CREAT. MATH. INFORM.
Volume **26** (2017), No. 3,
Pages 331 - 338

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2017.03.10

Some convergence results for nonexpansive mappings in uniformly convex hyperbolic spaces

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ABSTRACT. In this paper, we establish some strong and $\triangle$-convergence theorems of an iteration process for approximating a common fixed point of three nonexpansive mappings in a uniformly convex hyperbolic space. The results presented here extend and improve various results in the existing literature.

1. INTRODUCTION

Khan *et al.* [8] introduced the following iteration process in a Banach space:

$$\begin{cases} x_1 \in K, \\ y_n = (1-\beta_n)x_n + \beta_n Q x_n, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n S y_n, \ \ n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$.

The following iteration process is a translation of the iteration process (1.1) from Banach space to hyperbolic space:

$$\begin{cases} x_1 \in K, \\ y_n = W(x_n, Qx_n, \beta_n), \\ x_{n+1} = W(Tx_n, Sy_n, \alpha_n), \ \ n \geq 1. \end{cases} \tag{1.2}$$

It is worth mentioning that this iteration process coincides with the iteration process (1.1) when $W(x,y,\alpha) = (1-\alpha)x + \alpha y$ and X is a uniformly convex Banach space. Moreover, the iteration process (1.2) is reduced to the S-iteration process of Khan and Abbas [9] in a CAT(0) space if $W(x,y,\alpha) = (1-\alpha)x \oplus \alpha y$ and $T = S = Q$. It is also reduced to Ishikawa iteration in [7] when $T = I, S = Q$, Mann iteration in [16] when $T = Q = I$ and Picard iteration when $T = S, Q = I$.

Note that the iteration process given in (1.2) has three nonexpansive mappings T, S and Q. The purpose of this paper is to get some results on strong and $\triangle$-convergence of this iteration process in a uniformly convex hyperbolic space. Our results generalize some recent results given in [9, 3].

2. PRELIMINARIES ON HYPERBOLIC SPACE

In 1970, Takahashi [20] introduced the concept of convex metric space as follows.

A mapping $W : X \times X \times [0,1] \to X$ is a convex structure in X if

$$d(u, W(x,y,\lambda)) \leq (1-\lambda)d(u,x) + \lambda d(u,y),$$

Received: 07.09.2016. In revised form: 07.03.2017. Accepted: 14.03.2017
2010 *Mathematics Subject Classification.* 47H09, 47H10, 54H25.
Key words and phrases. Δ-*convergence, common fixed point, nonexpansive mapping, iteration process, strong convergence, uniformly convex hyperbolic space.*
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for all $x, y, u \in X$ and $\lambda \in [0,1]$. A metric space (X, d) together with a convex structure W is called *convex metric space*. A subset K of a convex metric space X is *convex* if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0,1]$.

After that several authors extended this concept in many ways. One such convex structure is available in the hyperbolic space introduced by Kohlenbach [13], which is more restrictive than the hyperbolic type in [5] and more general than the hyperbolic space defined in [17].

A hyperbolic space (X, d, W) (see [13]) is a metric space (X, d) together with a mapping $W : X \times X \times [0,1] \to X$ satisfying

(W1) $d(z, W(x, y, \lambda)) \leq (1-\lambda)d(z, x) + \lambda d(z, y)$,

(W2) $d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2| \, d(x, y)$,

(W3) $W(x, y, \lambda) = W(y, x, (1-\lambda))$,

(W4) $d(W(x, z, \lambda), W(y, w, \lambda)) \leq (1-\lambda)d(x, y) + \lambda d(z, w)$,

for all $x, y, z, w \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0,1]$. This class of hyperbolic spaces contains all normed linear spaces and convex subsets thereof, $\mathbb{R}$-trees, the Hilbert ball with the hyperbolic metric (see [6]), Cartesian products of Hilbert balls, Hadamard manifolds and CAT(0) spaces (see [2, 21, 22]), as special cases.

