called the metric projection.

Assume that $a \in B$ and $z \in C$, then $z = P_C a$ iff $\langle z-t, J\,(a-z)\rangle \geq 0$, for all $t \in C$. In a real Hilbert space H, there is a $P_C : H \to C$ projection mapping, which is nonexpansive, but, such a $P_C : B \to C$ projection mapping does not provide the nonexpansive property in a Banach space B, where C is a nonempty, closed and convex subset of them; see [7].

Definition 1.2. [20] Let $C \subset D$, and C and D be subsets of Banach space B. A mapping $Q : C \to D$ is said to be sunny if $Q\,(\delta x + (1-\delta)\,Qx) = Qx$, for each $x \in B$ and $\delta \in [0, 1)$.

A mapping Q is said to be a retraction if and only if $Q^2 = Q$. A mapping Q is a sunny nonexpansive retraction if and only if it is sunny, nonexpansive and retraction; a nonexpansive retract of C if and only if there exists a nonexpansive retraction.

In the sequel, we shall need the following results.

Lemma 1.1. [29] *Let B be a Banach space with weakly sequentially continuous duality mapping J_ϕ. Then*

$$\psi\,(\|a+b\|) \leq \psi\,(\|a\|) + 2\,\langle b, j_\phi\,(a+b)\rangle$$

for $a, b \in B$. If we get J instead of J_ϕ, we have

$$\|a+b\|^2 \leq \|a\|^2 + 2\,\langle b, j_\phi\,(a+b)\rangle$$

for $a, b \in B$.

Lemma 1.2. [8] *Let B be a Banach space with weakly sequentially continuous duality mapping J_ϕ and C be a ccs of B. Let $T : C \to C$ be a nonexpansive operator having $F\,(T) \neq \varnothing$. Then, for each $u \in C$, there exists $a \in F\,(T)$ such that*

$$\langle u-a, J\,(b-a)\rangle \leq 0$$

for all $b \in F\,(T)$.

Lemma 1.3. [30] *Let B be a reflexive Banach space with weakly sequentially continuous duality mapping J_ϕ and C be a ccs of B. Assume that $T : C \to C$ is a nonexpansive operator. Let $z_t \in C$ be the unique solution in C to the equation $z_t = tu + (1-t)\,Tz_t$ such that $u \in C$ and $t \in (0,1)$. Then T has a fixed point if and only if $\{z_t\}_{t\in(0,1)}$ remains bounded as $t \to 0^+$, and in this case, $\{z_t\}_{t\in(0,1)}$ converges as $t \to 0^+$ strongly to a fixed point of T. If we get the sunny nonexpansive retraction defined by $Q : C \to F(T)$ such that*

$$Q(u) = \lim_{t\to 0} z_t,$$

then $Q(u)$ solves the variational inequality

$$\langle u - Q(u), J_\phi(b - Q(u))\rangle \le 0,\ u \in C \text{ and } b \in F(T).$$

One of the useful and remarkable results in the theory of nonexpansive mappings is demiclosedness principle. It is defined as follows.

Definition 1.3. [15] Let B be a Banach space, C a nonempty subset of B, and $T : C \to B$ a mapping. Then the mapping T is said to be demiclosed at origin, that is, for any sequence $\{a_n\}_{n\in N}$ in C which $a_n \rightharpoonup p$ and $\|Ta_n - a_n\| \to 0$ imply that $Tp = p$.

Lemma 1.4. [1] *Let B be a reflexive Banach space having weakly sequentially continuous duality mapping J_ϕ with a gauge function ϕ, C be a ccs of B and $T : C \to B$ be a nonexpansive mapping. Then $I - T$ is demiclosed at each $p \in B$, i.e., for any sequence $\{a_n\}_{n\in N}$ in C which converges weakly to a, and $(I-T)a_n \to p$ converges strongly imply that $(I-T)a = p$. (Here I is the identity operator of B into itself.) In paticular, assuming $p = 0$, it is obtained $a \in F(T)$.*

Lemma 1.5. [16] *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a nonnegative real sequence and suppose $\{\mu_n\}$ satisfies the following inequality*

$$\mu_{n+1} \le (1-\varphi_n)\,\mu_n + \varphi_n\epsilon_n,$$

assume that $\{\varphi_n\}_{n\in\mathbb{N}}$ and $\{\epsilon_n\}_{n\in\mathbb{N}}$ satisfy the following conditions:

(1) $\{\varphi_n\}_{n\in\mathbb{N}} \subset [0,1]$ and $\sum_{n=1}^{\infty} \varphi_n = \infty$; (2) $\limsup_{n\to\infty} \epsilon_n \le 0$, or (3) $\sum_{n=1}^{\infty} \varphi_n\epsilon_n < \infty$,

then $\lim_{n\to\infty} \mu_n = 0$.

