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Convergence of a Newton-like S-iteration process in $\mathbb{R}$

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ABSTRACT. The aim of this paper is to show that Newton-like S-iteration method converges to the unique solution of the scalar nonlinear equation $f(x) = 0$ under weaker conditions involving only f and f'. We also present numerical examples to support our analytical results.

1. INTRODUCTION

We are interested in approximating a solution x^* of the nonlinear operator equation $f(x) = 0$, where f is a differentiable operator. Newton's methods are the most commonly used methods for solving such equations. There are numerous generalizations of the classical Newton's method for solving nonlinear operator equation $f(x) = 0$.

Newton's method is defined by an iterative sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n \geq 0, \tag{1.1}$$

under suitable assumptions on f and f'. Note that (1.1) can be viewed as the sequence of successive approximations (Picard iteration) of the Newton iteration function given by

$$T(x) = x - \frac{f(x)}{f'(x)},\ \text{where } f' \neq 0.$$

Moreover, under appropriate conditions, x^* is a solution of $f(x) = 0$ if and only if x^* is a fixed point of the iteration function T.

This establishes a strong link of Newton's methods with fixed point theory.

There are several convergence results in literature for the Newton's method, see for example [13], [14], and [15], which to ensure a quadratic convergence for the iterative method (1.1) require strong smoothness assumptions which involve f, f' and f''.

Theorem 1.1. ([13]) *Let $f : [a, b] \longrightarrow \mathbb{R}$. Suppose that the following conditions hold:*

(1) $f(a)f(b) < 0$;
(2) $f \in C^2[a, b]$ and $f'(x)f''(x) \neq 0, x \in [a, b]$;

Then the sequence $\{x_n\}$ defined by (1.1) starting with an initial guess $x_0 \in [a, b]$ converges to x^*; the unique solution of $f(x) = 0$ in $[a, b]$. Moreover, we have the following estimation

$$|x_n - x^*| \leq \frac{M_2}{2m_1} |x_n - x_{n-1}|,\ n \geq 1, \tag{1.2}$$

holds, where

$$m_1 = \min_{x\in[a,b]} |f'(x)| \text{ and } M_2 = \max_{x\in[a,b]} |f''(x)|.$$

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For numerical point of view, Theorem 1.1 is widely applicable but there exist more general results based on weaker smoothness conditions. In a series of papers [1]-[11], Berinde obtained more general convergence results which extend Newton's method both scalar ([1]-[8], [10]-[11]) and n-dimensional [9] equations. These results can be applied to weakly smooth functions. The term extended Newton method was adopted in view of the fact that the iterative method (1.1) has been extended from $[a, b]$ to the whole real axis $\mathbb{R}$.

One of the scalar variant of these results is stated below.

Theorem 1.2. ([4]-[5]) *Let $f : [a, b] \longrightarrow \mathbb{R}$, where $a < b$. If the following conditions hold:*

(f_1) $f(a)f(b) < 0$;
(f_2) $f \in C^1[a, b]$ and $f'(x) \neq 0$, $x \in [a, b]$;
(f_3) $2m > M$, where

$$m = \min_{x\in[a,b]} |f'(x)| \text{ and } M = \max_{x\in[a,b]} |f'(x)| . \tag{1.3}$$

Then the Newton iteration $\{x_n\}$, defined by (1.1) starting with $x_0 \in [a, b]$ converges to x^*; the unique solution of $f(x) = 0$ in $[a, b]$. Moreover, the following estimation

$$|x_n - x^*| \leq \frac{M}{m} |x_n - x_{n+1}| , \; n \geq 0, \tag{1.4}$$

holds.

All the proofs in [2], [4]-[7] are based on a classical technique which focuses on the behavior of the sequence $\{x_n\}$ defined in (1.1). However, Berinde [3] proved Theorem 1.2 using an elegant fixed point technique.

Recently, Sen et al. [20] extended Theorem 1.2 to the case of a Newton-like iteration of the form given as:

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + M_1 f(x_n)}, \; n \geq 0, \tag{1.5}$$

with $M_1 f(x) = sgn f'(x) \cdot M$, where M is defined by (1.3).

