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q-Greens' s formula on the complex plane in the sense of Harman

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ABSTRACT. In this work, we proved the q-Green's identity in the sense of Harman for the discrete functions $f(z), g(z)$ which are defined on the square discrete set $D \subset \mathbb{C}$. We also presented q -analogue of Cauchy integral formula by using discrete q-analytic $f(z)$ and p-analytic $g(z)$ functions.

1. INTRODUCTION

Based on a discrete set in complex plane, interesting results for q-discrete functions were given by defining q-derivatives and q-integrals. For instance; Cauchy theorem which is well known in Complex Analysis, Cauchy integral formula and q-analogues of Green formula were obtained. For this, see [1], [2], [3], [6], [7] and [9]. For $0 < q < 1$, q-analyticity can be defined in various ways. See [5], [6] and [11]. Also, for $q = 1$ instead of q-analytic function, monodiffric discrete functions were studied. For this see [2] and [3].

C. J. Harman, for discrete function in [6] has defined q-line integral differently than Jackson integral, and with the help of this integral, for two discrete functions he has found q-analogues of Cauchy integral formula and Green formula discrete domain $D \subset \mathbb{C}$.

In this study, we obtained a different form of q-Green formula for discrete functions $f(z)$, $g(z)$ which are defined on the square discrete domain $D \subset \mathbb{C}$ with the help of multiple Jackson series by using the definition of Harman's q-line integral. In addition, by using discrete q-analytic $f(z)$ and p-analytic $g(z)$ functions, we presented q-analogue of Cauchy integral formula which is a little different from one in [6].

2. SOME NOTATIONS

Now, let's give some definitions and concepts of q-analysis that we are going to use in this study. Here, the parameter q will be $0 < q < 1$.

For $a \in \mathbb{R}$,

$$[a]_q = \frac{1-q^a}{1-q}$$

and it is called q-analogue of a.

For $x \in \mathbb{R}, n \in \mathbb{N}$

$$(1+x)_n = (1+x)(1+qx)...(1+q^{n-1}x)\ ,\ (1+x)_0 = 1$$

and for $m, n \in \mathbb{N}$

$$[n]_q! = \frac{(1-q)_n}{(1-q)^n}, \binom{n}{m}_q = \frac{(1-q)_n}{(1-q)_m(1-q)_{n-m}}; n \geq m$$

representations exist. See [4].

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Let's consider a fixed point $z = x + iy \in \mathbb{C}$ and define in complex plane the discrete set of Q as following;

$$Q = \{(q^m x, q^n y) = q^m x + i q^n y \in \mathbb{C} : m, n \in \mathbb{Z}\}. \tag{2.1}$$

For instance, let $z = x + iy$ be an initial point in Q. Then the discrete set Q is showed geometrically below and Figure 1:

$$Q = \left\{ \begin{array}{llll} z = (x, y), z_1 = (qx, y), & z_2 = (qx, qy), & z_3 = (x, qy), & z_4 = (q^{-1}x, qy), \\ z_5 = (q^{-1}x, y), & z_6 = (q^{-1}x, q^{-1}y), & z_7 = (x, q^{-1}y), & z_8 = (qx, q^{-1}y) \end{array} \right\}.$$

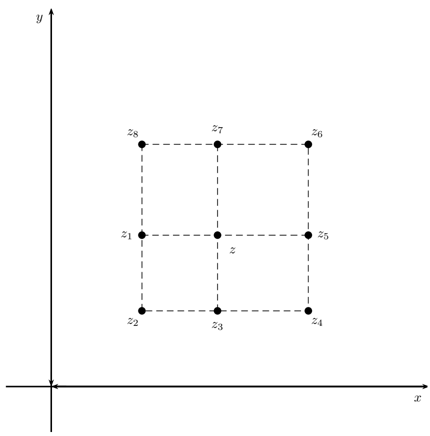

FIGURE 1. Q discrete set

Definition 2.1. [6] Let the point $z_i = (x_i, y_i) \in Q,\ i \in \mathbb{N}$ be given. If the point z_{i+1} is one of the following points

$$(qx_i, y_i), (q^{-1}x_i, y_i), (x_i, qy_i), (x_i, q^{-1}y_i)$$

then the points z_i and z_{i+1} are called adjacent.

For instance, in Figure 1, the points z_1, z_3, z_5, z_7 are adjacent points of z.

Definition 2.2. [6] Let the adjacent points $z_i, z_{i+1} \in Q$ be given. Then the following line γ is called a discrete q-curve in discrete set Q

$$\gamma =< z_0, z_1, ..., z_n > . \tag{2.2}$$

For $i \neq j$, if it is $z_i \neq z_j$ then, γ curve is called a simple q-curve in Q.

Definition 2.3. [6] If $\gamma =< z_0, z_1, ..., z_n >$ is a simple q-curve and $z_0 = z_n$, then this line is called simple closed q-curve.

Definition 2.4. For $z \in Q$, the following set

$$S(z) = \{z = (x, y), z_1 = (qx, y), z_2 = (qx, qy), z_3 = (x, qy)\} \tag{2.3}$$

is called simple basic set based on z.

