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# Properties of Restricted Cascade Product of Fuzzy Finite State Machines

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ABSTRACT. In this paper our sole aim is to introduce the Restricted cascade product of fuzzy finite state machines without the condition of completeness which was very much needed for Restricted cascade product of fuzzy finite switchboard state machines. The authors also aim to verify the properties of this product with all other products such as Full direct product, restricted direct product, cascade product, wreath product, cartesian product, direct sum and sum with respect to the notions of isomorphism and covering.

## 1. INTRODUCTION

After the introduction of fuzzy set by Lotfi Zadeh in 1965, fuzzy automaton was one of the prime concepts that has emerged as a topic of research due to its applications in various fields such as Computer Science, Electrical Engineering, Biological systems, Image and Pattern recognition, Medicine, Artificial Intelligence, Neural networks etc. [1, 5, 7, 21, 23, 30, 32, 33, 34]. Initially, Wee and Fu [34] applied the concept of fuzzy automaton in learning systems for automatic control and pattern recognition and discussed its advantages. The concept of Fuzzy grammar and languages was first discussed by Lee and Zadeh [13] and it was pointed out that the Context-sensitive fuzzy grammar is recursive. Thomason and Marinos [31] describe interplay between Fuzzy regular expression, Regular fuzzy language, and Fuzzy automaton. Santos [24, 25, 26, 27, 28]; Wechler [33]; Kandel and Lee [7] have developed the algebraic study of fuzzy finite state machines, automata and probabilistic automata. The concept of subsemiautomaton or fuzzy finite state machine was systematically developed by Malik et al.[16, 17, 18, 19], Mordeson and Malik [21] . The notion of product of finite automata is one of the important algebraic techniques that is used to design a new automaton for given ones that can carry out their works altogether. The construction of various types of products of fuzzy finite state machines such as restricted direct product, direct product, wreath product, cascade product, sum, direct sum, cartesian product etc was the motto of the papers by Malik et al. [18, 19]; Kim et al. [9] and Kumbhojkar and Chaudhari [11, 12].

Fuzzy languages viz. regular, context free, context sensitive etc are recognized by corresponding type of fuzzy finite automaton [2, 3, 4, 8, 13, 21]. The concept of products of fuzzy finite automata for fuzzy language recognition was studied by Malik et al.[18, 19] and Malik and Mordeson [21]; Kumbhojkar and Chaudhari [12] and many others [10, 14, 20, 24, 29]. A fuzzy automaton with output is known as Mealy type of fuzzy finite automaton. Various products for these types of fuzzy automata were discussed by Liu et al. [14] and they were further generalized by S. R. Chaudhari and S. A. Morye [22, 6].

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Recently, Kavikumar et al. [8] have introduced the concept of restricted cascade product of complete of fuzzy finite state switchboard machines and studied its relationship with their wreath product. Further they have illustrated a single pattern one-minute microwave as an example of restricted cascade product of fuzzy finite switchboard state machines. But the concept of restricted cascade product for fuzzy finite state machines is not yet introduced in literature.

In this paper we will introduce this concept for fuzzy finite state machines and discuss that the condition of completeness is not necessary while developing it for fuzzy finite state machines. We will also establish various algebraic properties of this restricted product of fuzzy finite state machines with all other products introduced in [11, 18, 19, 21] in terms of homomorphism and coverings.

## 2. Preliminaries

Here in this section, we will recall the preliminary concepts of fuzzy finite state machine and product of fuzzy finite state machines along with the newly introduce concept of restricted cascade product of fuzzy finite state machines. It is noted that the completeness of fuzzy finite state machines is not needed for products of fuzzy finite state machines and hence the authors will not impose it for the restricted cascade product of fuzzy finite state machines too.

**Definition 2.1.** [16] A fuzzy finite state machine (FFSM) is a triplet $M = (Q, X, \mu)$, where $Q$ is called the set of states, $X$ is called the set of input symbols and $\mu$ is a fuzzy subset of $Q \times X \times Q$, that is, $\mu : Q \times X \times Q \to [0, 1]$.

