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Lacunary statistical convergence of order (α, β) in topological groups

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ABSTRACT. In this paper, the concept of lacunary statistical convergence of order (α, β) is generalized to topological groups, and some inclusion relations between the set of all statistically convergent sequences of order (α, β) and the set of all lacunary statistically convergent sequences of order (α, β) are given.

1. INTRODUCTION

The idea of statistical convergence was given by Zygmund [20] in the first edition of his monograph puplished in Warsaw in 1935. The consept of statistical convergence was introduced by Steinhaus [19] and Fast [10] and later reintroduced by Schoenberg [17] independently. Later on it was further investigated from the sequence space point of view and linked with summability theory by Çakallı ([1], [2], [3], [4]), Et et al. ([7], [8], [9]), Fridy [12], Fridy and Orhan [13], Salat [15], Caserta and Kočinac [5] and many others.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. Throught this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r. Lacunary sequences have been studied in ([6], [11], [14], [16], [18]).

2. MAIN RESULTS

In this section, we give some inclusion relations between the set of all statistically convergent sequences and the set of all lacunary statistically convergent sequences of order (α, β) in topological groups.

Definition 2.1. Let X be an abelian topological Hausdorf group, $\theta = (k_r)$ be a lacunary sequence, $(x(k))$ be a sequence of real numbers, α and β be positive real numbers such that $0 < \alpha \leq \beta \leq 1$. The sequence $x = (x(k))$ is said to be $S_\alpha^\beta(\theta, X)$ −statistically convergent to l in X (or lacunary statistically convergent sequences of order (α, β) to l in X) if there is a real number l for each neighbourhood U of 0 such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : x(k) - l \notin U\}\right|^\beta = 0,$$

where $I_r = (k_{r-1}, k_r]$ and h_r^α denotes the αth power $(h_r)^\alpha$ of h_r, that is; $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, ..., h_r^\alpha, ...)$ and $\left|\{k \in I_r : x(k) - l \notin U\}\right|^\beta$ denotes the βth power of $|\{k \in I_r : x(k) - l \notin U\}|$. In this case we write $S_\alpha^\beta(\theta) - \lim x(k) = l$. The set of all $S_\alpha^\beta(\theta, X)$ −statistically convergent sequences in X will be denoted by $S_\alpha^\beta(\theta, X)$. If $\theta = (2^r)$, then we write $S_\alpha^\beta(X)$ instead of $S_\alpha^\beta(\theta, X)$. If $\alpha = \beta = 1$ and $\theta = (2^r)$, then we write $S(X)$ instead of $S_\alpha^\beta(\theta, X)$.

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The lacunary statistical convergence of order (α, β) in topological groups is well defined for $\alpha \leq \beta$, but it is not well defined for $\beta < \alpha$ in general. For this $\mathbf{x} = (x(k))$ be defined as follows:

$$x_k = \begin{cases} 3, & \text{if} \quad k = 2m \\ 2, & \text{if} \quad k \neq 2m \end{cases} \quad m = 1, 2, ...$$

Let U be $\frac{1}{2}$ neighbourhood of 0. We write for $\beta < \alpha$, $\ell = 1$ and $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : x(k) - 1 \notin U\}\right|^\beta = \lim_{r\to\infty} \frac{h_r^\beta}{2h_r^\alpha} = 0$$

and for $\ell = 0$

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : x(k) - 0 \notin U\}\right|^\beta = \lim_{r\to\infty} \frac{h_r^\beta}{2h_r^\alpha} = 0.$$

Thus $x(k) \to 1 \left(S_\alpha^\beta(\theta)\right)$ and $x(k) \to 0 \left(S_\alpha^\beta(\theta)\right)$ for $\beta < \alpha$. But this is impossible.

We note that every lacunary statistical convergent sequence of order (α, β) has only one limit, that is; if a sequence is lacunary statistically convergent of order (α, β) to l_1 and l_2 then $l_1 = l_2$. Suppose that $(x(k))$ has two different lacunary statistical limits of order (α, β), l_1, l_2 say. Since X is a Hausdorff space there exists a neighbourhood U of 0 such that $l_1 - l_2 \notin U$. Then we may choose a neighbourhood W of 0 such that $W + W \subset U$. Write $z(k) = l_1 - l_2$ for all $k \in \mathbb{N}$. Therefore for all $r \in \mathbb{N}$,

$$\{k \in I_r : z(k) \notin U\} \subset \{k \in I_r : l_1 - x(k) \notin W\} \cup \{k \in I_r : x(k) - l_2 \notin W\}.$$