According to this definition, any finite subset $D_1 \subset Q$ can be written as a combination of simple basic sets.

3. $q-$ANALYTIC FUNCTIONS

A function defined on a discrete set in complex plane is called a discrete function. Let $x_i \neq 0$ and $y_i \neq 0$ be. For $z_i \in Q$, let's take the set $T(z_i) = \{(x_i, y_i), (qx_i, y_i), (x_i, qy_i)\}$ into consideration. Let's define the discrete subset $D_1 = \bigcup_{i=1}^{n} T(z_i) \subset Q$. On this set a discrete function $f(z)$ is given.

Definition 3.5. [6], [8] Let $f(z)$ be a complex valued discrete function on D_1. The following operators $D_{q,x}$ and $D_{q,y}$ are called the complex partial q-derivative operators

$$D_{q,x}f(z) = \frac{f(z) - f(qx, y)}{(1-q)x},\ D_{q,y}f(z) = \frac{f(z) - f(x, qy)}{(1-q)iy}, x \neq 0, y \neq 0. \tag{3.4}$$

Definition 3.6. [6], [8] Let $f(z)$ be a discrete function defined on D_1. If following equation is satisfied at the point $z \in D_1$, then $f(z)$ is called $q-$analytic at z

$$D_{q,x}f(z) = D_{q,y}f(z). \tag{3.5}$$

If the equation in (3.5) is satisfied on each point of D_1, then $f(z)$ is called q-analytic in D_1.

Remark 3.1. In [11], the q-analyticity of a complex valued function $f(z)$ is defined with the following identity

$$D_{\overline{z}}f(z) = \frac{1}{2}(D^*_{q,x} + iM^y_{\frac{1}{q}}D^*_{q,y})f(z) \equiv 0. \tag{3.6}$$

Here

$$\begin{aligned} D^*_{q,x}f(z) &= \frac{f(z) - f(qx, y)}{(1-q)x},\ D^*_{q,y}f(z) = \frac{f(z) - f(x, qy)}{(1-q)y}, \\ M^y_q f(x, y) &= f(x, qy). \end{aligned} \tag{3.7}$$

Remark 3.2. It is clear that there is a relationship between operators in (3.4) and (3.7) such as $D_{q,x} = D^*_{q,x}$ and $D_{q,y} = -iD^*_{q,y}$.

Let $f(z)$ be a discrete function on D_1 and let's define operator L by

$$Lf(z) = \overline{z}f(z) - xf(x, qy) + iyf(qx, y). \tag{3.8}$$

Then, with a simple calculation, it can be seen that

$$f(z) \text{ is } q\text{-analytic on } D_1 \ \Leftrightarrow Lf(z) = 0. \tag{3.9}$$

Let $S(z)$ be as in (2.3), let's define the set D for $z_i \in Q$ as

$$D = \bigcup_{i=1}^{n} S(z_i). \tag{3.10}$$

Also let subset D_2 be as following

$$D_2 = \{z_j : z_j \in S(z_i);\ i = 1, 2, ..., n;\ z_i \in D\}. \tag{3.11}$$

If $f(z)$ is a q-analytic in D, then $D_{q,x}f(z) = D_{q,y}f(z)$ and the operator D_q can be used where $D_{q,x} = D_{q,y} = D_q$.

Remark 3.3. If $f(z)$ is q-analytic in D_2, then $D^n_q f(z)$ is also q-analytic. Here is $n = 1, 2, ...$.

Remark 3.4. If $u(x, y)$ is a real valued discrete function then the partial real q-derivatives of $u(x, y)$ with respect to x and y are given by

$$D^*_{q,x}u(x, y) = \frac{u(x, y) - u(qx, y)}{(1-q)x},\ D^*_{q,y}u(x, y) = \frac{u(x, y) - u(x, qy)}{(1-q)y}.$$

Let the function $f(z)$ is a q-analytic on discrete set D and $f(z) = u(x,y)+iv(x,y)$. Then from (3.9) the q-Cauchy-Riemann equations are

$$D^*_{q,x}u = D^*_{q,y}v \ , \ D^*_{q,x}v = -D^*_{q,y}u. \tag{3.12}$$

and q-Laplace equations are obtained as following

$$D^{*^2}_{q,x}u + D^{*^2}_{q,y}u = 0 \ , \ D^{*^2}_{q,x}v + D^{*^2}_{q,y}v = 0.$$

See [10].

Let $\gamma =< z_0, z_1, ..., z_n >$ be a simple closed curve in D and $f(z)$ is a function on γ. Then the line q-integral of $f(z)$ along γ is defined by

$$\int_\gamma f(t)d_qt = \int_{z_0}^{z_n} f(t)d_qt = \sum_{j=0}^{n-1} \int_{z_j}^{z_{j+1}} f(t)d_qt \tag{3.13}$$

where

$$\int_{z_j}^{z_{j+1}} f(t)d_qt = \begin{cases} (z_{j+1}-z_j)f(z_j); & z_{j+1} = (qx_j, y_j) \text{ or } (x_j, qy_j), \\ (z_{j+1}-z_j)f(z_{j+1}); & z_{j+1} = (q^{-1}x_j, y_j) \text{ or } (x_{j,}q^{-1}y_j). \end{cases} \tag{3.14}$$

See [6] for some properties of (3.13).