As usual $X^*$ denotes the set of all words of elements of $X$ of finite length. Let $\lambda$ denote the empty word in $X^*$ and $|x|$ denote the length of $x \in X^*$. $X^*$ is a free semi group with identity $\lambda$ with respect to the binary operation concatenation of two words. If we define $\mu^* : Q \times X^* \times Q \to [0, 1]$ by

$$\mu^*(q, \lambda, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

and

$$\mu^*(q, xa, p) = \bigvee_{r \in Q} \{\mu^*(q, x, r) \wedge \mu(r, a, p)\}$$

$\forall q, p \in Q$ and $\forall x \in X^*, a \in X$, then

$$\mu^*(q, xy, p) = \bigvee_{r \in Q} \{\mu^*(q, x, r) \wedge \mu^*(r, y, p)\}$$

$\forall q, p \in Q$ and $\forall x, y \in X^*$. Here onwards we shall denote $\mu^*$ by $\mu$ without any ambiguity. Malik et al.[18, 19] and Kim et al.[9] defined Full direct product, Restricted direct product, Cascade product, Wreath product and Cartesian composition for fuzzy finite state machines. Kumbhojkar and Chaudhari [11]. introduced Direct sum and sum for fuzzy finite state machines.

**Definition 2.2.** Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two FFSMs. Then the FFSM,

(1) $M_1 \times M_2 = (Q_1 \times Q_2, X_1 \times X_2, \mu_1 \times \mu_2)$ is called the **Full direct product** of $M_1$ and $M_2$, where $\mu_1 \times \mu_2 : (Q_1 \times Q_2) \times (X_1 \times X_2) \times (Q_1 \times Q_2) \to [0, 1]$ is defined as follows:

$$(\mu_1 \times \mu_2)((q_1, q_2), (a_1, a_2), (p_1, p_2)) = \mu_1(q_1, a_1, p_1) \wedge \mu_2(q_2, a_2, p_2),$$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, (a_1, a_2) \in X_1 \times X_2$.

(2) $M_1 \wedge M_2 = (Q_1 \times Q_2, X, \mu_1 \wedge \mu_2)$ is called the **Restricted direct product** of $M_1$ and $M_2$, where $X_1 = X_2 = X$ and $\mu_1 \wedge \mu_2 : (Q_1 \times Q_2) \times X \times (Q_1 \times Q_2) \to [0, 1]$ is defined as follows:

$$(\mu_1 \wedge \mu_2)((q_1, q_2), a, (p_1, p_2)) = \mu_1(q_1, a, p_1) \wedge \mu_2(q_2, a, p_2),$$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, a \in X$.

(3) $M_1 \omega M_2 = (Q_1 \times Q_2, X_2, \mu_1 \omega \mu_2)$ is called the **Cascade product** of $M_1$ and $M_2$, where $\omega : Q_2 \times X_2 \to X_1$ is a mapping and $\mu_1 \omega \mu_2 : (Q_1 \times Q_2) \times X_2 \times (Q_1 \times Q_2) \to [0, 1]$ is defined as follows:

$$(\mu_1 \omega \mu_2)((q_1, q_2), a_2, (p_1, p_2)) = \mu_1(q_1, \omega(q_2, a_2), p_1) \wedge \mu_2(q_2, a_2, p_2),$$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, a_2 \in X_2$.

(4) $M_1 \circ M_2 = (Q_1 \times Q_2, (X_1^{Q_2} \times X_2), \mu_1 \circ \mu_2)$ is called the **Wreath product** of $M_1$ and $M_2$, where $X_1^{Q_2} = \{f : Q_2 \to X_1\}$ and $\mu_1 \circ \mu_2 : (Q_1 \times Q_2) \times (X_1^{Q_2} \times X_2) \times (Q_1 \times Q_2) \to [0, 1]$ is defined as follows:

$$(\mu_1 \circ \mu_2)((q_1, q_2), (f, a_2), (p_1, p_2)) = \mu_1(q_1, f(q_2), p_1) \wedge \mu_2(q_2, a_2, p_2),$$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, a_2 \in X_2$.