Now it follows from this inclusion that, for all $r \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$,

$$\frac{1}{h_r^\alpha} \left|\{k \in I_r : z(k) \notin U\}\right|^\beta \leq \frac{1}{h_r^\alpha} \left|\{k \in I_r : l_1 - x(k) \notin W\}\right|^\beta + \frac{1}{h_r^\alpha} \left|\{k \in I_r : x(k) - l_2 \notin W\}\right|^\beta.$$

Since $S_\alpha^\beta(\theta) - \lim x(k) = l_1$ and $S_\alpha^\beta(\theta) - \lim x(k) = l_2$ we get

$$\begin{aligned} \lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : z(k) \notin U\}\right|^\beta \quad &\leq \quad \lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : l_1 - x(k) \notin W\}\right|^\beta \\ &+ \lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : x(k) - l_2 \notin W\}\right|^\beta. \end{aligned}$$

Hence $\lim_{r\to\infty} \frac{h_r^\beta}{h_r^\alpha} \leq 0 \left(\lim_{r\to\infty} \frac{h_r^\beta}{h_r^\alpha} \geq 1\right)$. This contradiction shows that $l_1 = l_2$.

Definition 2.2. Let X be an abelian topological Hausdorff group, $\theta = (k_r)$ be a lacunary sequence, $(x(k))$ be a sequence of real numbers, α and β be positive real numbers such that $0 < \alpha \leq \beta \leq 1$. The sequence $\mathbf{x} = (x(k))$ is said to be $S_\alpha^\beta(\theta, X)$ $-$Cauchy sequence if there is a subsequence $\left\{x\left(k^{'}(r)\right)\right\}$ of x such that $k'(r) \in I_r$ for each r, $\lim_r x\left(k^{'}(r)\right) = l$, and for each neighbourhood U of 0

$$\lim_r \frac{1}{h_r^\alpha} \left|\left\{k \in I_r : x(k) - x\left(k^{'}(r)\right) \notin U\right\}\right|^\beta = 0.$$

The proof of each of the following results is straightforward, so we choose to state these results without proof.

Theorem 2.1. Let α and β be positive real numbers such that $0 < \alpha \leq \beta \leq 1$. The sequence x is $S_\alpha^\beta(\theta, X)$ $-$convergent if and only if x is $S_\alpha^\beta(\theta, X)$ $-$Cauchy sequence.

Theorem 2.2. Let $\alpha_1, \alpha_2, \beta_1$ and β_2 be positive real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$ then $S_{\alpha_1}^{\beta_2}(X) \subseteq S_{\alpha_2}^{\beta_1}(X)$ and the inclusion is strict.

Theorem 2.2 yields the following corollary.

Corollary 2.1. If a sequence is $S_{\alpha}^{\beta}(\theta, X)-$statistically convergent of order (α, β) to l, then it is $S(\theta, X)-$statistically convergent to l.

Theorem 2.3. Let α and β be positive real numbers such that $0 < \alpha \leq \beta \leq 1$ and $\theta = (k_r)$ be a lacunary sequence. If $\liminf_r q_r > 1$, then $S_{\alpha}^{\beta}(X) \subset S_{\alpha}^{\beta}(\theta, X)$.

Proof. Suppose that $\liminf_r q_r > 1$; then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta} \Longrightarrow \left(\frac{h_r}{k_r}\right)^{\alpha} \geq \left(\frac{\delta}{1+\delta}\right)^{\alpha} \Longrightarrow \frac{1}{k_r^{\alpha}} \geq \frac{\delta^{\alpha}}{(1+\delta)^{\alpha}} \frac{1}{h_r^{\alpha}}.$$

If $x_k \to l\left(S_{\alpha}^{\beta}(\theta)\right)$, then for each neighbourhood U of 0 and for sufficiently large r, we have

$$\begin{aligned} \frac{1}{k_r^{\alpha}} \left|\{k \leq k_r : x(k) - l \notin U\}\right|^{\beta} &\geq \frac{1}{k_r^{\alpha}} \left|\{k \in I_r : x(k) - l \notin U\}\right|^{\beta} \\ &\geq \frac{\delta^{\alpha}}{(1+\delta)^{\alpha}} \frac{1}{h_r^{\alpha}} \left|\{k \in I_r : x(k) - l \notin U\}\right|^{\beta}, \end{aligned}$$

this completes the proof. □

Theorem 2.4. Let α and β be positive real numbers such that $0 < \alpha \leq \beta \leq 1$ and $\theta = (k_r)$ be a lacunary sequence. If

$$\lim_{r\to\infty} \inf \frac{h_r^{\alpha}}{k_r} > 0 \tag{2.1}$$

then $S(X) \subset S_{\alpha}^{\beta}(\theta, X)$.