Theorem 3.1. [6] *A function $f(z)$ is q-analytic in D if and only if the discrete line integral around every discrete closed curve in D is zero.*

Proof. Let $f(z)$ q-analytic in D, then for every $z \in D$ can be written

$$Lf(z) = \overline{z}f(z) - xf(x,qy) + iyf(qx,y) = 0.$$

Without loss of generality, we will chose the simple closed curve $\gamma =< z, z_1, z_2, z_3, z >$ as elements of set $S(z)$. Then we get

$$\begin{aligned} \int_\gamma f(\zeta)d_q\zeta &= z_1f(z) + (z_2 - z_1)f(z_1) + (z_3 - z_2)f(z_3) - z_3f(z) \\ &= (qx+iy)f(z) + (q-1)iyf(qx,y) + (1-q)xf(x,qy) - (x+iqy)f(z) \\ &= (q\overline{z} - \overline{z})f(z) + iqyf(qx,y) - iyf(qx,y) - qxf(x,qy) + xf(x,qy) \\ &= (q-1)[\overline{z}f(z) - xf(x,qy) + iyf(qx,y)] \\ &= (q-1)Lf(z) \\ &= 0. \end{aligned}$$

The same result can be found for the closed discrete curves γ that have more elements. □

4. A $q-$ANALOGUE OF GREEN FORMULA

Let's consider the sets D, D_2 which are respectively defined in (3.10) and (3.11), in previous sections and show that simple closed discrete boundary of D as $\partial D := \gamma =< z_0, z_1, ..., z_n = z_0 >$.
In this case, the following theorem is valid:

Theorem 4.2. *Let $f(z)$ be any discrete function on D, then*

$$\int_{\partial D} f(\zeta)d_q\zeta = (q-1)\sum_{z\in D_2} Lf(z).$$

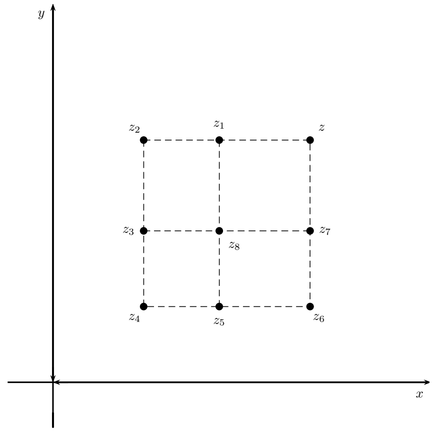

FIGURE 2. D discrete set

Proof. Without loss of generality, let us take set D specifically as $D = \{z, z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8\}$.

$$\begin{aligned}
\partial D &= \gamma =< z, z_1, z_2, z_3, z_4, z_5, z_6, z_7, z >, \\
S(z) &= \{z, z_1, z_7, z_8\}, \\
S(z_1) &= \{z_1, z_2, z_3, z_8\}, \\
S(z_8) &= \{z_3, z_4, z_5, z_8\}, \\
S(z_7) &= \{z_5, z_6, z_7, z_8\}, \\
D &= S(z) \cup S(z_1) \cup S(z_7) \cup S(z_8) = \{z, z_1, ..., z_7, z_8\}, \\
D_2 &= \{z, z_1, z_7, z_8\}.
\end{aligned}$$

Here it is clear that $D = S(z) \cup S(z_1) \cup S(z_7) \cup S(z_8) = \{z, z_1, ..., z_8\}$. Beside, $z = x+iy, z_1 = qx + iy, z_2 = q^2x + iy, z_3 = q^2x + iqy, z_4 = q^2x + iq^2y, z_5 = qx + iq^2y, z_6 = x + iq^2y, z_7 = x + iqy, z_8 = qx + iqy$. From (3.13) and (3.14), it can be written

$$\begin{aligned}
\int_{\partial D} f(\zeta) d_q\zeta &= (z_1 - z)f(z) + (z_2 - z_1)f(z_1) + (z_3 - z_2)f(z_2) + (z_4 - z_3)f(z_3) \\
&\quad +(z_5 - z_4)f(z_5) + (z_6 - z_5)f(z_6) + (z_7 - z_6)f(z_7) + (z - z_7)f(z) \\
&= (q-1)[xf(z) + qxf(z_1) + iyf(z_2) + iqyf(z_3) - qxf(z_5) - xf(z_6) - \\
&\quad iqyf(z_7) - iyf(z)]. \qquad (4.15)
\end{aligned}$$

On the other hand, from (3.8), for any discrete function $f(z)$, the following identities can be written;

$$\begin{aligned}
Lf(z) &= \overline{z}f(z) - xf(z_7) + iyf(z_1), \\
Lf(z_1) &= \overline{z_1}f(z_1) - qxf(z_8) + iyf(z_2), \\
Lf(z_7) &= \overline{z_7}f(z_7) - xf(z_6) + iqyf(z_8), \\
Lf(z_8) &= \overline{z_8}f(z_8) - qxf(z_5) + iqyf(z_3).
\end{aligned}$$