The following example accentuates the importance of hyperbolic space.

Let B_H be an open unit ball in a complex Hilbert space $(H, \langle . \rangle)$ w.r.t. the metric (also known as the Kobayashi distance)

$$k_{B_H}(x, y) = \arg\tanh\left(1 - \sigma(x, y)\right)^{\frac{1}{2}},$$

where

$$\sigma(x, y) = \frac{\left(1 - \|x\|^2\right)\left(1 - \|y\|^2\right)}{|1 - \langle x, y \rangle|^2} \quad \text{for all } x, y \in B_H.$$

Then (B_H, k_{B_H}, W) is a hyperbolic space where $W(x, y, \lambda)$ defines a unique point z in a unique geodesic segment $[x, y]$ for all $x, y \in B_H$.

A hyperbolic space (X, d, W) is said to be *uniformly convex* [19] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that $d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1-\delta)r$ whenever $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \varepsilon r$.

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is called *modulus of uniform convexity*. We call η *monotone* if it decreases with r (for a fixed ε).

The concept of $\triangle$-convergence in a metric space was introduced by Lim [14] and its analogue in a CAT(0) space has been investigated by Dhompongsa and Panyanak [3]. In [11], Khan *et al.* continued the investigation of $\triangle$-convergence in the general setup of hyperbolic spaces. Now, we collect some basic concepts.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, define a continuous functional $r(., \{x_n\}) : X \to [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in X\}.$$

The *asymptotic radius* $r_K(\{x_n\})$ of $\{x_n\}$ *with respect to a subset* K of X is given by

$$r_K(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in K\}.$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

The *asymptotic center* $A_K(\{x_n\})$ of $\{x_n\}$ *with respect to a subset* K of X is the set

$$A_K(\{x_n\}) = \{x \in K : r(x, \{x_n\}) = r_K(\{x_n\})\}.$$

Recall that a sequence $\{x_n\}$ in X is said to be $\triangle$-*convergent* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_{n\to\infty} x_n = x$ and call x as $\triangle$-limit of $\{x_n\}$.

In the sequel, we shall need the following results.

Lemma 2.1. [15, Proposition 3.3] *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*

Lemma 2.2. [11, Lemma 2.5] *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n, x) \le r, \ \limsup_{n\to\infty} d(y_n, x) \le r, \ \lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$$

for some $r \ge 0$, then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

3. Strong and $\triangle$-convergence theorems

Let K be a nonempty subset of a metric space (X, d) and T be a self-mapping on K. Then T is nonexpansive if $d(Tx, Ty) \le d(x, y)$ for all $x, y \in K$. From now onward, we denote F the set of all common fixed points of nonexpansive self mappings on K.

In this section, we prove some convergence theorems for nonexpansive mappings in uniformly convex hyperbolic spaces. First, we give the following key lemmas.

Lemma 3.3. *Let K be a nonempty, closed and convex subset of a hyperbolic space X and T, S, Q be three nonexpansive self mappings on K with $F \ne \emptyset$. Then for the sequence $\{x_n\}$ defined in (1.2), we have $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.*

Proof. For any $p \in F$, it follows from (1.2) that

$$\begin{aligned} d(y_n, p) &= d(W(x_n, Qx_n, \beta_n), p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n d(Qx_n, p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{3.3}$$

Using (3.3), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(Tx_n, Sy_n, \alpha_n), p) \\ &\le (1-\alpha_n)d(Tx_n, p) + \alpha_n d(Sy_n, p) \\ &\le (1-\alpha_n)d(x_n, p) + \alpha_n d(y_n, p) \\ &\le (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

Hence $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.

