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Principal functions of discrete Sturm-Liouville equations with hyperbolic eigenparameter

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ABSTRACT. In this study, we take under investigation principal functions corresponding to the eigenvalues and the spectral singularities of the boundary value problem (BVP) $a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n$, $n \in \mathbb{N}$ and $(\gamma_0 + \gamma_1\lambda) y_1 + (\beta_0 + \beta_1\lambda) y_0 = 0$ where (a_n) and (b_n) are complex sequences, λ is a hyperbolic eigenparameter and $\gamma_i, \beta_i \in \mathbb{C}$ for $i = 0, 1$.

1. INTRODUCTION

Spectral analysis of differential and discrete operators plays a crucial role on solutions of certain problems in various areas including mathematical physics, engineering, economics and quantum mechanics. Therefore, spectral analysis of differential and discrete operators have been main topic of various studies $[1-13]$.

Let us consider the discrete boundary value problem (BVP)

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N} = \{1, 2, ...\}, \tag{1.1}$$

$$y_0 = 0 \tag{1.2}$$

where (a_n) and (b_n) are complex sequences, $a_0 \neq 0$ and λ is a spectral parameter. In $[11]$, it has been proven that the spectrum of the BVP (1.1), (1.2) consists of a continuous spectrum, eigenvalues and spectral singularities.

Investigation of principal vectors corresponding to eigenvalues and spectral singularities of operators is an important research area in the sense that it helps to find spectral expansion of the operators and investigate effects of spectral singularities to this expansion. So, many authors investigated principal vectors of operators in their papers $[3, 8, 9]$.

In our previous study $[12]$, we showed that the BVP

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N} = \{1, 2, ...\}, \tag{1.3}$$

$$(\gamma_0 + \gamma_1\lambda) y_1 + (\beta_0 + \beta_1\lambda) y_0 = 0, \tag{1.4}$$

has a finite number of eigenvalues and spectral singularities with finite multiplicities if

$$\sup_{n\in\mathbb{N}} \left[\exp\left(\varepsilon n^{\delta}\right)\left(|1 - a_n| + |b_n|\right)\right] < \infty \tag{1.5}$$

for some $\varepsilon > 0$, $\frac{1}{2} \leq \delta \leq 1$ and $\lambda = 2\cosh z$.

Let us define the second order difference operator L in $l_2(\mathbb{N})$ by

$$(ly)_n := a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1}, \; n \in \mathbb{N} \cup \{0\},$$

where $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ are complex sequences and $a_n \neq 0$.

The set up of the study is as follows: In Section 2, we present some results obtaines in $[12]$. In the last part, we find principal vectors corresponding to the eigenvalues and spectral singularities of L and give some properties of them.

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2. Eigenvalues and spectral singularities of L

Assume (1.5). Then, the equation (1.3) has the solution

$$e_n(z) = \alpha_n e^{nz} \left(1 + \sum_{m=1}^{\infty} A_{nm} e^{mz}\right), \quad n \in \mathbb{N} \cup \{0\} \tag{2.6}$$

for $\lambda = 2\cos z$ where $z \in \overline{\mathbb{C}}_{left} := \{z : z \in \mathbb{C}, Re\, z \leq 0\}$ and A_{nm}, α_n are expressed in terms of (a_n) and (b_n) as

$$\begin{aligned} \alpha_n &= \left\{\prod_{k=n}^{\infty} a_k\right\}^{-1}, \\ A_{n1} &= -\sum_{k=n+1}^{\infty} b_k, \\ A_{n2} &= \sum_{m=1}^{\infty} \left\{\left(1-a_k^2\right) + b_k \sum_{s=k+1}^{\infty} b_s\right\}, \\ A_{nm} &= A_{n+1,m-2} + \sum_{k=n+1}^{\infty} \left\{\left(1-a_k^2\right) A_{k+1,m-2} - b_k A_{k,m-1}\right\}, \ m = 3, 4, ... \end{aligned} \tag{2.7}$$

Moreover A_{nm} satisfy

$$|A_{nm}| \leq C \sum_{k=n+\left[\frac{m}{2}\right]}^{\infty} (|1-a_k| + |b_k|) \tag{2.8}$$

where $C > 0$ is a constant and $\left[\frac{m}{2}\right]$ is integer part of $\frac{m}{2}$. Hence, $e_n(z)$ is analytic with respect to z in $\mathbb{C}_{left} := \{z : z \in \mathbb{C}, Re\, z < 0\}$ and continuous in $Re\, z = 0$.

