Generalized Functional Discriminating Measure For Finite Probability Distributions

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ABSTRACT. Analysts like Renyi (1961), Csiszar (1963, 1966), Bregman (1967), Burba-Rao (1982), Jain and Saraswat (2012), etc., have introduced and analyzed the different functional discriminating measures for comparing two discrete probability distributions. But in this article, a new functional discriminating measure has been proposed that will compare finite (more than two) discrete probability distributions simultaneously. Further, some intra-relations among the measures at different values of the parameters have been discussed. Also, an interesting connection with Csiszar's generalized functional measure has been created. Some new inequalities compared to variational discrimination and Chi-square discrimination have been discussed as well.

1. INTRODUCTION

Discriminating measures are fundamental measures of distance between two likelihoods of dissemination or comparing two likelihoods of dissemination or comparing two probability distributions. The concept of discriminating measure is working productively to resolve distinctive issues related to the likelihood hypothesis. The real information is approximated by the measures of likelihood and measurements. This data leads to data misfortune. The essential reason is to evaluate how much data is contained within the information.

Now a days, these measures are being connected in a few disciplines such as: color picture division [17], estimation of likelihood dispersions [4, 8], design acknowledgment [9, 23], 3D picture division and word arrangement [20], choice making [16, 22, 24, 25], attractive reverberation picture investigation [27], fetched- touchy classification for therapeutic conclusion [19], turbulence stream [5], fuzzy divergence and applications[3, 10, 15, 21, 26], etc.

Let $\Theta_l = \{U = (u_1, u_2, u_3, ..., u_l) : u_i > 0, \sum_{i=1}^l u_i = 1\}$, $l \geq 2$ be the set of all complete finite discrete probability distributions, where u_i is a probability mass function.

Definition 1.1. Convex function: A function $g(x)$ is said to be convex over an interval (a, b) if it has for every $x_1, x_2 \in (a, b)$ and $0 \leq \mu \leq 1$, we have

$$g[\mu x_1 + (1 - \mu) x_2] \leq \mu g(x_1) + (1 - \mu) g(x_2),$$

and if uniformity does not hold, i.e., $\mu \neq 0$ or $\mu \neq 1$, then it is said to be entirely convex. In a broader sense, we can write

$$g\left[\sum_{i=1}^m \mu_i x_i\right] \leq \sum_{i=1}^m \mu_i g(x_i), \tag{1.1}$$

for all $x_i \in (a, b)$ and $\mu_i \geq 0$ with $\sum_{i=1}^m \mu_i = 1$.

Convex functions have wide applications in immaculate and connected arithmetic. As of late numerous generalizations and expansions have been made for the convexity, like: strong convexity [28], s- convexity [2], GA- convexity [29], GG- convexity [12], preinvexity [11] and others.

2. NEW GENERALIZED FUNCTIONAL DISCRIMINATING MEASURE

Let $U_1 = (u_{11},..., u_{l1}),...,U_m = (u_{1m},..., u_{lm})$ and $W_1 = (w_{11},..., w_{l1}),...,W_m = (w_{1m}, ..., w_{lm})$ be discrete probability distributions such that $U_j, W_j \in \Theta_l \ \forall \ j = 1, 2, ..., m$. Then, the following new functional discriminating measure is being introduced.

$$S_g^m (U_1, U_2, ..., U_m, W_1, W_2, ..., W_m)$$

$$= \sum_{i=1}^{l} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1} w_{i2}...w_{im} g \left(\frac{\frac{u_{i1}+w_{i1}}{2w_{i1}} + \frac{u_{i2}+w_{i2}}{2w_{i2}} + ... + \frac{u_{im}+w_{im}}{2w_{im}}}{m} \right), \qquad (2.2)$$

where $g : (0, \infty) \to R$ (set of real no.) is real, continuous, and convex function. For $m = 1$, it will compare two probability distributions U_1 and W_1 at a time, for $m = 2$, it compares four probability distributions U_1, W_1, U_2, and W_2 at a time, and so on. This allows you to compare $2j$ probability distributions at the same time, where $j = 1, 2, ..., 2m$.
Jain Saraswat's discriminating measure [14] is a special case of (2.2) at $m = 1$, which is

$$S_g^1 (U_1, W_1) = \sum_{i=1}^{l} w_{i1} g \left(\frac{u_{i1} + w_{i1}}{2w_{i1}} \right) = \sum_{i=1}^{l} w_i g \left(\frac{u_i + w_i}{2w_i} \right). \qquad (2.3)$$