Lemma 1.6. [27] *Let B be a real Banach space, and let A be an $m-$accretive operator on B. For $t > 0$, let J_t be a resolvent operator related to A and t. Then*

$$\|J_k a - J_l a\| \le \frac{|k-l|}{k}\,\|a - J_k a\|\,, \text{for all } k, l > 0 \text{ and } a \in B.$$

Lemma 1.7. [13] *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $\{\mu_{n_i}\}_{i\in\mathbb{N}}$ of $\{\mu_n\}_{n\in\mathbb{N}}$ which satisfies $\mu_{n_i} < \mu_{n_{i+1}}$ for all $i \ge 0$. Also, we consider a subsequence $\left\{\eta_{(n)}\right\}_{n\ge n_0} \subset \mathbb{N}$ defined by*

$$\eta_{(n)} = \max\{k \le n : \mu_k \le \mu_{k+1}\}.$$

Then $\left\{\eta_{(n)}\right\}_{n\ge n_0}$ is a nondecreasing sequence providing $\lim_{n\to\infty} \eta_{(n)} = \infty$, for all $n \ge n_0$. Hence, it holds that $\mu_{\eta_{(n)}} \le \mu_{\eta_{(n)+1}}$, and we obtain $\mu_n \le \mu_{\eta_{(n)+1}}$.

Lemma 1.8. [2] *Let B be a uniformly convex Banach space and $t > 0$ be a constant. Then there exists a continuous, strictly increasing and convex function $g : [0,2t) \to [0,\infty)$ such that*

$$\left\|\sum_{i=1}^{\infty} \rho_i a_i\right\|^2 \le \sum_{i=1}^{\infty} \rho_i \|a_i\|^2 - \rho_k\rho_l g(\|a_k - a_l\|)$$

$\forall k, l \ge 0$, $a_i \in B_t = \{z \in B : \|z\| \le t\}$, $\rho_i \in (0,1)$ and $k \ge 0$ with $\sum_{i=0}^{\infty} \rho_i = 1$.

2. Main results

Theorem 2.3. *Let B be a real uniformly convex Banach space having the normalized duality mapping J and C be a ccs of B. Assume that $\{T_i\}_{i\in\mathbb{N}\cup\{0\}}$ is a infinite family of quasi nonexpansive mappings given in the form $T_i : C \to C$ such that $F = \bigcap_{i=0}^{\infty} F(T_i) \neq \varnothing$, and for each $i \geq 0$, $T_i - I$ is demiclosed at zero. Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} v_1,\ u \in C \text{ arbitrarily chosen}, \\ v_{n+1} = \xi_n u + (1-\zeta_n) T_0 v_n + (\zeta_n - \xi_n) T_0 w_n \\ w_n = \varphi_{n,0} v_n + \sum\limits_{i=1}^{\infty} \varphi_{n,i} T_i v_n, \quad n \geq 0, \end{cases} \tag{2.2}$$

where$\{\zeta_n\}_{n\in\mathbb{N}}$, $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{\varphi_{n,i}\}_{n\in\mathbb{N},i\in\mathbb{N}\cup\{0\}}$ are sequences in $[0,1]$ satisfying the following control conditions:

(1) $\lim_{n\to\infty} \xi_n = 0$; (2) $\sum\limits_{n=1}^{\infty} \xi_n = \infty$; (3) $\varphi_{n,0} + \sum\limits_{i=1}^{\infty} \varphi_{n,i} = 1$, *for all* $n \in \mathbb{N}$;
(4) $\liminf_{n\to\infty} \zeta_n \varphi_{n,0} \varphi_{n,i} > 0$, *for all* $n \in \mathbb{N}$.

Then $\{v_n\}_{n\in\mathbb{N}}$ converges strongly to $P_F u$, where the map $P_F : B \to F$ is the metric projection.

Proof. The proof consists of three parts.

Step 1. We prove that $\{v_n\}_{n\in\mathbb{N}}$, $\{w_n\}_{n\in\mathbb{N}}$ and $\{T_i v_n\}_{n\in\mathbb{N},i\in\mathbb{N}\cup\{0\}}$ are bounded. Firstly, we show that $\{v_n\}_{n\in\mathbb{N}}$ is bounded. Let $p \in F$ be fixed. By Lemma 1.8, we have the following inequality

$$\|w_n - p\|^2 = \left\| \varphi_{n,0} v_n + \sum_{i=1}^{\infty} \varphi_{n,i} T_i v_n - p \right\|^2$$

$$\begin{aligned} &\leq \varphi_{n,0} \|v_n - p\|^2 + \sum_{i=1}^{\infty} \varphi_{n,i} \|T_i v_n - p\|^2 - \varphi_{n,0}\varphi_{n,i} g(\|v_n - T_i v_n\|) \\ &\leq \varphi_{n,0} \|v_n - p\|^2 + \sum_{i=1}^{\infty} \varphi_{n,i} \|v_n - p\|^2 - \varphi_{n,0}\varphi_{n,i} g(\|v_n - T_i v_n\|) \\ &= \|v_n - p\|^2 - \varphi_{n,0}\varphi_{n,i} g(\|v_n - T_i v_n\|) \leq \|v_n - p\|^2 . \end{aligned} \tag{2.3}$$

which implies that

$$\begin{aligned} \|v_{n+1} - p\| &= \|\xi_n u + (1-\zeta_n) T_0 v_n + (\zeta_n - \xi_n) T_0 w_n - p\| \\ &\leq \xi_n \|u - p\| + (1-\zeta_n) \|T_0 v_n - p\| + (\zeta_n - \xi_n) \|T_0 w_n - p\| \\ &\leq \xi_n \|u - p\| + (1-\zeta_n) \|v_n - p\| + (\zeta_n - \xi_n) \|w_n - p\| \\ &\leq \xi_n \|u - p\| + (1-\xi_n) \|v_n - p\| \leq \max\{\|u - p\|, \|v_n - p\|\} . \end{aligned}$$

