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Hereditary Perfect Order Subset Groups

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ABSTRACT. A finite group G is said to be a POS-group if, for each $x \in G$, the cardinality of the set $\{y \in G : o(y) = o(x)\}$ is a divisor of the order of G. A POS-group G is said to be a Hereditary Perfect Order Subset group if all even order subgroups of G are POS-group. In this paper we study the structure of Hereditary perfect order subset groups.

1. INTRODUCTION

Let G be a finite group and $x \in G$. We denote the order of x, i.e. the smallest positive integer k such that $x^k = e$, as $o(x)$. Then the set of all elements in G having same order as x is called the order subset of G determined by x (see[4]). We say that G is a group with perfect order subsets or briefly a POS-group if the number of elements of each order subset of G is a divisor of $|G|$. Some of the recent works [1, 2, 5, 6, 8, 9, 10, 11] deal with the structure of POS-groups.

Let G be a finite POS-group. Motivated by the above notions, we say that G is Hereditary perfect order subset group, or briefly HPOS-group, if all even order subgroups of G are POS-group. In this paper we study the structure of HPOS-groups.

2. NOTATIONS AND BASIC RESULTS

Most of the notations, definitions and results mentioned here are standard and are as in [4, 3, 7]. Throughout the paper G denotes a finite group and $S(G) = \{o(x) : x \in G\}$. For each $k \in S(G)$, denote $S_k = \{x \in G : o(x) = k\}$. For a positive integer n, D_{2n} denotes the $n-$th dihedral group with $2n$ elements and $\varphi(n)$ denotes the number of non-negative integers less than n and relatively prime to n. Also, $\mathbb{Z}_n$ denotes the group of integers modulo n and $\mathbb{Z}_n^*$ denotes the group of relatively prime integers modulo n.

Definition 2.1. Let G be a finite group. Then the order class of G is defined as the set

$$\{(k, |S_k|) : k \in S(G)\}$$

and G is POS if $|S_k|$ divides G.

Example 2.1. Let $G = S_3$. Then $S(G) = \{1, 2, 3\}$. $S_1 = \{e\}$, $S_2 = \{(12), (23), (13)\}$ and $S_3 = \{(123), (132)\}$. Hence the order class of S_3 is $\{(k, |S_k|) : k \in S(G)\} = \{(1, 1), (2, 3), (3, 2)\}$. Since $|S_k|$ divide $|S_3|$ for all k, G is a POS-group.

Theorem 2.1. [2] *$\mathbb{Z}_n$ is a POS-group if and only if $n = 1$ or $n = 2^\alpha 3^\beta$ where $\alpha \geq 1$ and $\beta \geq 0$.*

Theorem 2.2. [2] *For each $k \in S(G)$, $|S_k|$ is a multiple of $\varphi(k)$.*

Theorem 2.3. [4] *Let G be a non trivial POS-group. Then $|G|$ is even.*

Theorem 2.4. [7] *If n is a positive divisor of $|G|$ and $X = \{g \in G : g^n = e\}$, then n divides $|X|$.*

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Theorem 2.5. [11] *D_{2n} is a POS-group if and only if $n = 3^{\alpha}$ where $\alpha \geq 0$*

Theorem 2.6. *For $n \geq 1$, $\varphi(n)$ divide n if and only if $n = 2^k 3^l$ where $k \geq 1$ and $l \geq 0$.*

Theorem 2.7. *A subgroup of D_{2n} is either isomorphic to $\mathbb{Z}_m$ or D_{2m} where m is a divisor of n.*

Theorem 2.8. *$Aut(D_{2n})$ is isomorphic to G_n for all n where $Aut(D_{2n})$ is the automorphism group of the Dihedral group D_{2n} and $G_n = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a \in \mathbb{Z}_n^*, b \in \mathbb{Z}_n \right\}$, a group of order $n\varphi(n)$ with respect to matrix multiplication.*

3. Main results

It is known, by Theorem 2.1, that the cyclic group $\mathbb{Z}_{2^n} (n \geq 1)$ is a POS-group in which all subgroups are also POS. However there exist groups in which all subgroups are not POS. For example, the cyclic group $\mathbb{Z}_6$ is a POS-group but its subgroup $\mathbb{Z}_3$ is not POS by Theorem 2.3. Hence in this work we characterize the groups where subgroups are also POS. This naturally raises the question as to which groups have the property that all of its subgroups are POS.