Proof. For each neighbourhood U of 0, we have

$$\{k \leq k_r : x(k) - l \notin U\} \supset \{k \in I_r : x(k) - l \notin U\}.$$

Therefore,

$$\begin{aligned} \frac{1}{k_r} \left|\{k \leq k_r : x(k) - l \notin U\}\right| &\geq \frac{1}{k_r} \left|\{k \in I_r : x(k) - l \notin U\}\right|^{\beta} \\ &= \frac{h_r^{\alpha}}{k_r} \frac{1}{h_r^{\alpha}} \left|\{k \in I_r : x(k) - l \notin U\}\right|^{\beta}. \end{aligned}$$

Taking limit as $r \to \infty$ and using (2.1), we get

$$x(k) \to l(S(X)) \Longrightarrow x(k) \to l\left(S_{\alpha}^{\beta}(\theta)\right).$$

□

Theorem 2.5. If $(x(k))$ belongs to both $S_{\alpha}^{\beta}(X)$ and $S_{\alpha}^{\beta}(\theta, X)$, then $S_{\alpha}^{\beta} - \lim_{k\to\infty} x(k) = S_{\alpha}^{\beta}(\theta) - \lim_{k\to\infty} x(k)$ for each $0 < \alpha \leq \beta \leq 1$.

Proof. Take any $(x(k)) \in S_{\alpha}^{\beta}(X) \cap S_{\alpha}^{\beta}(\theta, X)$ and $S_{\alpha}^{\beta} - \lim_{k\to\infty} x(k) = l_1$, $S_{\alpha}^{\beta}(\theta) - \lim_{k\to\infty} x(k) = l_2$, say. Suppose that $l_1 \neq l_2$. Since X is a Hausdorff space, there exists a symmetric neighbourhood U of 0 such that $l_1 - l_2 \notin U$. Then we may choose a symmetric neighbourhood W of 0 such that $W + W \subset U$. Then we obtain the following inequality:

$$\begin{aligned} \frac{1}{k_m^{\alpha}} \left|\{k \leq k_m : z(k) \notin U\}\right|^{\beta} &\leq \frac{1}{k_m^{\alpha}} \left|\{k \leq k_m : x(k) - l_1 \notin W\}\right|^{\beta} \\ &\quad + \frac{1}{k_m^{\alpha}} \left|\{k \leq k_m : l_2 - x(k) \notin W\}\right|^{\beta} \end{aligned}$$

where $z(k) = l_2 - l_1$ for all $k \in \mathbb{N}$. It follows from this inequality that

$$1 \leq \frac{1}{k_m^\alpha} \left|\{k \leq k_m : x(k) - l_1 \notin W\}\right|^\beta + \frac{1}{k_m^\alpha} \left|\{k \leq k_m : l_2 - x(k) \notin W\}\right|^\beta .$$

The second term on the right side of this inequality tends to 0 as $m \to \infty$. To see this write

$$\begin{aligned}
\frac{1}{k_m^\alpha} \left|\{k \leq k_m : l_2 - x(k) \notin W\}\right|^\beta &= \frac{1}{k_m^\alpha} \left|\left\{k \in \bigcup_{r=1}^m I_r : l_2 - x(k) \notin W\right\}\right|^\beta \\
&= \frac{1}{k_m^\alpha} \left(\sum_{r=1}^m \left|\{k \in I_r : l_2 - x(k) \notin W\}\right|\right)^\beta \\
&= \frac{1}{k_m^\alpha} \left(\sum_{r=1}^m h_r^\alpha t_r\right)^\beta \leq \frac{1}{\left(\sum_{r=1}^m h_r\right)^\alpha} \left(\sum_{r=1}^m h_r^\alpha t_r\right)
\end{aligned}$$

where $t_r = \frac{1}{h_r^\alpha} \left|\{k \in I_r : l_2 - x(k) \notin W\}\right|$ for $0 < \alpha \leq \beta \leq 1$. Since $S_\alpha^\beta(\theta) - \lim_{k\to\infty} x(k) = l_2$, we can write $S_\alpha(\theta) - \lim_{k\to\infty} x(k) = l_2$. We know that $\lim_{r\to\infty} t_r = 0$. Therefore

$$\lim_{m\to\infty} \frac{1}{k_m^\alpha} \left|\{k \leq k_m : l_2 - x(k) \notin W\}\right|^\beta = 0. \tag{2.2}$$