Lemma 3.4. *Let K be a nonempty, closed and convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and T, S, Q be three nonexpansive self mappings on K such that $d(x_n, Sx_n) \le d(Tx_n, Sx_n)$ and $F \ne \emptyset$. Let the sequence $\{x_n\}$ be as defined in (1.2) such that $\{\alpha_n\}, \{\beta_n\} \subset [a, b]$ for some $a, b \in (0, 1)$. Then*

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, Sx_n) = \lim_{n\to\infty} d(x_n, Qx_n) = 0.$$

Proof. Let $p \in F$. By Lemma 3.3, it follows that $\lim_{n\to\infty} d(x_n, p)$ exists. We may assume that

$$\lim_{n\to\infty} d(x_n, p) = r.$$

The case $r = 0$ is trivial. Next, we deal with the case $r > 0$. By (3.3) and the nonexpansiveness of S, we obtain

$$\begin{aligned}\limsup_{n\to\infty} d(Sy_n, p) &\leq \limsup_{n\to\infty} d(y_n, p) \\ &\leq \lim_{n\to\infty} d(x_n, p) = r.\end{aligned}$$

Moreover, we have

$$\limsup_{n\to\infty} d(Tx_n, p) \leq r.$$

Since

$$\lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d(W(Tx_n, Sy_n, \alpha_n), p) = r,$$

Lemma 2.2 gives

$$\lim_{n\to\infty} d(Tx_n, Sy_n) = 0. \tag{3.4}$$

Next

$$\begin{aligned}d(x_{n+1}, p) &\leq (1-\alpha_n)d(Tx_n, p) + \alpha_n d(Sy_n, p) \\ &\leq (1-\alpha_n)d(Tx_n, Sy_n) + (1-\alpha_n)d(Sy_n, p) + \alpha_n d(Sy_n, p) \\ &\leq d(y_n, p) + (1-\alpha_n)d(Tx_n, Sy_n)\end{aligned}$$

yields that $\liminf_{n\to\infty} d(y_n, p) \geq r$. But by (3.3), we have $\limsup_{n\to\infty} d(y_n, p) \leq r$. Hence

$$\lim_{n\to\infty} d(y_n, p) = \lim_{n\to\infty} d(W(x_n, Qx_n, \beta_n), p) = r.$$

Since $\limsup_{n\to\infty} d(Qx_n, p) \leq r$ and $\lim_{n\to\infty} d(x_n, p) = r$, Lemma 2.2 guarantees

$$\lim_{n\to\infty} d(x_n, Qx_n) = 0. \tag{3.5}$$

By virtue of (3.5), we get

$$\begin{aligned}d(Sx_n, Sy_n) &\leq d(x_n, y_n) \\ &= d(x_n, W(x_n, Qx_n, \beta_n)) \\ &\leq \beta_n d(x_n, Qx_n) \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.6}$$

From the hypothesis $d(x_n, Sx_n) \leq d(Tx_n, Sx_n)$, we have

$$\begin{aligned}d(x_n, Sx_n) &\leq d(Tx_n, Sx_n) \\ &\leq d(Tx_n, Sy_n) + d(Sy_n, Sx_n).\end{aligned}$$

It follows from (3.4) and (3.6) that

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0.$$

Since

$$d(x_n, Tx_n) \leq d(x_n, Sx_n) + d(Sx_n, Sy_n) + d(Sy_n, Tx_n),$$

we conclude that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

The proof is completed.

Now we prove the $\triangle$-convergence theorem of the iteration process defined by (1.2) in a uniformly convex hyperbolic space.