Later, this result was extended to the n-dimensional case [21]. However, in both cases an extended Newton-like algorithm was used.

In 2006, Berinde and Pacurar [12] obtained a convergence theorem for the iterative method (1.5) under the same assumptions as given in Theorem 1.2.

Theorem 1.3. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function such that the following conditions are satisfied*

(f_1) $f(a)f(b) < 0$;
(f_2) $f \in C^1[a, b]$ and $f'(x) \neq 0$, $x \in [a, b]$;
(f_3) $2m > M$, where

$$m = \min_{x\in[a,b]} |f'(x)| \text{ and } M = \max_{x\in[a,b]} |f'(x)| .$$

Then the Newton-like iteration $\{x_n\}$ defined by (1.5) starting with $x_0 \in [a, b]$ converges to the unique solution x^* of $f(x) = 0$ in $[a, b]$ with the following error estimate

$$|x_n - x^*| \leq \frac{2M}{m + M} |x_n - x_{n+1}| , \; n \geq 0, \tag{1.6}$$

holds.

It was pointed out that the error estimate (1.6) is better than the error estimate (1.4).

In 2011, Sahu [18] introduced the normal S-iteration method as follows:

Let E be a nonempty convex subset of a normed space X, $T : E \longrightarrow E$ be an operator and x_0 be an arbitrary point in E. The normal S-iteration method is defined an iterative sequence given by

$$x_{n+1} = T((1-\alpha_n)x_n + \alpha_n T x_n),\ n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$.

It was shown that the normal S-iteration method is faster than the Picard and Mann iteration methods for contraction operators [18].

In this paper, we are interested in employing the iteration method (1.7) for a real-valued function f. The iteration method was first introduced by Sahu [17] as follows:

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad z_n = (1-\alpha)x_n + \alpha y_n, \quad y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1.7}$$

for $\alpha \in (0,1)$ and $n \geq 0$.

Sahu et al. [19] have discussed the semilocal and local convergence analysis of the Newton-like S-iteration method in Banach spaces.

The purpose of this paper is to give a convergence result for Newton-like S-iteration method under weaker conditions involving only f and f'.

We show that the Newton-like S-iteration method is better than the Newton method (1.1) and extended Newton-like method (1.5). We present numerical examples to support our analytical results.

The following definitions and lemma will be needed in the sequel.

Definition 1.1. Let (X,d) be a metric space. An operator $T : X \longrightarrow X$ is said to be

(i) *contraction* if there exists a constant $k \in [0,1)$ such that for any $x, y \in X$, the following condition hold:

$$d(T(x), T(y)) \leq kd(x,y).$$

(ii) *quasi-contraction* [22] if there exist a constant $k \in [0,1)$ such that for any $x \in X$ and $x^* \in F(T)$, we have

$$d(T(x), x^*) \leq kd(x, x^*), \tag{1.8}$$

where, $F(T) = \{x \in X : Tx = x\} \neq \emptyset$.

Definition 1.2. [18] Let E be a nonempty convex subset of a normed space X and $T : E \to E$ be an operator. The operator $G : E \to E$ is said to be an S-operator generated by an $\alpha \in (0,1)$ and T if it has the following form:

$$G = T\left[(1-\alpha)I + \alpha T\right]$$

where I is an identity operator on E.

Note that G is a contraction operator with contractivity factor $k(1-\alpha(1-k))$ if T is a contraction operator with contractivity factor k.

Lemma 1.1. [12] *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ be a quasi-contractive operator with $x^* \in F(T)$. Then x^* is the unique fixed point of T and the Picard iteration $\{T^n(x_0)\}$ converges to x^* for each $x_0 \in X$.*

2. Main result

We start with the following result.

Theorem 2.4. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function such that the following conditions are satisfied*

(f_1) $f(a)f(b) < 0$;

(f_2) $f \in C^1[a,b]$ *and* $f'(x) \neq 0$, $x \in [a,b]$;

(f_3) $\sqrt{2m} \leq M < 2m$, *where*

$$m = \min_{x\in[a,b]} |f'(x)| \text{ and } M = \max_{x\in[a,b]} |f'(x)|.$$

Then

(i) Newton-like S-iteration method starting with an arbitrary point x_0 in $[a, b]$ converges the unique solution x^* of $f(x) = 0$ in $[a, b]$.