3. INTRA RELATION AMONG FUNCTIONAL DISCRIMINATING MEASURES

In this segment, an imperative and productive connection among new useful discriminations has been discussed. These discriminations are essentially uncommon cases of (2.2) agreeing to the number of likelihood dispersions or the number of discrete probability distributions.

Theorem 3.1. *Let $g : (0, \infty) \to R$ be a differentiable, convex function, i.e., $g''(x) \geq 0 \ \forall x > 0$. For $U_j, W_j \in \Theta_l \ \forall \ j = 1, 2..., m$, we have*

$$S_g^1 (U_1, W_1) \geq S_g^2 (U_1, U_2, W_1, W_2) \geq ... \geq S_g^m (U_1, U_2, ..., U_m, W_1, W_2, ..., W_m)$$

$$\geq S_g^{m+1} (U_1, U_2, ..., U_m, U_{m+1}, W_1, W_2, ..., W_m, W_{m+1}) \geq g (1), \qquad (3.4)$$

where $S_g^m (U_1, U_2, ..., U_m, W_1, W_2, ..., W_m)$ is given by (2.2).

Proof. By utilizing the disparity (1.1) for different summations, we get

$$\sum_{i=1}^{l} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1} w_{i2}...w_{im} w_{i(m+1)} g \left(\frac{\frac{u_{i1}+w_{i1}}{2w_{i1}} + \frac{u_{i2}+w_{i2}}{2w_{i2}} + ... + \frac{u_{im}+w_{im}}{2w_{im}} + \frac{u_{i(m+1)}+w_{i(m+1)}}{2w_{i(m+1)}}}{m+1} \right)$$

$$\geq g \left[\sum_{i=1}^{l} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1} w_{i2}...w_{im} w_{i(m+1)} \left(\frac{\frac{u_{i1}+w_{i1}}{2w_{i1}} + \frac{u_{i2}+w_{i2}}{2w_{i2}} + ... + \frac{u_{im}+w_{im}}{2w_{im}} + \frac{u_{i(m+1)}+w_{i(m+1)}}{2w_{i(m+1)}}}{m+1} \right) \right]$$

$$= g \left[\frac{1}{m+1} \left(\sum_{i=1}^{l} \frac{u_{i1} + w_{i1}}{2} \sum_{i=1}^{l} w_{i2} .. \sum_{i=1}^{l} w_{im} \sum_{i=1}^{l} w_{i(m+1)} \right) \right]$$

$$+ .. + g \left[\frac{1}{m+1} \left(\sum_{i=1}^{l} \frac{u_{i(m+1)} + w_{i(m+1)}}{2} \sum_{i=1}^{l} w_{i1} .. \sum_{i=1}^{l} w_{im} \right) \right]$$

$$= g\left[\frac{1}{m+1}\left(1+1+...+1\right)\right] = g\left(\frac{m+1}{m+1}\right) = g\left(1\right), \text{ i.e.,}$$

$$S_g^{m+1}\left(U_1, U_2, ..., U_m, U_{m+1}, W_1, W_2, ..., W_m, W_{m+1}\right) \geq g\left(1\right). \tag{3.5}$$

Consequently, it demonstrated the last inequality of the relation (3.4). $\qquad\square$

Now again apply the inequality (1.1) for $h_1, h_2, ..., h_{m+1}$, where $h_i \in (0, \infty) \ \forall \ i = 1, 2, ..., m+1$, we obtain

$$\frac{1}{m+1}\left[g\left(h_1\right) + g\left(h_2\right) + ... + g\left(h_m\right) + g\left(h_{m+1}\right)\right] \geq g\left[\frac{h_1 + h_2 + ... + h_m + h_{m+1}}{m+1}\right]. \tag{3.6}$$