If these identities are used in (4.15), at the end of an arrangement, we get

$$\int_{\partial D} f(\zeta) d_q\zeta = \frac{q-1}{q}\left[\overline{z_8}f(z) + \overline{z_3}f(z_1) + \overline{z_5}f(z_7) + \overline{z_4}f(z_8)\right]$$

$$+(q-1)[Lf(z)-\overline{z}f(z)+Lf(z_8)-\overline{z_8}f(z_8)+Lf(z_7)-\overline{z_7}f(z_7)+Lf(z_1)-\overline{z_1}f(z_1)].$$

When we consider $\frac{1}{q}\overline{z_8}=\overline{z}$, $\frac{1}{q}\overline{z_3}=\overline{z_1}$, $\frac{1}{q}\overline{z_5}=\overline{z_7}$, $\frac{1}{q}\overline{z_4}=\overline{z_8}$ and after some simplifications, it can be written as

$$\int_{\partial D} f(\zeta)d_q\zeta = (q-1)[Lf(z)+Lf(z_1)+Lf(z_7)+Lf(z_8)] = (q-1)\sum_{z\in D_2} Lf(z).$$

□

Let $f(z)$ be a discrete function in discrete set D and $p=q^{-1}$. The operators $D_{p,x}, D_{p,y}$ are defined by (as in (3.4))

$$D_{p,x}f(z)=\frac{f(z)-f(px,y)}{(1-p)x},\ D_{p,y}f(z)=\frac{f(z)-f(x,py)}{(1-p)iy};x\neq 0, y\neq 0. \tag{4.16}$$

Definition 4.7. [6] *Let $g(z)$ be a discrete function in D. If $g(z)$ satisfies*

$$D_{p,x}g(z)=D_{p,y}g(z), \tag{4.17}$$

then the function $g(z)$ is called $p-$analytic at z.

Remark 4.5. *Let $g(z)$ be a discrete function on D and let's define the operator B as*

$$Bg(z)=\overline{z}g(z)-xg(x,py)+iyg(px,y). \tag{4.18}$$

Then, it can be seen that $g(z)$ is $p-$analytic on D if and only if $Bg(z)=0$ in D.

Let the discrete curve $\gamma=<z_0,z_1,...,z_n>$ be in discrete set D and the discrete functions $f(z)$ and $g(z)$ be on this curve. Then, conjoint line integral along γ is defined as

$$\int_\gamma (f*g)(\zeta)d_q\zeta=\int_{z_0}^{z_n}(f*g)(\zeta)d_q\zeta=\sum_{j=0}^{n-1}\int_{z_j}^{z_{j+1}}(f*g)(\zeta)d_q\zeta, \tag{4.19}$$

where

$$\int_{z_j}^{z_{j+1}}(f*g)(\zeta)d_q\zeta=\begin{cases}(z_{j+1}-z_j)f(z_j)g(z_{j+1}); & z_{j+1}=(qx_j,y_j)\text{ or}(x_j,qy_j),\\ (z_{j+1}-z_j)f(z_{(j+1)})g(z_j); & z_{j+1}=(px_j,y_j)\text{ or }(x_j,py_j).\end{cases} \tag{4.20}$$

See [6].

Theorem 4.3. *Let's choose the simple closed discrete curve $\gamma=<z_0,z_1,...,z_{n-1},z_n=z_0>$ as boundary points of finite discrete set D, i.e. $\partial D=\gamma$. If $f(z)$ and $g(z)$ are discrete complex functions in D, then*

$$\int_\gamma (f*g)(\zeta)d_q\zeta=\frac{1-q}{q}\sum_{z\in D_2}[f(z)Bg(qz)-qg(qz)Lf(z)]. \tag{4.21}$$

Proof. Without loss of generality, let's take the set D as in Theorem 4.2. From (4.19), we can write

$$\int_\gamma (f*g)(\zeta)d_q\zeta = (z_1-z)f(z)g(z_1)+(z_2-z_1)f(z_1)g(z_2)+(z_3-z_2)f(z_2)g(z_3)$$
$$+(z_4-z_3)f(z_3)g(z_4)+(z_5-z_4)f(z_5)g(z_4)+(z_6-z_5)f(z_6)g(z_5)$$

$$\begin{aligned} & +(z_7-z_6)f(z_7)g(z_6)+(z-z_7)f(z)g(z_7) \\ = \;& (q-1)xf(z)g(z_1)+(q-1)qxf(z_1)g(z_2)+(q-1)iyf(z_2)g(z_3) \\ & +(q-1)iqyf(z_3)g(z_4)+(1-q)xf(z_6)g(z_5)+(1-q)iqyf(z_7)g(z_6) \\ & +(1-q)iyf(z)g(z_7). \end{aligned} \tag{4.22}$$