If we continue the way of induction, we have

$$\|v_{n+1} - p\| = \max\{\|u - p\|, \|v_1 - p\|\}, \forall n \in \mathbb{N}.$$

Hence, we conclude that $\|v_{n+1} - p\|$ is bounded and this implies that $\{v_n\}_{n\in\mathbb{N}}$ is bounded. Furthermore, it is easily show that $\{T_i v_n\}_{n\in\mathbb{N},i\in\mathbb{N}\cup\{0\}}$ and $\{w_n\}_{n\in\mathbb{N}}$ are bounded.

Step 2. We show that for any $n \in \mathbb{N}$,

$$\|v_{n+1} - z\|^2 \leq (1-\xi_n) \|v_n - z\|^2 + 2\xi_n \langle u - z, J(v_{n+1} - z)\rangle . \tag{2.4}$$

By (2.3), we have

$$\|w_n - z\|^2 = \|v_n - z\|^2 - \varphi_{n,0}\varphi_{n,i} g(\|v_n - T_i v_n\|) \tag{2.5}$$

which gives

$$
\begin{aligned}
\|v_{n+1}-z\|^2 &= \|\xi_n u+(1-\zeta_n)\,T_0v_n+(\zeta_n-\xi_n)\,T_0w_n\ -z\|^2 \\
&\le \xi_n\,\|u-z\|^2+(1-\zeta_n)\,\|T_0v_n-z\|^2+(\zeta_n-\xi_n)\,\|T_0w_n\ -z\|^2 \\
&\le \xi_n\,\|u-z\|^2+(1-\zeta_n)\,\|v_n-z\|^2 \\
&\quad +(\zeta_n-\xi_n)\left[\|v_n-z\|^2-\varphi_{n,0}\varphi_{n,i}g\,(\|v_n-T_iv_n\|)\right] \\
&= \xi_n\,\|u-z\|^2+(1-\xi_n)\,\|v_n-z\|^2-\zeta_n\varphi_{n,0}\varphi_{n,i}g\,(\|v_n-T_iv_n\|)+\xi_n\varphi_{n,0}\varphi_{n,i}g\,(\|v_n-T_iv_n\|)\,.
\end{aligned}
\tag{2.6}
$$

Assume that $K_1=\sup\left\{\left|\|u-z\|^2-\|v_n-z\|^2\right|+\xi_n\varphi_{n,0}\varphi_{n,i}g\,(\|v_n-T_iv_n\|)\right\}$.

By (2.6), we get that

$$
\zeta_n\varphi_{n,0}\varphi_{n,i}g\,(\|v_n-T_iv_n\|)\le\|v_n-z\|^2-\|v_{n+1}-z\|^2+\xi_nK_1. \tag{2.7}
$$

By Lemma 1.1 and (2.3), we have

$$
\begin{aligned}
\|v_{n+1}-z\|^2 &= \|\xi_n u+(1-\zeta_n)\,T_0v_n+(\zeta_n-\xi_n)\,T_0w_n\ -z\|^2 \\
&= \|\xi_n\,(u-z)+(1-\zeta_n)\,(T_0v_n-z)+(\zeta_n-\xi_n)\,(T_0w_n\ -z)\|^2 \\
&\le \|(1-\zeta_n)\,(T_0v_n-z)+(\zeta_n-\xi_n)\,(T_0w_n\ -z)\|^2 \\
&\quad +2\,\langle\xi_n\,(u-z)\,,J\,(v_{n+1}-z)\rangle\le(1-\zeta_n)\,\|T_0v_n-z\|^2+(\zeta_n-\xi_n)\,\|T_0w_n\ -z\|^2 \\
&\quad +2\,\langle\xi_n\,(u-z)\,,J\,(v_{n+1}-z)\rangle\le(1-\zeta_n)\,\|v_n-z\|^2+(\zeta_n-\xi_n)\,\|w_n\ -z\|^2 \\
&\quad +2\xi_n\,\langle u-z,J\,(v_{n+1}-z)\rangle\le(1-\zeta_n)\,\|v_n-z\|^2+(\zeta_n-\xi_n)\,\|v_n-z\|^2 \\
&\quad +2\xi_n\,\langle u-z,J\,(v_{n+1}-z)\rangle=(1-\xi_n)\,\|v_n-z\|^2+2\xi_n\,\langle u-z,J\,(v_{n+1}-z)\rangle\,.
\end{aligned}
$$

Step 3. We show that $v_n\to z$ as $n\to\infty$. To this end, we will examine two cases.