(ii) We have the following error estimate

$$|x_n - x^*| \leq \frac{mM}{(M^2 - \alpha m^2)} |x_n - x_{n+1}|, \tag{2.9}$$

for $n \geq 0$.

Proof. (i) It follows from (f_1) and (f_2) that the equation $f(x) = 0$ has a unique solution x^* in (a, b).

Suppose that $T : [a, b] \longrightarrow \mathbb{R}$ is the Newton-like iteration function associated with f, that is $T(x) = VU_\alpha(x)$, where $V, U_\alpha : [a, b] \longrightarrow \mathbb{R}$ are defined as:

$$V(x) = x - \frac{f(x)}{f'(x)} \text{ and } U_\alpha(x) = x - \alpha\frac{f(x)}{f'(x)}.$$

Note that, x^* is a solution of $f(x) = 0$ if and only if x^* is a fixed point of V and U_α, that is

$$V(x^*) = x^* \text{ and } U_\alpha(x^*) = x^*.$$

Thus, we obtain that

$$U_\alpha(x) - x^* = x - \alpha\frac{f(x)}{f'(x)} - x^* = x - x^* - \alpha\frac{f(x)}{f'(x)}.$$

Obviously,

$$f(x) = f(x) - 0 = f(x) - f(x^*).$$

From (f_2) and the mean value theorem, we get that

$$f(x) = f'(\bar{y})(x - x^*),$$

where $\bar{y} = x^* + \lambda(x - x^*)$, $0 < \lambda < 1$. Thus, for all $x \in [a, b]$ we have

$$U_\alpha(x) - x^* = (x - x^*)\left(1 - \alpha\frac{f'(\bar{y})}{f'(x)}\right). \tag{2.10}$$

By (f_2), f' preserves sign on $[a, b]$. Hence $f'(\bar{y})/f'(x) > 0$. Thus for any $x \in [a, b]$ and $\bar{y}$ between x^* and x, we have

$$1 - \alpha\frac{f'(\bar{y})}{f'(x)} < 1. \tag{2.11}$$

Also, by (f_3), we obtain that

$$\alpha\frac{f'(\bar{y})}{f'(x)} < \frac{f'(\bar{y})}{f'(x)} = \left|\frac{f'(\bar{y})}{f'(x)}\right| = \frac{|f'(\bar{y})|}{|f'(x)|} \leq \frac{M}{m} < 2,$$

which implies that

$$1 - \alpha\frac{f'(\bar{y})}{f'(x)} > -1, \ \forall x \in [a, b] \tag{2.12}$$

where, $\bar{y}$ between x^* and x. From (2.11), (2.12) and the continuity of f', we have

$$k = \max_{x,\bar{y}\in[a,b]} \left|1 - \alpha\frac{f'(\bar{y})}{f'(x)}\right| < 1 \text{ and } 0 < k < 1,$$

which together with (2.10) implies that

$$|U_\alpha(x) - x^*| \leq k |x - x^*|, \ \forall x \in [a, b].$$

Similarly, we obtain that

$$|V(x) - x^*| \leq l\,|x - x^*|\,,\ \forall x \in [a,b].$$

where

$$l = \max_{x,\bar{y}\in[a,b]} \left|1 - \frac{f'(\bar{y})}{f'(x)}\right| < 1 \text{ and } 0 < l < 1.$$

Note that we cannot apply Lemma 1.1 directly, as $[a,b]$ is generally not an invariant set under V and U_α. If so, we can use same arguments as given in the proof of Theorem 6 in [12] and obtain the following

$$x_{n+1} = V(x_n) \in [a,b]\,,$$

which means that $V : [a,b] \longrightarrow [a,b]$. Similarly, we have $U_\alpha : [a,b] \longrightarrow [a,b]$. Hence, $T : [a,b] \longrightarrow [a,b]\,, T(x) = VU_\alpha(x)$ is writen as Newton-like iteration function of iteration method (1.7). Since, $T(x^*) = VU_\alpha(x^*) = V(x^*) = x^*$, we get

$$|T(x) - x^*| = |VU_\alpha(x) - x^*| \leq k\,|U_\alpha(x) - x^*| \leq (kl)\,|x - x^*|\,. \tag{2.13}$$