(5) $M_1 \oplus M_2 = (Q_1 \cup Q_2, X_1 \cup X_2, \mu_1 \oplus \mu_2)$ is called the **Direct sum** of $M_1$ and $M_2$, where $Q_1 \cap Q_2 = \phi, X_1 \cap X_2 = \phi$ and $\mu_1 \oplus \mu_2 : (Q_1 \cup Q_2) \times (X_1 \cup X_2) \times (Q_1 \cup Q_2) \to [0, 1]$ is defined as follows:

$$(\mu_1 \oplus \mu_2)(q, a, p) = \begin{cases} \mu_1(q, a, p) & \text{if } q, p, \in Q_1 \text{ and } a \in X_1, \\ \mu_2(q, a, p) & \text{if } q, p, \in Q_2 \text{ and } a \in X_2, \\ 1 & \text{if either } (q, a) \in (Q_1 \times X_1) \text{ and } p \in Q_2, \\ & \text{or } (q, a) \in (Q_2 \times X_2) \text{ and } p \in Q_1, \\ 0 & \text{otherwise,} \end{cases}$$

for all $q, p \in Q_1 \cup Q_2, a \in X_1 \cup X_2$.

(6) $M_1 + M_2 = (Q_1 \cup Q_2, X_1 \cup X_2, \mu_1 + \mu_2)$ is called the **Sum** of $M_1$ and $M_2$, where $Q_1 \cap Q_2 = \phi, X_1 \cap X_2 = \phi$ and $\mu_1 + \mu_2 : (Q_1 \cup Q_2) \times (X_1 \cup X_2) \times (Q_1 \cup Q_2) \to [0, 1]$ is defined as follows:

$$(\mu_1 + \mu_2)(q, a, p) = \begin{cases} \mu_1(q, a, p) & \text{if } q, p, \in Q_1 \text{ and } a \in X_1, \\ \mu_2(q, a, p) & \text{if } q, p, \in Q_2 \text{ and } a \in X_2, \\ 0 & \text{otherwise,} \end{cases}$$

for all $q, p \in Q_1 \cup Q_2, a \in X_1 \cup X_2$.

(7) $M_1 \varpi M_2 = (Q_1 \times Q_2, X_2, \mu_1 \varpi \mu_2)$ is called the **Restricted cascade product** of $M_1$ and $M_2$ where $\varpi : X_2 \to X_1$ is a surjective mapping and $\mu_1 \varpi \mu_2 : (Q_1 \times Q_2) \times X_2 \times (Q_1 \times Q_2) \to [0, 1]$ is defined as follows:

$$(\mu_1 \varpi \mu_2)((q_1, q_2), a_2, (p_1, p_2)) = \mu_1(q_1, \varpi(a_2), p_1) \wedge \mu_2(q_2, a_2, p_2),$$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2, a_2 \in X_2$.

(8) $M_1 \bullet M_2 = (Q_1 \times Q_2, X_1 \cup X_2, \mu_1 \bullet \mu_2)$ is called the **Cartesian composition** of $M_1$ and $M_2$, where $X_1 \cap X_2 = \phi$ and $\mu_1 \bullet \mu_2 : (Q_1 \times Q_2) \times X_1 \cup X_2 \times (Q_1 \times Q_2) \to [0, 1]$ is defined as follows:

$$(\mu_1 \bullet \mu_2)((q_1, q_2), a, (p_1, p_2)) = \begin{cases} \mu_1(q_1, a, p_1) & \text{if } a \in X_1, \text{ and } q_2 = p_2 \\ \mu_2(q_2, a, p_2) & \text{if } a \in X_2, \text{ and } q_1 = p_1 \\ 0 & \text{otherwise,} \end{cases}$$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $a \in X_1 \cup X_2$.