Case 1. Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\|v_n-z\|\}_{n\ge n_0}$ is nonincreasing. Assume furthermore that the sequence $\{\|v_n-z\|\}_{n\in\mathbb{N}}$ is convergent. Thus, it is clear that $\|v_n-z\|^2-\|v_{n+1}-z\|^2\to0$ as $n\to\infty$. In view of condition (4) and (2.7), we have

$$
\lim_{n\to\infty}g\,(\|v_n-T_iv_n\|)=0 \text{ and hence } \lim_{n\to\infty}\|v_n-T_iv_n\|=0,
$$

from the properties of g. Also, we can construct the sequences (w_n-v_n) and $(v_{n+1}-w_n)$ as follows:

$$
w_n-v_n=\varphi_{n,0}v_n+\sum_{i=1}^{\infty}\varphi_{n,i}T_iv_n-v_n=\sum_{i=1}^{\infty}\varphi_{n,i}T_iv_n-v_n \text{ and } v_{n+1}-w_n=\xi_nu+(1-\zeta_n)\,T_0v_n+(\zeta_n-\xi_n)\,T_0 \tag{2.8}
$$

$$
\begin{aligned}
\|v_{n+1}-w_n\| &= \|\xi_n\,(u-T_0w_n)+\zeta_n\,(T_0v_n-T_0w_n)+(T_0v_n\ -w_n)\| \\
&\le \xi_n\,\|u-T_0w_n\|+\zeta_n\,\|T_0v_n-T_0w_n\|+\|T_0v_n\ -w_n\|\le\xi_n\,\|u-T_0w_n\|+\zeta_n\,\|v_n-w_n\|
\end{aligned}
$$

These imply that

$$
\lim_{n\to\infty}\|v_{n+1}-w_n\|=0 \qquad \text{and} \qquad \lim_{n\to\infty}\|w_n-v_n\ \|=0. \tag{2.10}
$$

By (2.10), we obtain

$$
\|v_{n+1}-v_n\|\le\|w_n-v_n\ \|+\|v_{n+1}-w_n\|
$$

which further yield

$$
\lim_{n\to\infty}\|v_{n+1}-v_n\|=0. \tag{2.11}
$$

Previously, we have shown that the sequence $\{v_n\}_{n\in\mathbb{N}}$ is bounded. Therefore, there exists a subsequence $\{v_{n_j}\}_{j\in\mathbb{N}}$ of $\{v_n\}_{n\in\mathbb{N}}$ such that $v_{n_j+1} \to l$ for all $j \in \mathbb{N}$. By principle of demi-closedness at zero, we conclude that $l \in F$. Considering the above facts and Definition (1.1), we obtain

$$\begin{aligned} \limsup_{n\to\infty} \langle u - z, J(v_{n+1}, z)\rangle &= \lim_{j\to\infty} \left\langle u - z, J\left(v_{n_j+1} - z\right)\right\rangle \\ &= \langle u - z, J(l - z)\rangle \\ &= \langle u - P_F u, J(l - P_F u)\rangle \\ &\leq 0. \end{aligned} \tag{2.12}$$

By Lemma (1.5), we have the desired result.

Case 2. Let $\{n_j\}_{j\in\mathbb{N}}$ be subsequence of $\{n\}_{n\in\mathbb{N}}$ such that

$$\left\|v_{n_j} - z\right\| \leq \left\|v_{n_j+1} - z\right\|, \text{ for all } j \in \mathbb{N}.$$

Then, in view of Lemma (1.7), there exists a nondecreasing sequence $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$, and hence

$$\|z - v_{m_k}\| < \|z - v_{m_k+1}\| \quad \text{and} \quad \|z - v_k\| \leq \|z - v_{m_k+1}\|, \forall k \in \mathbb{N}.$$

If we rewrite the equation (2.7) for this lemma, we have

$$\begin{aligned} \zeta_{m_k}\varphi_{m_k,0}\varphi_{m_k,i} g\left(\|v_{m_k} - T_i v_{m_k}\|\right) &\leq \|v_{m_k} - z\|^2 - \|v_{m_k+1} - z\|^2 + \xi_{m_k} K_1 \\ &\leq \xi_{m_k} K_1, \forall k \in \mathbb{N}. \end{aligned}$$

By (1) and (2), we obtain

$$\lim_{k\to\infty} g\left(\|v_{m_k} - T_i v_{m_k}\|\right) = 0 \text{ which implies } \lim_{k\to\infty} \|v_{m_k} - T_i v_{m_k}\| = 0.$$

Therefore, using the same argument as in Case 1, we have

$$\limsup_{n\to\infty} \langle u - z, J(v_{m_k}, z)\rangle = \limsup_{n\to\infty} \left\langle u - z, J\left(v_{v_{m_k}+1}, z\right)\right\rangle \leq 0.$$

Using (2.4), we get

$$\|v_{m_k+1} - z\|^2 \leq (1 - \xi_{m_k}) \|v_{m_k} - z\|^2 + 2\xi_{m_k} \langle u - z, J(v_{m_k+1} - z)\rangle.$$