On the other hand, we can write by using (3.8) and (4.18)

$$\begin{aligned} Bg(qz) &= Bg(z_8)=\overline{z_8}g(z_8)-qxg(z_1)+iqyg(z_7), \\ Bg(qz_1) &= Bg(z_3)=\overline{z_3}g(z_3)-q^2xg(z_2)+iqyg(z_8), \\ Bg(qz_7) &= Bg(z_5)=\overline{z_5}g(z_5)-qxg(z_8)+iq^2yg(z_6), \\ Bg(qz_8) &= Bg(z_4)=\overline{z_4}g(z_4)-q^2xg(z_3)+iqyg(z_5), \\ Lf(z) &= \overline{z}f(z)-xf(z_7)+iyf(z_1), \\ Lf(z_1) &= \overline{z_1}f(z_1)-qxf(z_8)+iyf(z_2), \\ Lf(z_7) &= \overline{z_7}f(z_7)-xf(z_6)+iqyf(z_8), \\ Lf(z_8) &= \overline{z_8}f(z_8)-qxf(z_5)+iqyf(z_3). \end{aligned}$$

And if these identities are used in (4.22), by simple calculations, we get

$$\begin{aligned} \int_\gamma (f*g)(\zeta)d_q\zeta &= \sum_{z\in D_2}\left\{\frac{1-q}{q}f(z)[Bg(qz)-(\overline{qz})g(qz)]-(1-q)g(qz)[Lf(z)-\overline{z}f(z)]\right\} \\ &= \frac{1-q}{q}\sum_{z\in D_2}[f(z)Bg(qz)-qg(qz)Lf(z)]. \end{aligned}$$

So, this completes the proof of theorem. □

Remark 4.6. Under the conditions of Theorem 4.3, (4.21) was given as

$$\int_\gamma (f*g)(\zeta)d_q\zeta=\sum_{z\in D_2}[f(z)Bg(qz)-g(qz)Lf(z)] \tag{4.23}$$

without proof in [6], and this result was used in the same paper. We think that this result (4.23) is incorrect. See [[6], Theorem 8.2].

Remark 4.7. Let $\gamma=z(t)=x(t)+iy(t)$ be a continuous curve on $\mathbb{C}$ with $t\in[0,a]$ and $f(z)=u(x,y)+iv(x,y)$ be a complex valued function on the curve γ. If $x(t)$ and $y(t)$ have q-derivative, then with the help of Jackson series the line q-integral of $f(z)$ along γ is given as

$$\int_\gamma f(z)d_qz=\int_0^a f(z(t))D_qz(t)d_qt=(1-q)a\sum_{n=0}^{\infty}q^nf(z(aq^n))D_qz(aq^n). \tag{4.24}$$

See [10].
Here is

$$D_qz(t)=\frac{z(t)-z(qt)}{(1-q)t}, t\neq 0.$$

We note that (3.13) and (4.24) q-line integrals are different.

Let's choose the discrete set D_3 where a is a constant $(a>0)$, as

$$D_3=\{z_{mn}=aq^m+iaq^n=(aq^m,aq^n):0\leq m,n<\infty;m,n\in\mathbb{N}\}. \tag{4.25}$$

In this case, the discrete boundary of D_3

$$\partial D_3=C_1\cup C_2\cup C_3\cup C_4, \tag{4.26}$$

where

$$\begin{aligned} C_1 &= \{aq^n : n = 0, 1, 2, ...\}, \\ C_2 &= \{a + iaq^n : n = 0, 1, 2, ...\}, \\ C_3 &= \{aq^n + ia : n = 0, 1, 2, ...\}, \\ C_4 &= \{iaq^n : n = 0, 1, 2, ...\}. \end{aligned} \tag{4.27}$$

Definition 4.8. Let $f(z)$ be given a discrete function in D_3 in (4.25). If the limits

$$\begin{aligned} \lim_{n\to\infty} f(x, aq^n) &= f(x, 0), 0 < x \leq a, \\ \lim_{n\to\infty} f(aq^n, y) &= f(0, y); 0 < y \leq a \end{aligned}$$

exist, then the $f(z)$ is called $q-$regular on γ_1 and γ_4, respectively.
Also, if the limit

$$\lim_{m,n\to\infty} f(q^m a, q^n a) := \lim_{m\to\infty}\left[\lim_{n\to\infty} f(q^m a, q^n a)\right] = \lim_{n\to\infty}\left[\lim_{m\to\infty} f(q^m a, q^n a)\right] = f(0, 0)$$

exists, then the $f(z)$ is called q-regular at the origin. See [10].

Lemma 4.1. *Let D_3 be as in (4.25). If the function $f(z)$ is q-analytic in D_3 and q-regular on ∂D_3, then*

$$\int_{\partial D_3} f(\zeta) d_q\zeta = 0. \tag{4.28}$$

Proof. From the definition (3.13), we get

$$\begin{aligned} \int_{\partial D_3} f(\zeta) d_q\zeta &= (1-q)a \sum_{n=0}^{\infty} q^n [f(aq^n, 0) - f(aq^n, a) + if(a, aq^n) - if(0, aq^n)] \\ &= (1-q)a \sum_{n=0}^{\infty} [q^n f(aq^n, 0) - q^n f(aq^n, a) + iq^n f(a, aq^n) - iq^n f(0, aq^n)]. \end{aligned} \tag{4.29}$$