## 3. Properties of restricted cascade product with isomorphism

In this section we recall the definition of homomorphism of fuzzy finite state machines given in [9, 21] and discuss the connection between restricted cascade product and all other products.

**Definition 3.3.** [9, 21] $M_i = (Q_i, X_i, \mu_i)$ be a FFSMs, $i = 1, 2$. Let $\alpha : Q_1 \to Q_2$ and $\beta : X_1 \to X_2$ be two mappings. Then the pair $(\alpha, \beta)$ is called a **fuzzy finite state homomorphism**, symbolically $(\alpha, \beta) : M_1 \to M_2$ , if $\mu_1(q, x, p) \leq \mu_2(\alpha(q), \beta(x), \alpha(p)), \forall q, p \in Q_1, x \in X_1$,

The homomorphism $(\alpha, \beta) : M_1 \to M_2$ is called **mono-morphism (epimorphism, isomorphism)**, if both the mappings $\alpha$ and $\beta$ are injective (surjective, bijective respectively). In case of isomorphism of $M_1$ and $M_2$ we shall denote it by $M_1 \cong M_2$.
In the following theorem we show that the restricted cascade product of fuzzy finite state machines is associative.

**Theorem 3.1.** *Let $M_i = (Q_i, X_i, \mu_i)$ be FFSMs, $i = 1, 2, 3$. Then $(M_1 \varpi^1 M_2)\varpi^2 M_3 \cong M_1 \varpi^3 (M_2 \varpi^4 M_3)$.*

*Proof.* Define $\alpha : (Q_1 \times Q_2) \times Q_3 \to Q_1 \times (Q_2 \times Q_3)$ by $\alpha((q_1, q_2), q_3) = (q_1, (q_2, q_3))$ and take $\beta : X_3 \to X_3$ as identity function. Also set $\varpi^3(x_3) = \varpi^1(\varpi^2(x_3))$ and $\varpi^4(x_3) = \varpi^2(x_3)$. Then, we have
$((\mu_1 \varpi^1 \mu_2)\varpi^2 \mu_3)(((q_1, q_2), q_3), x_3, ((p_1, p_2), p_3)) =$

$$\begin{aligned}
&= (\mu_1 \varpi^1 \mu_2)((q_1, q_2), \varpi^2(x_3), (p_1, p_2)) \wedge \mu_3(q_3, x_3, p_3)\\
&= \mu_1(q_1, \varpi^1(\varpi^2(x_3)), p_1) \wedge \mu_2(q_2, \varpi^2(x_3), p_2) \wedge \mu_3(q_3, x_3, p_3)\\
&= \mu_1(q_1, \varpi^3(x_3), p_1) \wedge \mu_2(q_2, \varpi^4(x_3), p_2) \wedge \mu_3(q_3, x_3, p_3)\\
&= \mu_1(q_1, \varpi^3(x_3), p_1) \wedge (\mu_2 \varpi^4 \mu_3)((q_2, q_3), x_3, (p_2, p_3))\\
&= (\mu_1 \varpi^3(\mu_2 \varpi^4 \mu_3))((q_1, (q_2, q_3)), x_3, (p_1, (p_2, p_3)))\\
&= (\mu_1 \varpi^3(\mu_2 \varpi^4 \mu_3))(\alpha((q_1, q_2), q_3), \beta(x_3), \alpha((p_1, p_2), p_3))
\end{aligned}$$

This proves that $(M_1 \varpi^1 M_2)\varpi^2 M_3 \cong M_1 \varpi^3 (M_2 \varpi^4 M_3)$. □

Now, authors have established the relation between restricted cascade product and all other products. Note that the property (3) of the following theorem is established by Kavikumar et al. [4] for restricted cascade product of switchboard state machines, Proposition 3.11, and we pointed out that the completeness is not necessary there.