Previously, we have shown that the inequality $\|v_{m_k} - z\| \leq \|v_{m_k+1} - z\|$, we obtain

$$\begin{aligned} \xi_{m_k} \|v_{m_k} - z\|^2 &\leq \|v_{m_k} - z\|^2 - \|v_{m_k+1} - z\|^2 + 2\xi_{m_k} \langle u - z, J(v_{m_k+1} - z)\rangle \\ &\leq 2\xi_{m_k} \langle u - z, J(v_{m_k+1} - z)\rangle. \end{aligned}$$

Hence, we get

$$\lim_{k\to\infty} \|v_{m_k} - z\| = 0. \tag{2.13}$$

considering the expressions (2.12) and (2.13), we obtain

$$\lim_{k\to\infty} \|v_{m_k+1} - z\| = 0.$$

Finaly, we get $\|v_k - z\| \leq \|v_{m_k+1} - z\|$, $\forall k \in \mathbb{N}$. It follows that $v_{m_k} \to z$ as $k \to \infty$. Then we have $v_k \to z$ as $n \to \infty$. □

Theorem 2.4. *Let B be a real uniformly convex Banach space having the weakly sequentially continuous duality mapping J_ϕ and C be a ccs of B such that $\overline{D(A_i)} \subset C \subset \bigcap_{r>0}^{\infty} R(I + rA_i)$ for each $i \in N$. Assume that $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ is an infinite family of accretive operators satisfying the* *range condition, and $r_n > 0$ and $r > 0$ be such that $lim_{n\to\infty} r_n = r$. Let $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$ be the resolvent of A. Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} v_1,\, u \in C \text{ arbitrarily chosen,} \\ v_{n+1} = \xi_n u + (1-\zeta_n)\, J_{r_n}^{A_0} v_n + (\zeta_n - \xi_n)\, J_{r_n}^{A_0} w_n \\ w_n = \varphi_{n,0} v_n + \sum\limits_{i=1}^{\infty} \varphi_{n,i} J_{r_n}^{A_i} v_n, \quad n \geq 0, \end{cases} \tag{2.14}$$

where$\{\zeta_n\}_{n\in\mathbb{N}}$, $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{\varphi_{n,i}\}_{n\in\mathbb{N}, i\in\mathbb{N}\cup\{0\}}$ are sequences in $[0,1]$ satisfing the following control conditions:

(1) $\lim_{n\to\infty} \xi_n = 0$;

(2) $\sum\limits_{n=1}^{\infty} \xi_n = \infty$;

(3) $\varphi_{n,0} + \sum\limits_{i=1}^{\infty} \varphi_{n,i} = 1$, *for all* $n \in \mathbb{N}$;

(4) $\liminf_{n\to\infty} \zeta_n \varphi_{n,0} \varphi_{n,i} > 0$, *for all* $n \in \mathbb{N}$.

If $Q_Z : B \to Z$ is the sunny nonexpansive retraction such that $Z = \bigcap\limits_{i=1}^{\infty} A_i^{-1}(0) \neq \varnothing$, then $\{v_n\}_{n\in\mathbb{N}}$ converges strongly to $Q_Z u$.

Proof. The proof consists of three parts.

We note that Z is closed and convex. Set $z = Q_Z u$.

Step 1. We prove that $\{v_n\}_{n\in\mathbb{N}}$, $\{w_n\}_{n\in\mathbb{N}}$ and $\left\{J_{r_n}^{A_i} v_n\right\}_{n\in\mathbb{N}, i\in\mathbb{N}\cup\{0\}}$ are bounded. Firstly, we show that $\{v_n\}_{n\in\mathbb{N}}$ is bounded. Let $p \in Z$ be fixed. By Lemma 1.8, we have the following inequality

$$\begin{aligned} \|w_n - p\|^2 &= \left\| \varphi_{n,0} v_n + \sum_{i=1}^{\infty} \varphi_{n,i} J_{r_n}^{A_i} v_n - p \right\|^2 \\ &\leq \varphi_{n,0} \|v_n - p\|^2 + \sum_{i=1}^{\infty} \varphi_{n,i} \left\| J_{r_n}^{A_i} v_n - p \right\|^2 - \varphi_{n,0}\varphi_{n,i} g\left(\left\| v_n - J_{r_n}^{A_i} v_n \right\| \right) \\ &\leq \varphi_{n,0} \|v_n - p\|^2 + \sum_{i=1}^{\infty} \varphi_{n,i} \|v_n - p\|^2 - \varphi_{n,0}\varphi_{n,i} g\left(\left\| v_n - J_{r_n}^{A_i} v_n \right\| \right) \\ &= \|v_n - p\|^2 - \varphi_{n,0}\varphi_{n,i} g\left(\left\| v_n - J_{r_n}^{A_i} v_n \right\| \right) \\ &\leq \|v_n - p\|^2 \end{aligned} \tag{2.15}$$

which implies that

$$\begin{aligned} \|v_{n+1} - p\| &= \left\| \xi_n u + (1-\zeta_n)\, J_{r_n}^{A_0} v_n + (\zeta_n - \xi_n)\, J_{r_n}^{A_0} w_n \ - p \right\| \\ &\leq \xi_n \|u - p\| + (1-\zeta_n) \left\| J_{r_n}^{A_0} v_n - p \right\| + (\zeta_n - \xi_n) \left\| J_{r_n}^{A_0} w_n \ - p \right\| \\ &\leq \xi_n \|u - p\| + (1-\zeta_n) \|v_n - p\| + (\zeta_n - \xi_n) \|w_n \ - p\| \\ &\leq \xi_n \|u - p\| + (1-\xi_n) \|v_n - p\| \\ &\leq \max\{\|u - p\|,\ \|v_n - p\|\ \} \end{aligned}$$