Theorem 3.9. *Let G be a finite POS-group. Then all subgroups of G are POS if and only if G isomorphic to $\mathbb{Z}_{2^n}$ for some $n \geq 0$.*

Proof. Suppose all subgroups of G are POS. Then by Theorem 2.3 either G is trivial or 2 is the only prime divisor of $|G|$ and hence $|G| = 2^n$ for some $n \geq 0$. Define $A_m = \{g \in G : g^{2^m} = e\}$ for all $0 \leq m \leq n$. Then clearly $A_{m-1} \subseteq A_m$ for all $1 \leq m \leq n$. Now we use mathematical induction to prove $|A_m| = 2^m$ for all $0 \leq m \leq n$.

Now, $|A_0| = 1 = 2^0$. Assume by induction,

$$|A_{m-1}| = 2^{m-1}, \ (1 \leq m < n) \tag{3.1}$$

Since $A_m - A_{m-1} = \{g \in G : o(g) = 2^m\}$ and G is a POS-group and by Theorem 2.2, we have

$$|A_m| - |A_{m-1}| = |A_m - A_{m-1}| = 0 \text{ or } 2^t \text{ for some } m-1 \leq t \leq n \tag{3.2}$$

By Theorem 2.4,

$$|A_m| = 2^m k \text{ for some } k \geq 1 \tag{3.3}$$

Hence by (3.1), (3.2) and (3.3), we have

$$\begin{aligned} k\,2^m &= 2^{m-1} + 2^t \text{ for some } m-1 \leq t \leq n \\ \implies k\,2^m &= 2^{m-1}(1 + 2^r) \text{ for some } 0 \leq r \leq n \\ \implies r &= 0 \text{ and } k = 1 \end{aligned}$$

Hence from (3.3), we have

$$|A_m| = 2^m \text{ for all } 0 \leq m \leq n$$

Therefore number of elements of order 2^n in G is $|A_n - A_{n-1}| = |A_n| - |A_{n-1}| = 2^n - 2^{n-1} = 2^{n-1} \geq 1$. Since G is a group of order 2^n and has atleast one element of order 2^n, we have

$$G \simeq \mathbb{Z}_{2^n} \text{ for some } n \geq 0$$

Conversely suppose $G \simeq \mathbb{Z}_{2^n}$ for some $n \geq 0$. Then every subgroup of G is isomorphic to $\mathbb{Z}_{2^k}$ for some $k \geq 0$ and which is POS-group by Theorem 2.1. □

The above theorem shows that $\mathbb{Z}_{2^n} (n \geq 1)$ is the only POS-group in which all subgroups are also POS. Many POS-groups are known to have the property that all of its even order subgroups are also POS. For example, consider the group D_{18}, Dihedral group of order 18. By Theorem 2.5, D_{18} is a POS group. Let X denote the collection of all subgroups of D_{18}. Then by Theorem 2.7, $X = \{Z_1, \mathbb{Z}_3, \mathbb{Z}_9, D_2 \cong \mathbb{Z}_2, D_6, D_{18}\}$. By Theorem 2.5 all even order subgroups of D_{18} are POS, but $\mathbb{Z}_3$ and $\mathbb{Z}_9$ are not.

However there also exist POS-groups whose even order subgroups are not POS. For example, consider the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. Then $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ is an even order subgroup of G, which is not POS, since the number of 2 order elements in H is 3 and 3 does not divide order of H. This leads to the question whether the POS-groups having the property that all even order subgroups are POS, can be characterized. In this connection we have the following.

Definition 3.2. Let G be a finite POS-group. G is said to be Hereditary perfect order subset group, or briefly HPOS-group if all even order subgroups of G are POS-group.

Firstly, we characterize all finite abelian Hereditary perfect order subset groups.

Theorem 3.10. *Let G be a finite abelian POS-group. Then G is HPOS if and only if G isomorphic to $\mathbb{Z}_{2^\alpha 3^\beta}$ for some $\alpha \geq 1$ and $\beta \geq 0$.*

Proof. Assume $G \simeq \mathbb{Z}_{2^\alpha 3^\beta}$ for some $\alpha \geq 1$ and $\beta \geq 0$. Then all even order subgroups of G are isomorphic to $\mathbb{Z}_{2^\gamma 3^\delta}$ for some $\gamma \geq 1$ and $\delta \geq 0$ and which is a POS-group by Theorem 2.1. Hence G is a HPOS group.