On the other hand, since $S_\alpha^\beta - \lim_{k\to\infty} x(k) = l_1$,

$$\lim_{m\to\infty} \frac{1}{k_m^\alpha} \left|\{k \leq k_m : x(k) - l_1 \notin W\}\right|^\beta = 0. \tag{2.3}$$

By (2.2) and (2.3) it follows that

$$\frac{1}{k_m^\alpha} \left|\{k \leq k_m : z(k) \notin U\}\right|^\beta = 0.$$

This contradiction completes the proof. □

Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let $\alpha_1, \alpha_2, \beta_1$ and β_2 be positive real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$. Now we shall give a general result for different $\alpha' s$, $\beta' s$ and $\theta' s$ which also include Theorem 2.2.

Theorem 2.6. Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$, let U be any neighbourhood of 0 and let $\alpha_1, \alpha_2, \beta_1$ and β_2 be positive real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$,
(i) If

$$\lim_{r\to\infty} \inf \frac{h_r^{\alpha_1}}{\ell_r^{\alpha_2}} > 0 \tag{2.4}$$

then $S_{\alpha_2}^{\beta_2}\left(\theta', X\right) \subseteq S_{\alpha_1}^{\beta_1}(\theta, X)$,
(ii) If

$$\lim_{r\to\infty} \frac{\ell_r}{h_r^{\alpha_2}} = 1 \tag{2.5}$$

then $S_{\alpha_1}^{\beta_2}(\theta, X) \subseteq S_{\alpha_2}^{\beta_1}\left(\theta', X\right)$, where $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $h_r = k_r - k_{r-1}$, $\ell_r = s_r - s_{r-1}$.

Proof. (i) Suppose that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let (2.4) be satisfied. For each neighbourhood W of 0 we have

$$\{k \in J_r : x(k) - l \notin W\} \supseteq \{k \in I_r : x(k) - l \notin U\}$$

and so

$$\frac{1}{\ell_r^{\alpha_2}}\left|\{k\in J_r : x(k)-l\notin W\}\right|^{\beta_2} \geq \frac{h_r^{\alpha_1}}{\ell_r^{\alpha_2}}\frac{1}{h_r^{\alpha_1}}\left|\{k\in I_r : x(k)-l\notin U\}\right|^{\beta_1}$$

for all $r\in\mathbb{N}$. Now taking the limit as $r\to\infty$ in the last inequality and using (2.4) we get $S_{\alpha_2}^{\beta_2}\left(\theta', X\right)\subseteq S_{\alpha_1}^{\beta_1}(\theta, X)$.

(ii) Let $x=(x(k))\in S_{\alpha_1}^{\beta_2}(\theta, X)$ and (2.5) be satisfied. Let W be any neighbourhood of 0. Then we may choose a neighbourhood $W_{1,}W_2, U$ of 0 such that $W_1+W_2+U\subseteq W$. Thus