If we continue the way of induction, we have

$$\|v_{n+1} - p\| = \max\{\|u - p\|,\ \|v_1 - p\|\ \},\ \forall n \in \mathbb{N}.$$

Hence, we conclude that $\|v_{n+1} - p\|$ is bounded and this implies that $\{v_n\}_{n\in\mathbb{N}}$ is bounded. Futhermore, it is easily show that $\left\{J_{r_n}^{A_i} v_n\right\}_{n\in\mathbb{N}, i\in\mathbb{N}\cup\{0\}}$ and $\{w_n\}_{n\in\mathbb{N}}$ are bounded.

Step 2. We show that for any $n \in \mathbb{N}$,

$$\|v_{n+1} - z\|^2 \leq (1 - \xi_n) \|v_n - z\|^2 + 2\xi_n \langle u - z, J_\phi (v_{n+1} - z) \rangle . \tag{2.16}$$

By (2.15), we have

$$\|w_n - z\|^2 = \|v_n - z\|^2 - \varphi_{n,0}\varphi_{n,i} g \left(\left\| v_n - J_{r_n}^{A_i} v_n \right\| \right) \tag{2.17}$$

which gives

$$\begin{aligned}
\|v_{n+1} - z\|^2 &= \left\| \xi_n u + (1 - \zeta_n) J_{r_n}^{A_0} v_n + (\zeta_n - \xi_n) J_{r_n}^{A_0} w_n - z \right\|^2 \\
&\leq \xi_n \|u - z\|^2 + (1 - \zeta_n) \left\| J_{r_n}^{A_0} v_n - z \right\|^2 + (\zeta_n - \xi_n) \left\| J_{r_n}^{A_0} w_n - z \right\|^2 \\
&\leq \xi_n \|u - z\|^2 + (1 - \zeta_n) \|v_n - z\|^2 \\
&\quad + (\zeta_n - \xi_n) \left[\|v_n - z\|^2 - \varphi_{n,0}\varphi_{n,i} g \left(\left\| v_n - J_{r_n}^{A_i} v_n \right\| \right) \right] \\
&= \xi_n \|u - z\|^2 + (1 - \xi_n) \|v_n - z\|^2 \\
&\quad - \zeta_n \varphi_{n,0}\varphi_{n,i} g \left(\left\| v_n - J_{r_n}^{A_i} v_n \right\| \right) + \xi_n \varphi_{n,0}\varphi_{n,i} g \left(\left\| v_n - J_{r_n}^{A_i} v_n \right\| \right) .
\end{aligned} \tag{2.18}$$

Assume that $K_2 = \sup \left\{ \left| \|u - z\|^2 - \|v_n - z\|^2 \right| + \xi_n \varphi_{n,0}\varphi_{n,i} g \left(\left\| v_n - J_{r_n}^{A_i} v_n \right\| \right) \right\}$.
By (2.18), we get that

$$\zeta_n \varphi_{n,0}\varphi_{n,i} g \left(\left\| v_n - J_{r_n}^{A_i} v_n \right\| \right) \leq \|v_n - z\|^2 - \|v_{n+1} - z\|^2 + \xi_n K_2. \tag{2.19}$$

By Lemma 1.1 and (2.15), we have

$$\begin{aligned}
\|v_{n+1} - z\|^2 &= \left\| \xi_n u + (1 - \zeta_n) J_{r_n}^{A_0} v_n + (\zeta_n - \xi_n) J_{r_n}^{A_0} w_n - z \right\|^2 \\
&= \left\| \xi_n (u - z) + (1 - \zeta_n) \left(J_{r_n}^{A_0} v_n - z \right) + (\zeta_n - \xi_n) \left(J_{r_n}^{A_0} w_n - z \right) \right\|^2 \\
&\leq \left\| (1 - \zeta_n) \left(J_{r_n}^{A_0} v_n - z \right) + (\zeta_n - \xi_n) \left(J_{r_n}^{A_0} w_n - z \right) \right\|^2 \\
&\quad + 2 \langle \xi_n (u - z), J_\phi (v_{n+1} - z) \rangle \\
&\leq (1 - \zeta_n) \left\| J_{r_n}^{A_0} v_n - z \right\|^2 + (\zeta_n - \xi_n) \left\| J_{r_n}^{A_0} w_n - z \right\|^2 \\
&\quad + 2 \langle \xi_n (u - z), J_\phi (v_{n+1} - z) \rangle \\
&\leq (1 - \zeta_n) \|v_n - z\|^2 + (\zeta_n - \xi_n) \|w_n - z\|^2 \\
&\quad + 2\xi_n \langle u - z, J_\phi (v_{n+1} - z) \rangle \\
&\leq (1 - \zeta_n) \|v_n - z\|^2 + (\zeta_n - \xi_n) \|v_n - z\|^2 \\
&\quad + 2\xi_n \langle u - z, J_\phi (v_{n+1} - z) \rangle \\
&= (1 - \xi_n) \|v_n - z\|^2 + 2\xi_n \langle u - z, J_\phi (v_{n+1} - z) \rangle .
\end{aligned}$$