Let

$$h_1 = \frac{z_1 + z_2 + ... + z_m}{m}, h_2 = \frac{z_2 + z_3 + ... + z_m + z_{m+1}}{m}, ..., h_{m+1} = \frac{z_{m+1} + z_1 + ... + z_{m-1}}{m},$$

where $z_i \in (0, \infty) \ \forall \ i = 1, 2, ..., m+1$.
Then by using the inequality (3.6), we get

$$\frac{1}{m+1}\left[g\left(\frac{z_1 + z_2 + ... + z_m}{m}\right) + ... + g\left(\frac{z_{m+1} + z_1 + ... + z_{m-1}}{m}\right)\right]$$

$$\geq g\left[\frac{1}{m+1}\left(\frac{z_1 + z_2 + ... + z_m}{m} + ... + \frac{z_{m+1} + z_1 + ... + z_{m-1}}{m}\right)\right] = g\left(\frac{m\left(z_1 + z_2 + ... + z_m + z_{m+1}\right)}{m\left(m+1\right)}\right)$$

$$= g\left(\frac{z_1 + z_2 + ... + z_m + z_{m+1}}{m+1}\right). \tag{3.7}$$

Now put $z_j = \frac{u_{ij} + w_{ij}}{2w_{ij}}$ in (3.7), multiply with $w_{ij} \ \forall \ j = 1, ..., m+1$ and for each $i = 1, ..., l$ and then summation $m+1$ times from $i=1$ to $i=l$, we get

$$\frac{1}{m+1}\left[\sum_{i=1}^{l}\cdots\sum_{i=1}^{l}w_{i1}...w_{i(m+1)}g\left(\frac{\frac{u_{i1}+w_{i1}}{2w_{i1}} + ... + \frac{u_{im}+w_{im}}{2w_{im}}}{m}\right)\right]$$

$$+...+\frac{1}{m+1}\left[\sum_{i=1}^{l}\cdots\sum_{i=1}^{l}w_{i1}...w_{i(m+1)}g\left(\frac{\frac{u_{i(m+1)}+w_{i(m+1)}}{2w_{i(m+1)}} + \frac{u_{i1}+w_{i1}}{2w_{i1}} + ... + \frac{u_{i(m-1)}+w_{i(m-1)}}{2w_{i(m-1)}}}{m}\right)\right]$$

$$\geq \sum_{i=1}^{l}\cdots\sum_{i=1}^{l}w_{i1}...w_{i(m+1)}g\left(\frac{\frac{u_{i1}+w_{i1}}{2w_{i1}} + ... + \frac{u_{i(m+1)}+w_{i(m+1)}}{2w_{i(m+1)}}}{m+1}\right), \text{ i.e.,}$$

$$S_g^m\left(U_1, U_2, ..., U_m, W_1, W_2, ..., W_m\right) \geq S_g^{m+1}\left(U_1, U_2, ..., U_{m+1}, W_1, W_2, ..., W_{m+1}\right). \tag{3.8}$$

As a result, the second last inequality of the relation (3.4) is established for all m, and the theorem is established.

Remark 3.1. If g is normalized, i.e., $g\left(1\right) = 0$, then we get the following set of relations

$$S_g^1\left(U_1, W_1\right) \geq S_g^2\left(U_1, U_2, W_1, W_2\right) \geq ... \geq S_g^m\left(U_1, U_2, ..., U_m, W_1, W_2, ..., W_m\right)$$

$$\geq S_g^{m+1}\left(U_1, U_2, ..., U_m, U_{m+1}, W_1, W_2, ..., W_m, W_{m+1}\right) \geq 0. \tag{3.9}$$