Theorem 3.1. *Let K, X, T, S, Q and $\{x_n\}$ be the same as in Lemma 3.4. Then the sequence $\{x_n\}$ $\triangle$-converges to a point in F.*

Proof. It follows from Lemma 3.3 that the sequence $\{x_n\}$ is bounded. Therefore by Lemma 2.1, $\{x_n\}$ has a unique asymptotic center, that is, $A_K(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A_K(\{u_n\}) = \{u\}$. By Lemma 3.4, we have

$$\lim_{n\to\infty} d(u_n, Tu_n) = \lim_{n\to\infty} d(u_n, Su_n) = \lim_{n\to\infty} d(u_n, Qu_n) = 0. \tag{3.7}$$

We claim that $u \in F$. So, we calculate

$$\begin{aligned} d(Tu, u_n) &\leq d(Tu, Tu_n) + d(Tu_n, u_n) \\ &\leq d(u, u_n) + d(Tu_n, u_n). \end{aligned}$$

Taking lim sup on both sides of the above inequality and using (3.7), we have

$$r(Tu, \{u_n\}) = \limsup_{n\to\infty} d(Tu, u_n) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

The uniqueness of asymptotic center implies that $Tu = u$. A similar argument shows that $Su = u$ and $Qu = u$. This means that $u \in F$. Since $\lim_{n\to\infty} d(x_n, u)$ exists (by Lemma 3.3) and considering the uniqueness of asymptotic center, we have

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u) \end{aligned}$$

a contradiction. Hence $x = u$. Thus $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$, that is, $\{x_n\}$ $\triangle$-converges to $x \in F$.

A sequence $\{x_n\}$ in a metric space X is said to be *Fejér monotone with respect to* K (a subset of X) if $d(x_{n+1}, p) \leq d(x_n, p)$ for all $p \in K$ and $n \in \mathbb{N}$. □

For further development, we need the following technical result.

Lemma 3.5. [1] *Let K be a nonempty closed subset of a complete metric space (X, d) and let $\{x_n\}$ be Fejér monotone with respect to K. Then $\{x_n\}$ converges to some $p \in K$ if and only if $\lim_{n\to\infty} d(x_n, K) = 0$.*

Next we discuss the strong convergence of the iteration process defined by (1.2) in a uniformly convex hyperbolic space.

Theorem 3.2. *Let K, X, T, S, Q and $\{x_n\}$ be the same as in Lemma 3.4. Then $\{x_n\}$ converges strongly to some $p \in F$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$ where $d(x, F) = inf\{d(x, p) : p \in F\}$.*

Proof. If $\{x_n\}$ converges to $p \in F$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \leq d(x_n, F) \leq d(x_n, p)$, we have $\liminf_{n\to\infty} d(x_n, F) = 0$. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. It follows from Lemma 3.3 that $\lim_{n\to\infty} d(x_n, F)$ exists. Thus by hypothesis, $\lim_{n\to\infty} d(x_n, F) = 0$. Again by Lemma 3.3, $\{x_n\}$ is Fejér monotone with respect to F. Thus Lemma 3.5 implies that $\{x_n\}$ converges strongly to a point p in F. □

Remark 3.1. In Theorem 3.2, the condition $\liminf_{n\to\infty} d(x_n, F) = 0$ may be replaced with $\limsup_{n\to\infty} d(x_n, F) = 0$.

Example 3.1. Let $\mathbb{R}$ be the real line with the usual metric $|.|$ and $T, S, Q : \mathbb{R} \to \mathbb{R}$ be three mappings defined by $T(x) = 1 - x$, $S(x) = \frac{2x+1}{4}$ and $Q(x) = \frac{1}{2}$. It is noticed in [8, p.10] that T and S satisfy the condition $d(x_n, Sx_n) \leq d(Tx_n, Sx_n)$. Additionally T, S and Q are nonexpansive mappings. Clearly, $F = \left\{\frac{1}{2}\right\}$. Set $\alpha_n = \frac{n}{2n+1}$ and $\beta_n = \frac{2n}{3n+1}$ for all $n \in \mathbb{N}$.

Thus, the conditions of Lemma 3.4 are fulfilled. Therefore the results of Theorem 3.1 and Theorem 3.2 can be easily seen.

Following Senter and Dotson [18], Khan and Fukhar-ud-din [10] introduced the so-called condition $(A^{'})$ for two mappings and gave an improved version of it in [4] as follows.