Conversely suppose G is HPOS-group. Since G is a finite abelian POS-group, we have

$$G \simeq \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_r^{n_r}} \tag{3.4}$$

where $\alpha \geq 1$ and $n_i \geq 0$ for $1 \leq i \leq r$ and p_i's are prime.

Suppose $n_i \geq 1$ for some i and $p_i \neq 3$. If $p_i = 2$ then $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ is an even order group which is isomorphic to a subgroup of G containing 3 elements of order 2 and 3 does not divides $|H|$. Therefore H is not a POS-group and hence G is not HPOS. If $p_i > 3$, then $H = \mathbb{Z}_2 \times \mathbb{Z}_{p_i} \simeq \mathbb{Z}_{2\times p_i}$ is an even order group which is isomorphic to a subgroup of G. By Theorem 2.1 H is not a POS-group and hence G is not HPOS. Therefore if $n_i \neq 0$ then $p_i = 3$.

Suppose $n_i \neq 0$ and $n_j \neq 0$ for some $i \neq j$. Then $H = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ is an even order group which is isomorphic to a subgroup of G. The number of 3 order elements in H is 8 and 8 does not divide $|H|$. Hence H is not a POS-group and so G is not HPOS. Thus from (3.4) we have

$$\begin{aligned} G &\simeq \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{3^\beta} \quad \text{whre } \alpha \geq 1 \text{ and } \beta \geq 0 \\ &\simeq \mathbb{Z}_{2^\alpha 3^\beta} \quad \text{where } \alpha \geq 1 \text{ and } \beta \geq 0 \end{aligned}$$

□

Now we characterize those Dihedral groups which are HPOS.

Theorem 3.11. *Let n be a positive integer. Then D_{2n} is HPOS if and only if $n = 3^\alpha$ where $\alpha \geq 0$.*

Proof. Assume D_{2n} is HPOS. Since D_{2n} is even order POS-group, by Theorem 2.5, we have $n = 3^\alpha$ for some $\alpha \geq 0$.

Conversly, suppose $n = 3^\alpha$ where $\alpha \geq 0$. Let H be an even order subgroup of D_{2n}. Then $|H| = 2 \times 3^\beta$ where $\beta \geq 0$. Then by theorem 2.7, either $H \simeq \mathbb{Z}_{2\times 3^\beta}$ or $H \simeq D_{2\times 3^\beta}$ where $\beta \geq 0$. In both cases, H is a POS-groups. Hence D_{2n} is a HPOS-group if $n = 3^\alpha$ when $\alpha \geq 0$. □

Next we discuss the automorphism groups of Dihedral groups which are POS but not HPOS.

Lemma 3.1. *Let p be a prime number. Then*

$$1 + z + z^2 + \ldots + z^{k-1} \equiv 0 \, mod(p)$$

for all $z \in \mathbb{Z}_p^*, z \neq 1$ and $o(z) = k$ in $\mathbb{Z}_p^*$.

Proof. Since $o(z) = k$ in $\mathbb{Z}_p^*$, $z^k \equiv 1 \, mod(p)$.

Now,

$$\begin{aligned}(1 + z + z^2 + \ldots + z^{k-1})(z-1) &= z^k - 1 \\ &\equiv 0 \, mod(p) \qquad (3.5)\end{aligned}$$

Since $1 < z < p$, we have $z - 1$ is not congruent to $0 \, mod(p)$. Hence by (3.5),

$$1 + z + z^2 + \ldots + z^{k-1} \equiv 0 \, mod(p)$$

for all $z \in \mathbb{Z}_p^*, z \neq 1$ and $o(z) = k$ in $\mathbb{Z}_p^*$. □

Lemma 3.2. *Let p be a prime number. Then the order class of $Aut(D_{2p})$ is*

$$\{(1,1), (p, p-1), (k, p\varphi(k)) \, : k \in D(p-1), k \neq 1\}$$

Proof. By Theorem 2.8, $Aut(D_{2p})$ is isomorphic to G_p.
Let $y \in \mathbb{Z}_p$. Then for any $m \in \mathbb{N}$,

$$\begin{aligned}\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}^m &= \begin{bmatrix} 1 & my \\ 0 & 1 \end{bmatrix} \\ \implies o\left(\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\right) \text{ in } G_p &= o(y) \text{ in } \mathbb{Z}_p \qquad (3.6)\end{aligned}$$