On the other hand, because the $f(\zeta)$ is q-analytic,

$$\begin{aligned} Lf(a, aq^n) &= (a - iaq^n) f(a, aq^n) - f(a, aq^{n+1}) + iaq^n f(aq, aq^n) = 0, \\ Lf(aq^n, a) &= (aq^n - ia) f(aq^n, a) - aq^n f(aq^n, aq) + iaf(aq^{n+1}, a) = 0. \end{aligned}$$

can be written. From these equations, we get

$$iq^n f(a, aq^n) = f(a, aq^n) - f(a, aq^{n+1}) + iq^n f(aq, aq^n), \tag{4.30}$$

$$-q^n f(aq^n, a) = -if(aq^n, a) - q^n f(aq^n, aq) + if(aq^{n+1}, a). \tag{4.31}$$

By using (4.30) and (4.31) in (4.29), we get

$$\begin{aligned} \int_{\partial D_3} f(\zeta) d_q\zeta &= (1-q)a \sum_{n=0}^{\infty} [f(a, aq^n) - f(a, aq^{n+1})] - (1-q)ia \sum_{n=0}^{\infty} [f(aq^n, a) - f(aq^{n+1}, a)] \\ &\quad + (1-q)a \sum_{n=0}^{\infty} [q^n f(aq^n, 0) - iq^n f(0, aq^n) + iq^n f(aq, aq^n) - q^n f(aq^n, aq)] \\ &= (1-q)a \lim_{n\to\infty} [f(a, a) - f(a, aq^{n+1})] - (1-q)ia \lim_{n\to\infty} [f(a, a) - f(aq^{n+1}, a)] \\ &\quad + (1-q)a \sum_{n=0}^{\infty} [q^n f(aq^n, 0) - iq^n f(0, aq^n) + iq^n f(aq, aq^n) - q^n f(aq^n, aq)] \end{aligned}$$

$$= (1-q)a[f(a,a)-f(a,0)-if(a,a)+if(0,a)+f(a,0)-if(0,a)+if(aq,a)-f(a,aq)]$$
$$+ (1-q)a\sum_{n=1}^{\infty}[q^n f(aq^n,0) - iq^n f(0,aq^n) + iq^n f(aq,aq^n) - q^n f(aq^n,aq)]$$
$$= (1-q)aLf(a,a)+(1-q)a\sum_{n=1}^{\infty}[q^n f(aq^n,0)-iq^n f(0,aq^n)+iq^n f(aq,aq^n)-q^n f(aq^n,aq)]$$
$$= (1-q)a\sum_{n=1}^{\infty}[q^n f(aq^n,0) - iq^n f(0,aq^n) + iq^n f(aq,aq^n) - q^n f(aq^n,aq)]. \tag{4.32}$$

Beside, from hypothesis

$$\begin{aligned} Lf(aq^n,aq) &= (aq^n - iaq)f(aq^n,aq) - aq^n f(aq^n,aq^2) + iaqf(aq^{n+1},aq) = 0, \\ Lf(aq,aq^n) &= (aq - iaq^n)f(aq,aq^n) - aqf(aq,aq^{n+1}) + iaq^n f(aq^2,aq^n) = 0 \end{aligned}$$

and from here

$$q^n f(aq^n,aq) = iqf(aq^n,aq) + q^n f(aq^n,aq^2) - iqf(aq^{n+1},aq), \tag{4.33}$$
$$iq^n f(aq,aq^n) = qf(aq,aq^n) - qf(aq,aq^{n+1}) + iq^n f(aq^2,aq^n) \tag{4.34}$$

can be written. By rewriting (4.33) and (4.34) in (4.32) and if the same operations are repeated, then it follows

$$\int_{\partial D_3} f(\zeta)d_q\zeta = (1-q)a\sum_{n=2}^{\infty}[q^n f(aq^n,0) - iq^n f(0,aq^n) + iq^n f(aq^2,aq^n) - q^n f(aq^n,aq^2)].$$

If these steps are repeated k times, finally we get

$$\int_{\partial D_3} f(\zeta)d_q\zeta = (1-q)a\sum_{n=k}^{\infty} q^n[f(aq^n,0) - if(0,aq^n) + if(aq^k,aq^n) - f(aq^n,aq^k)].$$

Since $n \to \infty$ for $k \to \infty$, so we get

$$\int_{\partial D_3} f(\zeta)d_q\zeta = 0.$$

□

Theorem 4.4. *Let D_3 and ∂D_3 be as in (4.25) and (4.26) respectively. Also, let the discrete functions $f(z)$ and $g(z)$ are q-regular on γ_1, γ_4 and at the origin. Then*

$$\int_{\partial D_3} (f*g)(\zeta)d_q\zeta = \frac{1-q}{q}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[f(z_{mn})Bg(qz_{mn}) - qg(qz_{mn})Lf(z_{mn})]. \tag{4.35}$$