Using (2.6) and boundary condition (1.4), we define the function f,

$$f(z) = (\gamma_0 + 2\gamma_1 \cosh z)\, e_1(z) + (\beta_0 + 2\beta_1 \cosh z) e_0(z). \tag{2.9}$$

The function f is analytic in $\mathbb{C}_{left}$, continuous in $\overline{\mathbb{C}}_{left}$ and $f(z) = f(z + 2\pi i)$.

Let us define the semi-strips

$$P_0 := \left\{z : z \in \mathbb{C}, z = \xi + i\tau, -\frac{\pi}{2} \leq \tau \leq \frac{3\pi}{2}, \xi < 0\right\}$$

and

$$P := P_0 \cup \left\{z : z \in \mathbb{C}, z = \xi + i\tau, -\frac{\pi}{2} \leq \tau \leq \frac{3\pi}{2}, \xi = 0\right\}.$$

We show the set of eigenvalues and spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. From the definition of eigenvalues and spectral singularities [7] , we get

$$\begin{aligned} \sigma_d(L) &= \{\lambda : \lambda = 2\cosh z, z \in P_0, f(z) = 0\}, \\ \sigma_{ss}(L) &= \left\{\lambda : \lambda = 2\cosh z, z = \xi + i\tau, \xi = 0, \tau \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], f(z) = 0\right\} \setminus \{0\}. \end{aligned} \tag{2.10}$$

From (2.6) and (2.9), we get

$$
\begin{aligned}
f(z) &= [\gamma_0 + \gamma_1 (e^z + e^{-z})] \left[\alpha_1 e^z \left(1 + \sum_{m=1}^{\infty} A_{1m} e^{mz}\right)\right] \\
&\quad + [\beta_0 + \beta_1 (e^z + e^{-z})] \left[\alpha_0 \left(1 + \sum_{m=1}^{\infty} A_{0m} e^{mz}\right)\right] \\
&= \alpha_0 \beta_1 e^{-z} + \alpha_1 \gamma_1 + \alpha_0 \beta_0 + (\alpha_1 \gamma_0 + \alpha_0 \beta_1) e^z + \alpha_1 \gamma_1 e^{2z} \\
&\quad + \sum_{m=1}^{\infty} \alpha_0 \beta_1 A_{0m} e^{(m-1)z} + \sum_{m=1}^{\infty} (\alpha_1 \gamma_1 A_{1m} + \alpha_0 \beta_0 A_{0m}) e^{mz} \\
&\quad + \sum_{m=1}^{\infty} [\alpha_1 \gamma_0 A_{1m} + \alpha_0 \beta_1 A_{0m}] e^{(m+1)z} + \sum_{m=1}^{\infty} [\alpha_1 \gamma_1 A_{1m}] e^{(m+2)z}.
\end{aligned} \tag{2.11}
$$

Let

$$F(z) := f(z) e^z. \tag{2.12}$$

Then, the function F is analytic in $\mathbb{C}_{left}$, continuous in $\overline{\mathbb{C}}_{left}$ and $F(z) = F(z + 2\pi i)$.

$$
\begin{aligned}
F(z) &= \alpha_0 \beta_1 + (\alpha_1 \gamma_1 + \alpha_0 \beta_0) e^z + [\alpha_1 \gamma_0 + \alpha_0 \beta_1] e^{2z} \\
&\quad + (\alpha_1 \gamma_1) e^{3z} + \sum_{m=1}^{\infty} \alpha_0 \beta_1 A_{0m} e^{mz} + \sum_{m=1}^{\infty} (\alpha_1 \gamma_1 A_{1m} + \alpha_0 \beta_0 A_{0m}) e^{(m+1)z} \\
&\quad + \sum_{m=1}^{\infty} [\alpha_1 \gamma_0 A_{1m} + \alpha_0 \beta_1 A_{0m}] e^{(m+2)z} + \sum_{m=1}^{\infty} [\alpha_1 \gamma_1 A_{1m}] e^{(m+3)z}.
\end{aligned} \tag{2.13}
$$