Let $x \in \mathbb{Z}_p^*, x \neq 1$ and $y \in \mathbb{Z}_p$. Then for any $m \in \mathbb{N}$,

$$\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}^m = \begin{bmatrix} x^m & (1 + x + \ldots + x^{m-1})y \\ 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{aligned}\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}^m &= I \\ \implies x^m &\equiv 1 \, mod(p) \\ \implies m &\geq o(x) \text{ in } \mathbb{Z}_p^* \qquad (3.7)\end{aligned}$$

Let $o(x) = k$ in $\mathbb{Z}_p^*$. Then

$$\begin{aligned}\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}^k &= \begin{bmatrix} x^k & (1 + x + \ldots + x^{k-1})y \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad ; \text{ by theorem 3.1} \\ \implies o\left(\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\right) &\leq k = o(x) \text{ in } \mathbb{Z}_p^* \qquad (3.8)\end{aligned}$$

From (3.6) and (3.7), we get

$$o\left(\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\right) = o(x) \text{ in } \mathbb{Z}_p^* \qquad (3.9)$$

for all $x(\neq 1) \in \mathbb{Z}_p^*$ and $y \in \mathbb{Z}_p$. From (3.5) and (3.8)

$$S(G_p) = S(\mathbb{Z}_p^*) \cup S(\mathbb{Z}_p) = \{p, k \,:\, k \in D(p-1)\}$$

Also, $|S_1| = 1, |S_p| = p-1$ and $S_k = p\varphi(k)$ for all $k \in D(p-1)$ and $k \neq 1$. Hence the order class of $Aut(D_{2p})$ is

$$\{(1,1), (p, p-1), (k, p\varphi(k)) \,:\, k \in D(p-1), k \neq 1\}$$

□

Lemma 3.3. *$Aut(D_{2p})$ is a POS-group if and only if $p = 1 + 2^k 3^l$ for some $k \geq 1$ and $l \geq 0$.*

Proof. We have $|Aut(_{2p})| = p(p-1)$. Hence by the above theorem, $Aut(D_{2p})$ is a POS-group if and only if $\varphi(k)$ divide $p-1$ for all $k \in D(p-1)$. Hence by Theorem 2.6 $Aut(D_{2p})$ is a POS-group if and only if $p = 1 + 2^k 3^l$ for some $k \geq 1$ and $l \geq 0$. □

Theorem 3.12. *Let p be a prime number of the form $1 + 2^k$. Then $Aut(D_{2p})$ is a non-abelain POS-group but not HPOS.*

Proof. By Corollary 3.3, $Aut(D_{2p})$ is a non-abelian POS-group. Since $|Aut(D_{2p})| = p(p-1)$, by Theorem 3.2, the number of p-order elements in $Aut(D_{2p})$ is $p-1$. Therefore $Aut(D_{2p})$ has a unique Sylow p-subgroup, let denote it by H. Then H is normal subgroup of order p. Since $Aut(D_{2p})$ is of even order, it has a subgroup K of order 2. Then HK is a subgroup of $Aut(D_{2p})$ of order $2p$. Therefore HK is either isomorphic to $\mathbb{Z}_{2p}$ or D_{2p} whre p is prime ≥ 5 and none of them is POS. Hence $Aut(D_{2p})$ is not HPOS.

□

4. Conclusion

In this paper, firstly we have proved that all subgroups of a finite POS-group G are POS if and only if G isomorphic to $\mathbb{Z}_{2^n}$ for some $n \geq 0$. We introduced the concept of Hereditary perfect order subset group and characterized all finite abelian Hereditary perfect order subset groups. Also, we classified those Dihedral groups which are HPOS and the automorphism groups of Dihedral groups which are POS but not HPOS. We conclude by stating a conjecture:

Conjecture 4.1. *If G is HPOS and order of G is not a power of 2, then 3 divides the order of G.*

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