Thus T is quasi contractive operator with constant kl. On the other hand, from (2.13) we obtain $|T^n(x) - x^*| =$

$$= |(VU_\alpha)^n(x) - x^*| \leq (kl)\left|(VU_\alpha)^{n-1}(x) - x^*\right| \cdots \leq (kl)^n\,|x - x^*|\,.$$

Since $0 < kl < 1$, on taking limit as $n \longrightarrow \infty$ on the both sides of the above inequality, we have, $T^n(x_0) \longrightarrow x^*$ for each $x_0 \in [a,b]$.Therefore, all the conditions of Lemma 1.1 are satisfied and hence x^* is the unique fixed point of T.

(ii) By (1.7), we have

$$x_n - x_{n+1} = \alpha\frac{f(x_n)}{f'(x_n)} + \frac{f(y_n)}{f'(y_n)}. \tag{2.14}$$

Using mean value theorem in (2.14), we have

$$\begin{aligned} x_n - x_{n+1} &= \alpha\frac{f'(c_n)}{f'(x_n)}(x_n - x^*) + \frac{f'(\bar{c}_n)}{f'(y_n)}(x_n - x^*)\left(1 - \alpha\frac{f'(c_n)}{f'(x_n)}\right) \qquad (2.15)\\ &= \left[\alpha\frac{f'(c_n)}{f'(x_n)} + \frac{f'(\bar{c}_n)}{f'(y_n)}\left(1 - \alpha\frac{f'(c_n)}{f'(x_n)}\right)\right](x_n - x^*) \end{aligned}$$

where $c_n = x^* + \mu(x_n - x^*)$ and $\bar{c}_n = x^* + \mu(y_n - x^*)$ for $0 < \mu < 1$. From (2.15), we get

$$\frac{x_n - x^*}{x_n - x_{n+1}} = \frac{1}{\left[\alpha\frac{f'(c_n)}{f'(x_n)} + \frac{f'(\bar{c}_n)}{f'(y_n)}\left(1 - \alpha\frac{f'(c_n)}{f'(x_n)}\right)\right]} \leq \frac{1}{\left(\alpha\frac{m}{M} - \frac{M}{m}\right)}.$$

Thus, we have

$$\left|\frac{x_n - x^*}{x_n - x_{n+1}}\right| \leq \frac{mM}{(M^2 - \alpha m^2)} \text{ and hence } |x_n - x^*| \leq \frac{mM}{(M^2 - \alpha m^2)}\,|x_n - x_{n+1}|\,,$$

which is a required error estimation. □

Remark 2.1. Note that the error estimate (2.9) is better than the error estimate (1.6) and consequently the error estimate (1.4).

3. NUMERICAL EXAMPLES

We now present some numerical examples to show the efficiency of Newton-like S-iteration method. We compare the Newton method (NM) (1.1), the extended Newton-like method (ENLM) (1.5) and the Newton-like S-iteration method (SNLM) (1.7). All computations are done using the MATLAB.

We consider the following nonlinear equations in order to compare the above methods (1.1), (1.5) and (1.7).

$$f_1 : \left[\frac{9}{10}, \frac{6}{5}\right] \to \mathbb{R},\ f_1(x) = x^3 - 1; f_2 : \left[7, \frac{42}{5}\right] \to \mathbb{R},\ f_2(x) = x^2 - 11x + 24;$$

$$f_3 : \left[\frac{21}{10}, 4\right] \to \mathbb{R},\ f_3(x) = x^2 - 9; f_4 : [-1, 1] \to \mathbb{R},\ f_4(x) = \arctan x.$$

Table 3.1. Comparison of iterative methods. (f_1, $x_0 = 1.2$, $\alpha = 0.5$)

n	NM (1.1)	ENLM (1.5)	SNLM (1.7)
1	1.031481481481481	1.031481481481481	1.025703603467683
2	1.000951060058530	1.005536144775204	1.000486236297209
3	1.000000903369599	1.000993822096082	1.000000177257022
4	1.000000000000816	1.000179068011932	1.000000000000024
5	1.000000000000000	1.000032286214456	1.000000000000000