Proof. From the additivity property of q-line integral in (4.19) [see [6]], we can write

$$\begin{aligned} \int_{\partial D_3} (f*g)(\zeta)d_q\zeta &= \int_{C_1}(f*g)(\zeta)d_q\zeta + \int_{C_2}(f*g)(\zeta)d_q\zeta \\ &\quad + \int_{C_3}(f*g)(\zeta)d_q\zeta + \int_{C_4}(f*g)(\zeta)d_q\zeta \\ &= (1-q)a\sum_{n=0}^{\infty} q^n f(aq^n,0)g(aq^{n+1},0) + (1-q)ia\sum_{n=0}^{\infty} q^n f(a,aq^n)g(a,aq^{n+1}) \\ &\quad +(q-1)a\sum_{n=0}^{\infty} q^n f(aq^n,a)g(aq^{n+1},a) + (q-1)ia\sum_{n=0}^{\infty} q^n f(0,aq^n)g(0,aq^{n+1}) \end{aligned}$$

$$= (1-q)a\sum_{n=0}^{\infty}q^n[f(aq^n,0)g(aq^{n+1},0)+if(a,aq^n)g(a,aq^{n+1})$$
$$-f(aq^n,a)g(aq^{n+1},a)-if(0,aq^n)g(0,aq^{n+1})]. \quad (4.36)$$

Also, from (3.8) and (4.18),

$$\begin{aligned}
Lf(z_{mn}) &= Lf(aq^m,aq^n)\\
&= (aq^m-iaq^n)f(aq^m,aq^n)-aq^m f(aq^m,aq^{n+1})\\
&\quad +iaq^n f(aq^{m+1},aq^n)\\
&= aq^m f(aq^m,aq^n)-iaq^n f(aq^m,aq^n)\\
&\quad -aq^m f(aq^m,aq^{n+1})+iaq^n f(aq^{m+1},aq^n),\\
Bg(qz_{mn}) &= Bg(aq^{m+1},aq^{n+1})\\
&= (aq^{m+1}-iaq^{n+1})g(aq^{m+1},aq^{n+1})-aq^{m+1}g(aq^{m+1},aq^n)\\
&\quad +iaq^{n+1}g(aq^m,aq^{n+1})\\
&= aq^{m+1}g(aq^{m+1},aq^{n+1})-iaq^{n+1}g(aq^{m+1},aq^{n+1})\\
&\quad -aq^{m+1}g(aq^{m+1},aq^n)+iaq^{n+1}g(aq^m,aq^{n+1})
\end{aligned}$$

can be written.
If these identities are used on the right side of the equation in (4.35), it follows

$$\begin{aligned}
&\frac{1-q}{q}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[f(z_{mn})Bg(qz_{mn})-qg(qz_{mn})Lf(z_{mn})]\\
=\ &(1-q)a\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f(aq^m,aq^n)\left[q^m g(aq^{m+1},aq^{n+1})-iq^n g(aq^{m+1},aq^{n+1})\right.\\
&\left.-q^m g(aq^{m+1},aq^n)+iq^n g(aq^m,aq^{n+1})\right]\\
&-(1-q)a\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}g(aq^{m+1},aq^{n+1})\left[q^m f(aq^m,aq^n)-iq^n f(aq^m,aq^n)\right.\\
&\left.-q^m f(aq^m,aq^{n+1})+iq^n f(aq^{m+1},aq^n)\right]\\
=\ &(1-q)a\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\{q^m[f(aq^m,aq^{n+1})g(aq^{m+1},aq^{n+1})-f(aq^m,aq^n)g(aq^{m+1},aq^n)]\\
&+iq^n[f(aq^m,aq^n)g(aq^m,aq^{n+1})-f(aq^{m+1},aq^n)g(aq^{m+1},aq^{n+1})]\}. \qquad (4.37)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q^m[f(aq^m,aq^{n+1})g(aq^{m+1},aq^{n+1})-f(aq^m,aq^n)g(aq^{m+1},aq^n)]\\
=\ &\sum_{m=0}^{\infty}q^m[f(aq^m,aq)g(aq^{m+1},aq)-f(aq^m,a)g(aq^{m+1},a)\\
&+f(aq^m,aq^2)g(aq^{m+1},aq^2)-f(aq^m,aq)g(aq^{m+1},aq)\\
&+f(aq^m,aq^3)g(aq^{m+1},aq^3)-f(aq^m,aq^2)g(aq^{m+1},aq^2)\\
&+f(aq^m,aq^4)g(aq^{m+1},aq^4)-f(aq^m,aq^3)g(aq^{m+1},aq^3)\\
&\vdots\\
&+f(aq^m,aq^k)g(aq^{m+1},aq^k)-f(aq^m,aq^{k-1})g(aq^{m+1},aq^{k-1})\\
&+f(aq^m,aq^{k+1})g(aq^{m+1},aq^{k+1})-f(aq^m,aq^k)g(aq^{m+1},aq^k)\\
&+f(aq^m,aq^{k+2})g(aq^{m+1},aq^{k+2})-f(aq^m,aq^{k+1})g(aq^{m+1},aq^{k+1})+...]
\end{aligned}$$

$$= \sum_{m=0}^{\infty} q^m [f(aq^m, 0)g(aq^{m+1}, 0) - f(aq^m, a)g(aq^{m+1}, a)]. \tag{4.38}$$