Step 3. We show that $v_n \to z$ as $n \to \infty$. For this , we will examine two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|v_n - z\|\}_{n \geq n_0}$ is nonincreasing. Furthermore, the sequence $\{\|v_n - z\|\}_{n \in \mathbb{N}}$ is convergent. Thus, it is clear that $\|v_n - z\|^2 - \|v_{n+1} - z\|^2 \to 0$ as $n \to \infty$. In view of condition (4) and (2.19), we have

$$\lim_{n \to \infty} g \left(\left\| v_n - J_{r_n}^{A_i} v_n \right\| \right) = 0 \text{ and hence } \lim_{n \to \infty} \left\| v_n - J_{r_n}^{A_i} v_n \right\| = 0,$$

from the properties of g. Also, we can construct the sequences $(w_n - v_n)$ and $(v_{n+1} - w_n)$, as follows:

$$w_n - v_n = \varphi_{n,0} v_n + \sum_{i=1}^{\infty} \varphi_{n,i} J_{r_n}^{A_i} v_n - v_n = \sum_{i=1}^{\infty} \varphi_{n,i} \left(J_{r_n}^{A_i} v_n - v_n \right) . \tag{2.20}$$

and

$$v_{n+1}-w_n=\xi_n u+(1-\zeta_n)\,J_{r_n}^{A_0}v_n+(\zeta_n-\xi_n)\,J_{r_n}^{A_0}w_n\;-w_n$$

$$\begin{aligned}\|v_{n+1}-w_n\| &= \left\|\begin{array}{c}\xi_n(u-w_n)+(1-\zeta_n)\left(J_{r_n}^{A_0}v_n-w_n\right)\\ +(\zeta_n-\xi_n)\left(J_{r_n}^{A_0}w_n\;-w_n\right)\end{array}\right\| \\ &\le \xi_n\|u-w_n\|+(1-\zeta_n)\left\|J_{r_n}^{A_0}v_n-w_n\right\| \\ &\quad +(\zeta_n-\xi_n)\left\|J_{r_n}^{A_0}w_n\;-w_n\right\|. \end{aligned}\tag{2.21}$$

These imply that

$$\lim_{n\to\infty}\|v_{n+1}-w_n\|=0 \qquad \text{and} \qquad \lim_{n\to\infty}\|w_n-v_n\;\|=0. \tag{2.22}$$

By (2.22), we obtain

$$\|v_{n+1}-v_n\|\le\|w_n-v_n\;\|+\|v_{n+1}-w_n\|$$

which gives

$$\lim_{n\to\infty}\|v_{n+1}-v_n\|=0. \tag{2.23}$$

By Lemma 1.6 and (2.20), we have

$$\begin{aligned}\left\|v_n-J_r^{A_i}v_n\right\| &\le \left\|v_n-J_{r_n}^{A_i}v_n\right\|+\left\|J_{r_n}^{A_i}v_n-J_r^{A_i}v_n\right\| \\ &\le \left\|v_n-J_{r_n}^{A_i}v_n\right\|+\frac{|r_n-r|}{r_n}\left\|v_n-J_{r_n}^{A_i}v_n\right\|\end{aligned}$$

which gives

$$\lim_{n\to\infty}\left\|v_n-J_r^{A_i}v_n\right\|=0, \text{ for all } i\in\mathbb{N}.$$

Previously, we have shown that the sequence $\{v_n\}_{n\in\mathbb{N}}$ is bounded. Therefore, there exists a subsequence $\{v_{n_j}\}_{j\in\mathbb{N}}$ of $\{v_n\}_{n\in\mathbb{N}}$ such that $v_{n_j+1}\to l\in F\left(J_r^{A_i}v_n\right)$ for all $j\in\mathbb{N}$. This, together with Lemma 1.1 implies that

$$\begin{aligned}&\limsup_{n\to\infty}\langle u-z,J_\phi(v_{n+1},z)\rangle= \\ &=\lim_{j\to\infty}\left\langle u-z,J_\phi\left(v_{n_j+1}-z\right)\right\rangle=\langle u-z,J_\phi(l-z)\rangle\le 0.\end{aligned}\tag{2.24}$$

By Lemma (1.5), we obtain the desired result.