4. Relation With Csiszar's Generalized Functional Discrimination

In 2000, S.S. Dragomir [7] presented the following important discrimination measure:

$$C_g^m (U_1, U_2, ..., U_m, W_1, W_2, ..., W_m) = \sum_{i=1}^{l} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1} w_{i2} ... w_{im} g \left(\frac{\frac{u_{i1}}{w_{i1}} + \frac{u_{i2}}{w_{i2}} + ... + \frac{u_{im}}{w_{im}}}{m} \right),$$
(4.10)

where $g : (0, \infty) \to R$ (set of real no.) is real, continuous, and convex function. Ciszar's discriminating measure [1, 6] is a special case of this measure, which is

$$C_g^1 (U_1, W_1) = \sum_{i=1}^{l} w_{i1} g \left(\frac{u_{i1}}{w_{i1}} \right) = \sum_{i=1}^{l} w_i g \left(\frac{u_i}{w_i} \right).$$
(4.11)

Now, a vital connection among measures (2.2) and (4.10) is being evaluated by the following theorem.

Theorem 4.2. *Let* $g : (0, \infty) \to R$ *be a differentiable, convex and normalized function, i.e.,* $g'' (x) \geq 0 \, \forall x > 0$ *and* $g (1) = 0$ *respectively. For* $U_j, W_j \in \Theta_l \, \forall \, j = 1, 2..., m$, *we have*

$$S_g^m (U_1, U_2, ..., U_m, W_1, W_2, ..., W_m) \leq \frac{1}{2} C_g^m (U_1, U_2, ..., U_m, W_1, W_2, ..., W_m).$$
(4.12)

Proof. Apply the inequality (1.1) for the domain $I \subset (0, \infty)$, by putting $\mu_1 = \mu_2 = \frac{1}{2}, \mu_3 = ... = \mu_m = 0$, we obtain

$$g \left(\frac{x_1 + x_2}{2} \right) \leq \frac{1}{2} [g (x_1) + g (x_2)].$$
(4.13)

Now put $x_1 = x$ and $x_2 = 1$ in above inequality, we obtain

$$g \left(\frac{x + 1}{2} \right) \leq \frac{1}{2} g (x).$$
(4.14)

Now in the inequality (4.14), take $x = \frac{\sum_{j=1}^{m} \frac{u_{ij}}{w_{ij}}}{m}$, multiply with $\prod_{j=1}^{m} w_{ij}$ for each i and then summation over m times from $i = 1$ to $i = l$, we obtain the required relation (4.12). $\square$

5. Important Deductions

Now, some curious results are going to be determined by using the defined discrimination (2.2) in the inequalities (3.9).

Proposition 5.1. *Let* $U_j, W_j \in \Theta_l \, \forall \, j = 1, 2, ..., m$, *then we have*

$$V_m (U_1, U_2, ..., U_m, W_1, W_2, ..., W_m) \leq \sum_{j=1}^{m} V (U_j, W_j)$$
(5.15)

and

$$V_1 (U_1, W_1) = V (U, W) \geq \frac{1}{2} V_2 (U_1, U_2, W_1, W_2) \geq ... \geq \frac{1}{m} V_m (U_1, ..., U_m, W_1, ..., W_m)$$

$$\geq \frac{1}{m + 1} V_{m+1} (U_1, ..., U_m, U_{m+1}, W_1, ..., W_m, W_{m+1}) \geq 0.$$
(5.16)

Proof. Let $g(x) = |x-1|, x > 0$. Here $g(x)$ is convex and normalized function because $g''(x) \geq 0 \ \forall \ x > 0$ and $g(1) = 0$ respectively, but $g(x)$ is not differentiable at $x = 1$.
Put $g(x)$ in (2.2), we have

$$S_g^m(U_1,...,U_m,W_1,...,W_m) = \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{im} \left| \frac{\frac{u_{i1}+w_{i1}}{2w_{i1}} + ... + \frac{u_{im}+w_{im}}{2w_{im}}}{m} - 1 \right|$$

$$= \frac{1}{m} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{im} \left| \left(\frac{u_{i1}+w_{i1}}{2w_{i1}} - 1 \right) + ... + \left(\frac{u_{im}+w_{im}}{2w_{im}} - 1 \right) \right|$$