Two mappings $T, S : K \to K$ with $F \neq \emptyset$ are said to satisfy the *condition* $(A^{'})$ if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that either $d(x, Tx) \geq f(d(x, F))$ or $d(x, Sx) \geq f(d(x, F))$ for all $x \in K$.

This condition becomes condition (A) of Senter and Dotson [18] whenever $S = T$.

We can modify this definition for three mappings as follows.

Let T, S and Q be three nonexpansive self mappings on K with $F \neq \emptyset$. These mappings are said to satisfy *condition* (B) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F))$ or $d(x, Sx) \geq f(d(x, F))$ or $d(x, Qx) \geq f(d(x, F))$ for all $x \in K$.

The condition (B) is reduced to the condition (A') when $Q = T$.

We use the condition (B) to study strong convergence of $\{x_n\}$ defined in (1.2).

Theorem 3.3. *Under the assumptions of Lemma 3.4, if T, S, Q satisfy the condition (B), then $\{x_n\}$ converges strongly to a point in F.*

Proof. By Lemma 3.3, $\lim_{n\to\infty} d(x_n, F)$ exists. Also, by Lemma 3.4, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, Sx_n) = \lim_{n\to\infty} d(x_n, Qx_n) = 0.$$

Then, by using the condition (B), we get $\lim_{n\to\infty} f\left(d\left(x_n, F\right)\right) = 0$. Since f is a non-decreasing function with $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore Theorem 3.2 implies that $\{x_n\}$ converges strongly to a point in F.

Recall that a mapping T from a subset K of a metric space (X, d) into itself is *semi-compact* if every bounded sequence $\{x_n\} \subset K$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a strongly convergent subsequence.

By using this definition, we obtain the following strong convergence theorem. □

Theorem 3.4. *Under the assumptions of Lemma 3.4, if one of the mappings T, S and Q is semi-compact or K is compact, then $\{x_n\}$ converges strongly to a point in F.*

Proof. It is clear that the condition (B) is weaker than both the compactness of K and the semi-compactness of one of the nonexpansive mappings T, S and Q. Therefore we have the result of above theorem. □

Remark 3.2. (i) Theorems 3.1-3.3 extend the corresponding results of Khan and Abbas [9] from CAT(0) space to the general setup of uniformly convex hyperbolic space.

(ii) Theorems 3.1–3.4 contain the corresponding theorems proved for the Ishikawa iteration when $T = I, S = Q$, for the Mann iteration when $T = Q = I$ and for the Picard iteration when $T = S, Q = I$. Then these theorems improve and generalize some results of Dhompongsa and Panyanak [3].

If we take $Q = T$ in Theorems 3.1–3.4, we get the following corollary, which is single-valued case of the Theorems 2.4-2.7 in [12].

Corollary 3.1. *Let K be a nonempty, closed and convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and T, S be two nonexpansive self* *mappings on K such that $F \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by*

$$\begin{cases} x_1 \in K, \\ y_n = W(x_n, Tx_n, \beta_n), \\ x_{n+1} = W(Tx_n, Sy_n, \alpha_n), \quad n \in \mathbb{N}. \end{cases} \tag{3.8}$$

(i) Then the sequence $\{x_n\}$ $\triangle$-converges to some $p \in F$.

(ii) Then $\{x_n\}$ converges strongly to some $p \in F$ if and only if

$$\liminf_{n\to\infty} d(x_n, F) = 0 \text{ or } \limsup_{n\to\infty} d(x_n, F) = 0.$$

(iii) If T and S satisfy the condition $(A^{'})$, then $\{x_n\}$ converges strongly to a point in F.

(iv) If one of the mappings T and S is semi-compact or K is compact, then $\{x_n\}$ converges strongly to a point in F.

Remark 3.3. Note that the iteration process (3.8) has two nonexpansive mappings T, S and the condition $d(x_n, Sx_n) \leq d(Tx_n, Sx_n)$ is not needed to get convergence of this iteration.

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