Case 2. Let $\{n_j\}_{j\in\mathbb{N}}$ be subsequence of $\{n\}_{n\in\mathbb{N}}$ such that

$$\left\|v_{n_j}-z\right\|\le\left\|v_{n_j+1}-z\right\|, \text{ for all } j\in\mathbb{N}.$$

Then, in view of Lemma (1.7), there exists a nondecreasing sequence $\{m_k\}_{k\in\mathbb{N}}\subset\mathbb{N}$, and hence

$$\|z-v_{m_k}\|<\|z-v_{m_k+1}\| \qquad \text{and} \qquad \|z-v_k\|\le\|z-v_{m_k+1}\|, \forall k\in\mathbb{N}.$$

If we rewrite the equation (2.7) for this Lemma, we have

$$\begin{aligned}\zeta_{m_k}\varphi_{m_k,0}\varphi_{m_k,i}g\left(\left\|v_{m_k}-J_{r_n}^{A_i}v_{m_k}\right\|\right) &\le \|v_{m_k}-z\|^2-\|v_{m_k+1}-z\|^2+\xi_{m_k}K_2 \\ &\le \xi_{m_k}K_2, \forall k\in\mathbb{N}.\end{aligned}$$

By (1) and (2), we obtain

$$\lim_{k\to\infty}g\left(\left\|v_{m_k}-J_{r_n}^{A_i}v_{m_k}\right\|\right)=0, \text{ which gives } \lim_{k\to\infty}\left\|v_{m_k}-J_{r_n}^{A_i}v_{m_k}\right\|=0.$$

Therefore, using the same argument as Case 1, we have

$$\limsup_{n\to\infty}\langle u-z,J_\phi(v_{m_k},z)\rangle=\limsup_{n\to\infty}\left\langle u-z,J_\phi\left(v_{v_{m_k}+1},z\right)\right\rangle\le 0.$$

Using (2.16), we get

$$\|v_{m_k+1}-z\|^2 \le (1-\xi_{m_k})\|v_{m_k}-z\|^2 + 2\xi_{m_k}\langle u-z, J_\phi(v_{m_k+1}-z)\rangle.$$

Previously, we have shown that the inequality $\|v_{m_k}-z\| \le \|v_{m_k+1}-z\|$, we obtain

$$\begin{aligned}\xi_{m_k}\|v_{m_k}-z\|^2 &\le \|v_{m_k}-z\|^2 - \|v_{m_k+1}-z\|^2 + 2\xi_{m_k}\langle u-z, J_\phi(v_{m_k+1}-z)\rangle \\ &\le 2\xi_{m_k}\langle u-z, J_\phi(v_{m_k+1}-z)\rangle.\end{aligned}$$

Hence, we get

$$\lim_{k\to\infty}\|v_{m_k}-z\| = 0. \tag{2.25}$$

By (2.24) and (2.25), we obtain

$$\lim_{k\to\infty}\|v_{m_k+1}-z\| = 0.$$

Finaly, we get $\|v_k-z\| \le \|v_{m_k+1}-z\|$, $\forall k \in \mathbb{N}$. It follows that $v_{m_k} \to z$ as $k \to \infty$. Then we have $v_k \to z$ as $n \to \infty$. □

Theorem 2.5. *Let B be a real uniformly convex Banach space having a Gâteaux differentiable norm. and C be a ccs of B such that $\overline{D(A_i)} \subset C \subset \bigcap_{r>0}^{\infty} R(I+rA_i)$ for each $i \in N$. Assume that $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ is an infinite family of accretive operators satisfying the range condition, and $r_n > 0$ and $r > 0$ be such that $lim_{n\to\infty}r_n = r$. Let $J_{r_n}^{A_i} = (I+r_nA_i)^{-1}$ be the resolvent of A. Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} v_1,\, u \in C \text{ arbitrarily chosen}, \\ v_{n+1} = \xi_n u + (1-\zeta_n) J_{r_n}^{A_0} v_n + (\zeta_n - \xi_n) J_{r_n}^{A_0} w_n \\ w_n = \varphi_{n,0} v_n + \sum_{i=1}^{\infty} \varphi_{n,i} J_{r_n}^{A_i} v_n, \quad n \ge 0, \end{cases} \tag{2.26}$$

where$\{\zeta_n\}_{n\in\mathbb{N}}$, $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{\varphi_{n,i}\}_{n\in\mathbb{N},i\in\mathbb{N}\cup\{0\}}$ are sequences in $[0,1]$ satisfing the following control conditions:

(1) $\lim_{n\to\infty}\xi_n = 0$;

(2) $\sum_{n=1}^{\infty}\xi_n = \infty$;

(3) $\varphi_{n,0} + \sum_{i=1}^{\infty}\varphi_{n,i} = 1$, *for all* $n \in \mathbb{N}$;

(4) $\liminf_{n\to\infty}\zeta_n\varphi_{n,0}\varphi_{n,i} > 0$, *for all* $n \in \mathbb{N}$.

If $Q_Z : B \to Z$ is the sunny nonexpansive retraction such that $Z = \bigcap_{i=1}^{\infty} A_i^{-1}(0) \ne \varnothing$, then $\{v_n\}_{n\in\mathbb{N}}$ converges strongly as $n \to \infty$ to $Q_Z u$.

Proof. The proof can be done simply using similar arguments as in the proof of Theorem 2.4. □

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