Similarly, we can find

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} iq^n[f(aq^m, aq^n)g(aq^m, aq^{n+1}) - f(aq^{m+1}, aq^n)g(aq^{m+1}, aq^{n+1})] \\
= &\sum_{n=0}^{\infty} iq^n[f(a, aq^n)g(a, aq^{n+1}) - f(aq\ , aq^n)g(aq\ , aq^{n+1}) \\
&+f(aq, aq^n)g(aq, aq^{n+1}) - f(aq^2, aq^n)g(aq^2, aq^{n+1}) \\
&+f(aq^2, aq^n)g(aq^2, aq^{n+1}) - f(aq^3, aq^n)g(aq^3, aq^{n+1}) \\
&+f(aq^3, aq^n)g(aq^3, aq^{n+1}) - f(aq^4, aq^n)g(aq^4, aq^{n+1}) \\
&\vdots \\
&+f(aq^{k-1}, aq^n)g(aq^{k-1}, aq^{n+1}) - f(aq^k, aq^n)g(aq^k, aq^{n+1}) \\
&+f(aq^k, aq^n)g(aq^k, aq^{n+1}) - f(aq^{k+1}, aq^n)g(aq^{k+1}, aq^{n+1}) \\
&+f(aq^{k+1}, aq^n)g(aq^{k+1}, aq^{n+1}) - f(aq^{k+2}, aq^n)g(aq^{k+2}, aq^{n+1}) + ...] \\
= &\sum_{n=0}^{\infty} iq^n[f(a, aq^n)g(a, aq^{n+1}) - f(0, aq^n)g(0, aq^{n+1})].
\end{aligned} \tag{4.39}$$

If (4.38) and (4.39) are replaced in (4.37) and by considering (4.36), we get

$$\begin{aligned}
&\frac{1-q}{q} a \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[f(z_{mn})Bg(qz_{mn}) - qg(qz_{mn})Lf(z_{mn})] \\
= \ &(1-q)a\sum_{n=0}^{\infty} q^n \left[f(aq^n, 0)g(aq^{n+1}, 0) + if(a, aq^n)g(a, aq^{n+1})\right. \\
&\left.-f(aq^n, a)g(aq^{n+1}, a) - if(0, aq^n)g(0, aq^{n+1})\right] \\
= \ &\int_{\partial D_3} (f * g)(\zeta) d_q\zeta.
\end{aligned} \tag{4.40}$$

So this completes the theorem. □

Remark 4.8. If the discrete function f is q-analytic and the discrete function g is p-analytic in D_3, then $Lf(z_{mn}) = 0$, $Bg(qz_{mn}) = 0$ and from (4.35)

$$\int_{\partial D_3} (f * g)(\zeta) d_q\zeta = 0.$$

Remark 4.9. *(4.35) is called a q-analogue of Green identity on the complex plane.*

Corollary 4.1. *If the function $f(z)$ is q-analytic in D_3, then $Lf(z_{mn}) = 0$ and*

$$\int_{\partial D_3} (f * g)(\zeta) d_q\zeta = \frac{1-q}{q} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f(z_{mn})Bg(qz_{mn});\ 0 < q < 1.$$

Definition 4.9. [6] Let D_3 be a discrete set as (4.25) and B the operator in (4.18). For $z, \zeta \in D_3$, the discrete function G_ζ is called q-singularity function if it satisfies the following condition:

$$B[G_z(\zeta)] = \begin{cases} 1; & \zeta = z, \\ 0; & \zeta \neq z. \end{cases} \tag{4.41}$$

As an example for (4.41), the following example is given in [6] for $z = (x, y) \in Q$

$$G_z(\zeta) = G_z(q^m x, q^n y) = \begin{cases} \frac{\binom{m+n}{n}_q x^n y^m (y+ix)}{(x-iy)_{n+1}(y+ix)_{m+1}}; & m \geq 0, n \geq 0, \\ 0, & m, n = -1, -2, ..., \end{cases}$$

where Q discrete set defined as in (2.1).
Then, for $m \geq 0, n \geq 0$, it follows $BG_z(z) = 1$ and for $\zeta \neq z$, $BG_z(\zeta) = 0$. From here, by using (4.35) we get

$$\int_{\partial D_3} (f * G_z)(\zeta) d_q\zeta = \frac{1-q}{q} f(pz) \tag{4.42}$$

and it is a q-analogue of classical Cauchy integral formula.

Corollary 4.2. *If the function $f(z)$ is q-analytic on discrete domain D_3 and $G_z(\zeta)$ is a q-singularity function, then*

$$\int_{\partial D_3} (f * G_{qz})(\zeta) d_q\zeta = \begin{cases} \frac{1-q}{q} f(z); & z \in D_3, \\ 0; & z \notin D_3. \end{cases} \tag{4.43}$$

5. Conclusion

In [6], Harman defined a discrete contour integral and found analogues for Cauchy integral theorem. Harman obtained the results in [6] for a set which is in the form as (3.10). In this paper, differently from Harman, we obtained the results for set which is in the form as (4.25). For these results, we used q-regularity which is defined in Definition 4.8.

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