Tablo 3.2. Comparison of iterative methods. (f_1, $x_0 = 1.2$, $\alpha = 1.0$)

n	NM (1.1)	ENLM (1.5)	SNLM (1.7)
1	1.031481481481481	1.031481481481481	1.008923866607488
2	1.000951060058530	1.005536144775204	1.000001379921697
3	1.000000903369599	1.000993822096082	1.000000000000000
4	1.000000000000816	1.000179068011932	1.000000000000000
5	1.000000000000000	1.000032286214456	1.000000000000000

Tablo 3.3. Comparison of iterative methods. (f_2, $x_0 = 7.5$, $\alpha = 0.5$)

n	NM (1.1)	ENLM (1.5)	SNLM (1.7)
1	8.062500000000000	8.011363636363637	8.010488013698630
2	8.000762195121951	7.999768625636278	8.000005511337729
3	8.000000116152869	8.000004722148766	8.000000000001519
4	8.000000000000002	7.999999903629711	7.999999999999998
5	7.999999999999999	8.000000001966741	8.000000000000000

Tablo 3.4. Comparison of iterative methods. (f_2, $x_0 = 7.5$, $\alpha = 1.0$)

n	NM (1.1)	ENLM (1.5)	SNLM (1.7)
1	8.062500000000000	8.011363636363637	8.000762195121951
2	8.000762195121951	7.999768625636278	8.000000000000002
3	8.000000116152869	8.000004722148766	7.999999999999999
4	8.000000000000002	7.999999903629711	7.999999999999999
5	7.999999999999999	8.000000001966741	7.999999999999999

Tablo 3.5. Comparison of iterative methods. (f_3, $x_0 = 4$, $\alpha = 0.5$)

n	NM (1.1)	ENLM (1.5)	SNLM (1.7)
1	3.125000000000000	3.125000000000000	3.044407894736842
2	3.002500000000000	3.017543859649123	3.000082755197912
3	3.000001040799334	3.002500000000000	3.000000000285355
4	3.000000000000180	3.000357015351660	3.000000000000000
5	3.000000000000000	3.000050999592003	3.000000000000000

Tablo 3.6. Comparison of iterative methods. (f_3, $x_0 = 4$, $\alpha = 1.0$)

n	NM (1.1)	ENLM (1.5)	SNLM (1.7)
1	3.125000000000000	3.125000000000000	3.002500000000000
2	3.002500000000000	3.017543859649123	3.000000000000180
3	3.000001040799334	3.002500000000000	3.000000000000000
4	3.000000000000180	3.000357015351660	3.000000000000000
5	3.000000000000000	3.000050999592003	3.000000000000000

Tablo 3.7. Comparison of iterative methods. (f_4, $x_0 = 1$, $\alpha = 0.5$)

n	NM (1.1)	ENLM (1.5)	SNLM (1.7)
1	0.738200612200851	0.607300918301276	0.607805063038586
2	0.481414019366476	0.267743757940761	0.255409851684756
3	0.252831297040562	0.061386831843207	0.045568155214680
4	0.088357772945637	0.003622880946457	0.001167667461501
5	0.013463797786353	0.000013093679810	0.000000684170713
6	0.000353833278588	0.000000000171443	0.000000000000234
7	0.000000250233662	0.000000000000000	0.000000000000000

Tablo 3.8. Comparison of iterative methods. (f_4, $x_0 = 1$, $\alpha = 1.0$)

n	NM (1.1)	ENLM (1.5)	SNLM (1.7)
1	0.738200612200851	0.607300918301276	0.481414019366476
2	0.481414019366476	0.267743757940761	0.088357772945637
3	0.252831297040562	0.061386831843207	0.000353833278588
4	0.088357772945637	0.003622880946457	0.000000000000125
5	0.013463797786353	0.000013093679810	0.000000000000000
6	0.000353833278588	0.000000000171443	0.000000000000000
7	0.000000250233662	0.000000000000000	0.000000000000000

4. Conclusion

In this paper, we used the Newton-like S-iteration method for solving nonliner equations. This method is then compared with some other methods to show its better performance than comparable methods.

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