$$= \frac{1}{2m} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{im} \left| \frac{u_{i1}-w_{i1}}{w_{i1}} + ... + \frac{u_{im}-w_{im}}{w_{im}} \right| = \frac{1}{2m} V_m(U_1,...,U_m,W_1,...,W_m),$$
(5.17)

and for $m = 1$

$$\frac{1}{2} \sum_{i=1}^{l} |u_i - w_i| = \frac{1}{2} V_1(U_1, W_1) = \frac{1}{2} V(U, W), \tag{5.18}$$

where $V(U,W)$ is the Variational distance [13], a special case of the $V_m(U_1,..,U_m,W_1,..,W_m)$ for comparing two probability distributions.
Now, equation (5.17) can be written as

$$\frac{1}{2m} V_m(U_1,...,U_m,W_1,...,W_m) \leq \frac{1}{2m} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{im} \left[\left| \frac{u_{i1}-w_{i1}}{w_{i1}} \right| + ... + \left| \frac{u_{im}-w_{im}}{w_{im}} \right| \right]$$

$$= \frac{1}{2m} \left[\sum_{i=1}^{l} |u_{i1}-w_{i1}| \sum_{i=1}^{l} w_{i2}... \sum_{i=1}^{l} w_{im} + ... + \sum_{i=1}^{l} |u_{im}-w_{im}| \sum_{i=1}^{l} w_{i1}... \sum_{i=1}^{l} w_{i(m-1)} \right]$$

$$= \frac{1}{2m} \left[\sum_{i=1}^{l} |u_{i1}-w_{i1}| + ... + \sum_{i=1}^{l} |u_{im}-w_{im}| \right] = \frac{1}{2m} \sum_{j=1}^{m} \sum_{i=1}^{l} |u_{ij}-w_{ij}| = \frac{1}{2m} \sum_{j=1}^{m} V(U_j, W_j).$$

$$\Rightarrow V_m(U_1,...,U_m,W_1,...,W_m) \leq \sum_{j=1}^{m} V(U_j, W_j).$$

Hence, the result (5.15) is obtained. Also, the sequence of inequalities (5.16) can be gotten by utilising (3.9). We are overlooking the points of interest. □

Proposition 5.2. *Let $U_j, W_j \in \Theta_l \ \forall \ j = 1, 2, ..., m$, then we have*

$$\chi_m^2(U_1, U_2, ..., U_m, W_1, W_2, ..., W_m) = \sum_{j=1}^{m} \chi^2(U_j, W_j) \tag{5.19}$$

and

$$\chi_1^2(U_1, W_1) = \chi^2(U, W) \geq \frac{1}{2^2} \chi_2^2(U_1, U_2, W_1, W_2) \geq ... \geq \frac{1}{m^2} \chi_m^2(U_1, ..., U_m, W_1, ..., W_m)$$

$$\geq \frac{1}{(m+1)^2} \chi_{m+1}^2(U_1, ..., U_m, U_{m+1}, W_1, ..., W_m, W_{m+1}) \geq 0. \tag{5.20}$$

Proof. Let $g(x) = (x-1)^2, x > 0$. Here $g(x)$ is convex and normalized function because $g''(x) \geq 0 \ \forall \ x > 0$ and $g(1) = 0$ respectively.
Put $g(x)$ in (2.2), we obtain

$$S_g^m(U_1,...,U_m,W_1,...,W_m) = \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{im} \left[\frac{\frac{u_{i1}+w_{i1}}{2w_{i1}} + ... + \frac{u_{im}+w_{im}}{2w_{im}}}{m} - 1 \right]^2$$

$$= \frac{1}{m^2} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{im} \left[\left(\frac{u_{i1} + w_{i1}}{2w_{i1}} - 1 \right) + ... + \left(\frac{u_{im} + w_{im}}{2w_{im}} - 1 \right) \right]^2$$

$$= \frac{1}{4m^2} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{im} \left[\frac{u_{i1} - w_{i1}}{w_{i1}} + ... + \frac{u_{im} - w_{im}}{w_{im}} \right]^2 = \frac{1}{4m^2} \chi_m^2 \left(U_1, ..., U_m, W_1, ..., W_m \right),$$