It follows from (2.10)-(2.13) that

$$
\begin{aligned}
\sigma_d(L) &= \{\lambda : \lambda = 2\cosh z,\ z \in P_0,\ F(z) = 0\}, \quad (2.14) \\
\sigma_{ss}(L) &= \left\{\lambda : \lambda = 2\cosh z,\ z = \xi + i\tau, \xi = 0,\ \tau \in \left[-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right]\ ,\ F(z) = 0\right\} \diagdown \{0\}
\end{aligned}
$$

Definition 2.1. The multiplicity of a zero of F in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (1.3), (1.4).

3. Principal functions

Let $\lambda_1, \lambda_2, ..., \lambda_p$ and $\lambda_{p+1}, \lambda_{p+2}, ..., \lambda_q$ denote the zeros of F in P_0 with multiplicities $m_1, m_2, ..., m_p$ and $m_{p+1}, m_{p+2}, ..., m_q$, respectively.

Definition 3.2. Let $\lambda = \lambda_0$ be an eigenvalue of L. If the vectors $y^{(0)}, y^{(1)}, ..., y^{(s)}$; $y^{(k)} = \left\{y_n^{(k)}\right\}_{n \in \mathbb{N}}$ $k = 0, 1, ..., s$ satisfy the equations

$$
\begin{aligned}
\left(ly^{(0)}\right)_n - \lambda_0 y_n^{(0)} &= 0, \quad (3.15) \\
\left(ly^{(k)}\right)_n - \lambda_0 y_n^{(k)} - y_n^{(k-1)} &= 0,\ k = 1, 2, ..., s;\ n \in \mathbb{N}
\end{aligned}
$$

then the vector $y^{(0)}$ is called the eigenvector corresponding to the eigenvalue $\lambda = \lambda_0$ of L. The vectors $y^{(1)}, ..., y^{(s)}$ are called the associated vectors corresponding to $\lambda = \lambda_0$. The eigenvector and the associated vectors corresponding to $\lambda = \lambda_0$ are called the principal vectors of the eigenvalue $\lambda = \lambda_0$.

The principal vectors of the spectral singularities of L are defined similarly.

Let us introduce the vectors

$$
\begin{aligned}
U_n^{(k)}(\lambda_j) &= \frac{1}{k!} \left\{\frac{d^k}{d\lambda^k} E_n(\lambda)\right\}_{\lambda = \lambda_j},\ k = 0, 1, ..., m_j - 1;\ j = 1, 2, ..., p, \quad (3.16) \\
U_n^{(k)}(\lambda_j) &= \frac{1}{k!} \left\{\frac{d^k}{d\lambda^k} E_n(\lambda)\right\}_{\lambda = \lambda_j},\ k = 0, 1, ..., m_j - 1;\ j = p+1, p+2, ..., q,
\end{aligned}
$$

where $\lambda = 2\cosh z$, $z \in P_0$ and

$$\{E_n(\lambda)\} := \left\{ e_n \left(\arccos h \frac{\lambda}{2} \right) \right\},\ n \in \mathbb{N}. \tag{3.17}$$

Also, note that if $y(\lambda) = \{y_n(\lambda)\}_{n\in\mathbb{N}}$ is a solution of (1.2), then $\left(d^k/d\lambda^k\right) y(\lambda) = \left\{ \left(d^k/d\lambda^k\right) y_n(\lambda) \right\}_{n\in\mathbb{N}}$ satisfies

$$a_{n-1}\frac{d^k}{d\lambda^k}y_{n-1}(\lambda) + b_n\frac{d^k}{d\lambda^k}y_n(\lambda) + a_n\frac{d^k}{d\lambda^k}y_{n+1}(\lambda) = \lambda\frac{d^k}{d\lambda^k}y_n(\lambda) + k\frac{d^{k-1}}{d\lambda^{k-1}}y_n(\lambda). \tag{3.18}$$