$$\begin{aligned}
\frac{1}{\ell_r^{\alpha_2}}\left|\{k\in J_r : x_k-l\notin W\}\right|^{\beta_1} &= \frac{1}{\ell_r^{\alpha_2}}\left|\{s_{r-1}<k\leq k_{r-1} : x(k)-l\notin W_1\}\right|^{\beta_1}\\
&\quad+\frac{1}{\ell_r^{\alpha_2}}\left|\{k_r<k\leq s_r : x(k)-l\notin W_2\}\right|^{\beta_1}\\
&\quad+\frac{1}{\ell_r^{\alpha_2}}\left|\{k_{r-1}<k\leq k_r : x(k)-l\notin U\}\right|^{\beta_1}\\
&\leq \frac{(k_{r-1}-s_{r-1})^{\beta_1}}{\ell_r^{\alpha_2}}+\frac{(s_r-k_r)^{\beta_1}}{\ell_r^{\alpha_2}}\\
&\quad+\frac{1}{\ell_r^{\alpha_2}}\left|\{k\in I_r : x(k)-l\notin U\}\right|^{\beta_2}\\
&\leq \frac{k_{r-1}-s_{r-1}}{\ell_r^{\alpha_2}}+\frac{s_r-k_r}{\ell_r^{\alpha_2}}+\frac{1}{\ell_r^{\alpha_2}}\left|\{k\in I_r : x(k)-l\notin U\}\right|^{\beta_2}\\
&= \frac{\ell_r-h_r}{\ell_r^{\alpha_2}}+\frac{1}{\ell_r^{\alpha_2}}\left|\{k\in I_r : x(k)-l\notin U\}\right|^{\beta_2}\\
&\leq \frac{\ell_r-h_r^{\alpha_2}}{h_r^{\alpha_2}}+\frac{1}{h_r^{\alpha_2}}\left|\{k\in I_r : x(k)-l\notin U\}\right|^{\beta_2}\\
&\leq \left(\frac{\ell_r}{h_r^{\alpha_2}}-1\right)+\frac{1}{h_r^{\alpha_1}}\left|\{k\in I_r : x(k)-l\notin U\}\right|^{\beta_2}
\end{aligned}$$

for all $r\in\mathbb{N}$. This implies that $S_{\alpha_1}^{\beta_2}(\theta, X)\subseteq S_{\alpha_2}^{\beta_1}\left(\theta', X\right)$. □

From Theorem 2.6 we have the following.

Corollary 2.2. Let $\theta=(k_r)$ and $\theta'=(s_r)$ be two lacunary sequences such that $I_r\subset J_r$ for all $r\in\mathbb{N}$, let U be any neighbourhood of 0 and let $\alpha_1, \alpha_2, \beta_1$ and β_2 be such that $0<\alpha_1\leq\alpha_2\leq\beta_1\leq\beta_2\leq 1$.

If (2.4) holds then,

(i) $S_{\alpha_2}^{\beta_1}\left(\theta', X\right)\subseteq S_{\alpha_1}^{\beta_1}(\theta, X)$, for $\beta_1=\beta_2$,

(ii) $S_{\alpha_2}\left(\theta', X\right)\subseteq S_{\alpha_1}(\theta, X)$, for $\beta_1=\beta_2=1$,

(iii) $S_{\alpha_1}^{\beta_2}\left(\theta', X\right)\subseteq S_{\alpha_1}^{\beta_1}(\theta, X)$, for $\alpha_1=\alpha_2$,

(iv) $S_{\alpha_2}^{\beta_2}\left(\theta', X\right)\subseteq S_{\alpha_1}^{\alpha_2}(\theta, X)$, for $\alpha_2=\beta_1$,

(v) $S_{\alpha_2}^{\alpha_2}\left(\theta', X\right)\subseteq S_{\alpha_1}^{\alpha_2}(\theta, X)$, for $\alpha_2=\beta_1=\beta_2$,

(vi) $S\left(\theta', X\right)\subseteq S_{\alpha_1}(\theta, X)$, for $\alpha_2=\beta_1=\beta_2=1$.

If (2.5) holds then,

(i) $S_{\alpha_1}^{\beta_1}(\theta, X)\subseteq S_{\alpha_2}^{\beta_1}\left(\theta', X\right)$, for $\beta_1=\beta_2$,

(ii) $S_{\alpha_1}(\theta, X)\subseteq S_{\alpha_2}\left(\theta', X\right)$, for $\beta_1=\beta_2=1$,

$(iii)\ S_{\alpha_1}^{\beta_2}(\theta, X) \subseteq S_{\alpha_1}^{\beta_1}\left(\theta^{'}, X\right)$, for $\alpha_1 = \alpha_2$,

$(iv)\ S_{\alpha_1}^{\beta_2}(\theta, X) \subseteq S_{\alpha_2}^{\alpha_2}\left(\theta^{'}, X\right)$, for $\alpha_2 = \beta_1$,

$(v)\ S_{\alpha_1}^{\alpha_2}(\theta, X) \subseteq S_{\alpha_2}^{\alpha_2}\left(\theta^{'}, X\right)$, for $\alpha_2 = \beta_1 = \beta_2$,

$(vi)\ S_{\alpha_1}(\theta, X) \subseteq S\left(\theta^{'}, X\right)$, for $\alpha_2 = \beta_1 = \beta_2 = 1$.

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