$$(5.21)$$

and for $m = 1$

$$\frac{1}{4} \sum_{i=1}^{l} \frac{(u_i - w_i)^2}{w_i} = \frac{1}{4} \chi_1^2 \left(U_1, W_1 \right) = \frac{1}{4} \chi^2 \left(U, W \right), \tag{5.22}$$

where $\chi^2 (U, W)$ is the well known Chi-square discriminating measure [18], an uncommon case of the

$$\chi_m^2 \left(U_1, ..., U_m, W_1, ..., W_m \right),$$

for comparing two probability distributions.
Now, the above equation (5.21) can be written as

$$\frac{1}{4m^2} \chi_m^2 \left(U_1, ..., U_m, W_1, ..., W_m \right)$$

$$= \frac{1}{m^2} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}..w_{im} \left[\sum_{j=1}^{m} \left(\frac{u_{ij} + w_{ij}}{2w_{ij}} - 1 \right)^2 + 2 \sum_{1 \le j < k \le m} \left(\frac{u_{ij} + w_{ij}}{2w_{ij}} - 1 \right) \left(\frac{u_{ik} + w_{ik}}{2w_{ik}} - 1 \right) \right]$$

$$= \frac{1}{m^2} \sum_{j=1}^{m} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{im} \left(\frac{u_{ij} + w_{ij}}{2w_{ij}} - 1 \right)^2$$

$$+ \frac{2}{m^2} \sum_{1 \le j < k \le m} \sum_{i=1}^{l} ... \sum_{i=1}^{l} w_{i1}...w_{im} \left(\frac{u_{ij} + w_{ij}}{2w_{ij}} - 1 \right) \left(\frac{u_{ik} + w_{ik}}{2w_{ik}} - 1 \right)$$

$$= \frac{1}{m^2} \sum_{j=1}^{m} \left[\sum_{i=1}^{l} w_{i1} ... \sum_{i=1}^{l} w_{ij} \left(\frac{u_{ij} + w_{ij}}{2w_{ij}} - 1 \right)^2 ... \sum_{i=1}^{l} w_{im} \right]$$

$$+ \frac{2}{m^2} \sum_{1 \le j < k \le m} \left[\sum_{i=1}^{l} w_{i1} ... \sum_{i=1}^{l} w_{ij} \left(\frac{u_{ij} + w_{ij}}{2w_{ij}} - 1 \right) ... \sum_{i=1}^{l} w_{ik} \left(\frac{u_{ik} + w_{ik}}{2w_{ik}} - 1 \right) ... \sum_{i=1}^{l} w_{im} \right]$$

$$= \frac{1}{m^2} \sum_{j=1}^{m} \sum_{i=1}^{l} w_{ij} \left(\frac{u_{ij} + w_{ij}}{2w_{ij}} - 1 \right)^2 + \frac{2}{m^2} \sum_{1 \le j < k \le m} \sum_{i=1}^{l} \left(\frac{u_{ij} - w_{ij}}{2} \right) \sum_{i=1}^{l} \left(\frac{u11_{ik} - w_{ik}}{2} \right)$$

$$= \frac{1}{4m^2} \sum_{j=1}^{m} \chi^2 \left(U_j, W_j \right).$$

$$\Rightarrow \chi_m^2 \left(U_1, ..., U_m, W_1, ..., W_m \right) = \sum_{j=1}^{m} \chi^2 \left(U_j, W_j \right).$$

Hence, prove the result (5.19). Also, the sequence of inequalities (5.20) can be obtained by using (3.9). We are omitting the details.   □

6. CONCLUSION

The main conclusion of this article is to present a new composite functional discriminating measure for analysing more than two discrete probability distributions at a time. The obtained results (in sections 3, 4, and 5) are important in information theory and are original to the best of the author's knowledge. Also, the same is valid for continuous probability distributions, like Normal, Exponential, Uniform, Chi-square, etc. The application of obtained results in signal processing and pattern recognition is in the works, and these will be provided in the next article as soon as possible.

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