From (3.16) and (3.18), it is seen that

$$\begin{aligned} \left(lU^{(0)}(\lambda_j)\right)_n - \lambda_j U_n^{(0)}(\lambda_j) &= 0, \qquad (3.19)\\ \left(lU^{(k)}(\lambda_j)\right)_n - \lambda_j U_n^{(k)}(\lambda_j) - U_n^{(k-1)}(\lambda_j) &= 0,\ k = 0,1,...,m_j-1;\ j = 1,2,...,q. \end{aligned}$$

Hence, the vectors $U_n^{(k)}(\lambda_j)$; $k = 0,1,...,m_j-1$; $j = 1,2,...,p$ and $U_n^{(k)}(\lambda_j)$; $k = 0,1,...,m_j-1$; $j = p+1,p+2,...,q$ are the principal vectors of eigenvalues and spectral singularities of L, respectively.

Theorem 3.1. *Under the condition* (1.5),

$$\begin{aligned} U_n^{(k)}(\lambda_j) &\in l_2(\mathbb{N}),\ k = 0,1,...,m_j-1;\ j = 1,2,...,p,\\ U_n^{(k)}(\lambda_j) &\notin l_2(\mathbb{N}),\ k = 0,1,...,m_j-1;\ j = p+1,p+2,...,q. \end{aligned}$$

Proof. By using the function $E_n(\lambda) = e_n(\arccos h(\lambda/2))$, we get that

$$\left\{\frac{d^k}{d\lambda^k}E_n(\lambda)\right\}_{\lambda=\lambda_j} = \sum_{v=0}^{k} C_v \left\{\frac{d^v}{d\lambda^v}e_n(z)\right\}_{z=z_j},\ n \in \mathbb{N},$$

where $\lambda_j = 2\cosh z_j$, $z_j \in P$, $j = 1,2,...,q$; C_v is a constant depending on λ_j. From (2.6), we obtain that

$$\begin{aligned} \left\{\frac{d^v}{d\lambda^v}e_n(z)\right\}_{z=z_j} &= \alpha_n e^{nz_j}\left\{ n^v + \sum_{m=1}^{\infty}(n+m)^v A_{nm}e^{mz_j} \right\} \qquad (3.20)\\ &= \alpha_n e^{nz_j} n^v + \alpha_n e^{nz_j}\sum_{m=1}^{\infty}(n+m)^v A_{nm}e^{mz_j}. \end{aligned}$$

We see that the following equation is satisfied:

$$\left\{\frac{d^k}{d\lambda^k}E_n(\lambda)\right\}_{\lambda=\lambda_j} = \sum_{v=0}^{k} C_v \left\{ \alpha_n e^{nz_j} n^v + \alpha_n e^{nz_j}\sum_{m=1}^{\infty}(n+m)^v A_{nm}e^{mz_j} \right\}; \tag{3.21}$$

for the principal vectors $U_n^{(k)}(\lambda_j) = \left\{U_n^{(k)}(\lambda_j)\right\}_{n\in\mathbb{N}}$, $k = 0,1,...,m_j-1$; $j = 1,2,...,p$, corresponding to the eigenvalues $\lambda_j = 2\cosh z_j$, $j = 1,2,...,p$ of L.

So that

$$U_n^{(k)}(\lambda_j) = \frac{1}{k!}\left\{ \sum_{v=0}^{k} C_v \left[\alpha_n e^{nz_j} n^v + \alpha_n e^{nz_j}\sum_{m=1}^{\infty}(n+m)^v A_{nm}e^{mz_j} \right] \right\} \tag{3.22}$$

for $k = 0,1,...,m_j-1$; $j = 1,2,...,p$.

By using (3.22) and $Rez_j < 0$, $j = 1, 2, ..., p$, we get that

$$\sum_{n=1}^{\infty}\left|\frac{1}{k!}\sum_{v=0}^{k}C_v\alpha_n e^{nz_j}n^v\right|^2 \leq \frac{1}{(k!)^2}\left[\sum_{n=1}^{\infty}\sum_{v=0}^{k}|C_v|\,|\alpha_n|\,e^{n\,Rez_j}n^v\right]^2 \leq \frac{A}{(k!)^2}\left[\sum_{n=1}^{\infty}e^{nRez_j}\left(1+n+n^2+...+n^k\right)\right]^2 \leq \frac{A}{(k!)^2}(k+1)^2\left(\sum_{n=1}^{\infty}e^{nRez_j}n^k\right)^2 < \infty, \tag{3.23}$$

where A is constant.
Let us define the function

$$g_n(z) = \frac{1}{k!}\sum_{v=0}^{k}\alpha_n e^{nz_j}\sum_{m=1}^{\infty}(n+m)^v A_{nm}e^{mz_j},\ j = 1, 2, ..., p. \tag{3.24}$$

From the inequality (2.8), we get that

$$|g_n(z)| \leq \sum_{v=0}^{k}|\alpha_n|\,e^{nRez_j}\sum_{m=1}^{\infty}|n+m|^v\,|A_{nm}|\,e^{m\,Rez_j} \leq |\alpha_n|\,e_j^{Rez}\left[\sum_{m=1}^{\infty}|A_{nm}|\,e^{m\,Rez_j}+\sum_{m=1}^{\infty}(n+m)\,|A_{nm}|\,e^{mRez_j} +...+\sum_{m=1}^{\infty}(n+m)^k\,|A_{nm}|\,e^{mRez_j}\right] < Be^{n\,Rez_j}, \tag{3.25}$$

where $B = |\alpha_n|\sum\limits_{m=1}^{\infty}\sum\limits_{v=0}^{k}|A_{nm}|\,e^{mRez_j}(n+m)^v$. Hence, we obtain

$$\sum_{n=1}^{\infty}|g_n(z)|^2 \leq B^2e^{2nRe\,z_j} < \infty,\ j = 1, 2, ..., p. \tag{3.26}$$

From (3.23) and (3.26), we get $U_n^{(k)}(\lambda_j) \in l_2(\mathbb{N})$, $k = 0, 1, ..., m_j - 1$; $j = 1, 2, ..., p$.

Now, we will use (3.22) for the principal vectors corresponding to the spectral singularities $\lambda_j = 2\cosh z_j$, $j = p+1, p+2, ..., q$ of L and consider that $Rez_j = 0$ for the spectral singularities, then we obtain

$$U_n^{(k)}(\lambda_j) = \frac{1}{k!}\left\{\sum_{v=0}^{k}C_v\alpha_n e^{nz_j}n^v + \alpha_n e^{nz_j}\sum_{v=0}^{k}\sum_{m=1}^{\infty}(n+m)^v A_{nm}e^{mz_j}\right\} \tag{3.27}$$

for $k = 0, 1, ..., m_j - 1$; $j = p + 1, p + 2, ..., q$.
It follows from $\{Rez_j = 0,\ j = p + 1, p + 2, ..., q\}$ and (3.27) that

$$\frac{1}{k!}\sum_{n=1}^{\infty}\left|\sum_{v=0}^{k}C_v\alpha_n e^{nz_j}n^v\right|^2 = \infty. \tag{3.28}$$

Now, let us introduce the function $t_n(z) = \sum\limits_{v=0}^{k}\sum\limits_{m=1}^{\infty}(n+m)^v A_{nm}e^{mz_j}$ and use (2.8). Then, we get

$$\begin{aligned}|t_n(z)| &\le \sum_{v=0}^{k}\sum_{m=1}^{\infty}\left|(n+m)^v A_{nm}\right| \\ &\le \sum_{v=0}^{k}\sum_{m=1}^{\infty}(n+m)^v C \sum_{k=n+[m/2]}^{\infty}(|1-a_k|+|b_k|) \\ &\le C\sum_{v=0}^{k}\sum_{m=1}^{\infty}(n+m)^v \sum_{k=n+[m/2]}^{\infty}\exp(-\varepsilon k)\exp(\varepsilon k)(|1-a_k|+|b_k|) \\ &\le C\sum_{v=0}^{k}\sum_{m=1}^{\infty}(n+m)^v \exp\left[\frac{-\varepsilon}{4}(n+m)\right]\sum_{k=n+[m/2]}^{\infty}\exp(\varepsilon k)(|1-a_k|+|b_k|) \\ &\le C_1\sum_{v=0}^{k}\sum_{m=1}^{\infty}(n+m)^v \exp\left[\frac{-\varepsilon}{4}(n+m)\right] \\ &= C_1\exp\left[\frac{-\varepsilon}{4}n\right]\sum_{m=1}^{\infty}\sum_{v=0}^{k}(n+m)^v\exp\left[\frac{-\varepsilon}{4}m\right] \\ &= A\exp\left[\frac{-\varepsilon}{4}n\right]\end{aligned} \tag{3.29}$$

where

$$A = C_1\sum_{m=1}^{\infty}\sum_{v=0}^{k}(n+m)^v\exp\left[\frac{-\varepsilon}{4}m\right]. \tag{3.30}$$

Using (3.29), we find that

$$\begin{aligned}\frac{1}{k!}\sum_{n=1}^{\infty}\left|\alpha_n e^{nz_j}\sum_{v=0}^{k}\sum_{m=1}^{\infty}(n+m)^v A_{nm}e^{mz_j}\right|^2 &\le \frac{1}{k!}\sum_{n=1}^{\infty}\alpha_n^2 A^2 e^{-\varepsilon n/2} \\ &< \infty.\end{aligned} \tag{3.31}$$

So, $U_n^{(k)}(\lambda_j) \notin l_2(\mathbb{N})$, $k=0,1,...,m_j-1$; $j=p+1,p+2,...,q$. □

Let us introduce Hilbert spaces

$$\begin{aligned}H_k(\mathbb{N}) &= \left\{y=\{y_n\}_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}}(1+|n|)^{2k}|y_n|^2<\infty\right\}, \\ H_{-k}(\mathbb{N}) &= \left\{u=\{u_n\}_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}}(1+|n|)^{-2k}|u_n|^2<\infty\right\},\ k=0,1,2,...,\end{aligned} \tag{3.32}$$

with $\|y\|_k^2 = \sum_{n\in\mathbb{N}}(1+|n|)^{2k}|y_n|^2$, $\|u\|_{-k}^2 = \sum_{n\in\mathbb{N}}(1+|n|)^{-2k}|u_n|^2$, respectively. It is obvious that $H_0(\mathbb{N}) = l_2(\mathbb{N})$ and

$$H_{k+1}(\mathbb{N}) \subsetneqq H_k(\mathbb{N}) \subsetneqq l_2(\mathbb{N}) \subsetneqq H_{-k}(\mathbb{N}) \subsetneqq H_{-(k+1)}(\mathbb{N}),\ k=1,2,.... \tag{3.33}$$

Theorem 3.2. $U_n^{(k)}(\lambda_j) \in H_{-(k+1)}(\mathbb{N})$, $k=0,1,...,m_j-1$, $j=p+1,...,q$.

Proof. Using the expression of $U_n^{(k)}(\lambda_j)$ in (3.27), we obtain

$$\sum_{n=1}^{\infty}(1+|n|)^{-2(k+1)}\left|\frac{1}{k!}\sum_{v=0}^{k}C_v\alpha_n e^{nz_j}n^v\right|^2 < \infty,$$

$$\sum_{n=1}^{\infty}(1+|n|)^{-2(k+1)}\left|\frac{1}{k!}\sum_{v=0}^{k}\alpha_n e^{nz_j}\sum_{m=1}^{\infty}(n+m)^v A_{nm}e^{mz_j}\right|^2 < \infty,$$

for $k=0,1,...,m_j-1$, $j=p+1,...,q$. From the above inequalities, it is clear to see that $U_n^{(k)}(\lambda_j)\in H_{-(k+1)}(\mathbb{N})$, $k=0,1,...,m_j-1$, $j=p+1,...,q$. □

Let us choose $m_0=\max\{m_{p+1},m_{p+2},...,m_q\}$. Now, we can present the following theorem as a consequence of previous theorems.

Theorem 3.3. $U_n^{(k)}(\lambda_j)\in H_{-m_0}(\mathbb{N})$, $k=0,1,...,m_j-1$, $j=p+1,...,q$.

Proof. Proof of theorem is obvious. □

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