**Theorem 3.2.** *Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two FFSMs. Then,*

*(1)* $M_1 \times M_2 \cong M_1 \varpi M_2$
*(2)* $M_1 \varpi M_2 \cong M_1 \wedge M_2$
*(3)* $M_1 \omega M_2 \cong M_1 \varpi M_2$
*(4)* $M_1 \varpi M_2 \cong M_1 \circ M_2$

*Proof.* Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two FFSMs.

(1) Take $\alpha : Q_1 \times Q_2 \to Q_1 \times Q_2$ as natural function and define $\beta$ as identity function on $X_2$. Also set $\varpi(x_2) = x_1$. Then,

$$\begin{aligned}(\mu_1 \times \mu_2)((q_1, q_2), (x_1, x_2), (p_1, p_2)) &= \mu_1(q_1, x_1, p_1) \wedge \mu_2(q_2, x_2, p_2)\\ &= \mu_1(q_1, \varpi(x_2), p_1) \wedge \mu_2(q_2, x_2, p_2)\\ &= (\mu_1 \varpi \mu_2)((q_1, q_2), x_2, (p_1, p_2))\\ &= (\mu_1 \varpi \mu_2)(\alpha((q_1, q_2)), \beta(x_2), \alpha((p_1, p_2)))\end{aligned}$$

(2) Take $\alpha : Q_1 \times Q_2 \to Q_1 \times Q_2$ as natural function and define $\beta$ as identity function on $X$. Also set $\varpi(x) = x$. Then,

$$\begin{aligned}(\mu_1 \varpi \mu_2)((q_1, q_2), x, (p_1, p_2)) &= \mu_1(q_1, \varpi(x), p_1) \wedge \mu_2(q_2, x, p_2)\\ &= \mu_1(q_1, x, p_1) \wedge \mu_2(q_2, x, p_2)\\ &= (\mu_1 \wedge \mu_2)((q_1, q_2)), x, (p_1, p_2))\\ &= (\mu_1 \wedge \mu_2)(\alpha((q_1, q_2)), \beta(x), \alpha((p_1, p_2)))\end{aligned}$$

(3) Define $\alpha : Q_1 \times Q_2 \to Q_1 \times Q_2$ as natural function and define $\beta$ as identity function on $X_2$. Also set $\omega(q_2, x_2) = \varpi(x_2)$ Then,

$$\begin{aligned}(\mu_1 \omega \mu_2)((q_1, q_2)), x_2, (p_1, p_2)) &= \mu_1(q_1, \omega(q_2, x_2), p_1) \wedge \mu_2(q_2, x_2, p_2)\\ &= \mu_1(q_1, \varpi(x_2), p_1) \wedge \mu_2(q_2, x_2, p_2)\\ &= (\mu_1 \varpi \mu_2)((q_1, q_2), x_2, (p_1, p_2))\\ &= (\mu_1 \varpi \mu_2)(\alpha((q_1, q_2)), \beta(x_2), \alpha((p_1, p_2)))\end{aligned}$$

(4) Define $\alpha : Q_1 \times Q_2 \to Q_1 \times Q_2$ as natural function and define $\beta : X_2 \to (X_1^{Q_2} \times X_2)$ by $\beta(x_2) = (f, x_2)$. Also set $f(q_2) = \varpi(x_2)$ Then,

$$\begin{aligned}(\mu_1 \varpi \mu_2)((q_1, q_2), x_2, (p_1, p_2)) &= \mu_1(q_1, \varpi(x_2), p_1) \wedge \mu_2(q_2, x_2, p_2)\\ &= \mu_1(q_1, f(q_2), p_1) \wedge \mu_2(q_2, x_2, p_2)\\ &= (\mu_1 \circ \mu_2)((q_1, q_2)), (f, x_2), (p_1, p_2))\\ &= (\mu_1 \circ \mu_2)(\alpha((q_1, q_2)), \beta(x_2), \alpha((p_1, p_2)))\end{aligned}$$

□

**Theorem 3.3.** *Let $M_i = (Q_i, X_i, \mu_i)$ be FFSMs, $i = 1, 2, 3$. Then $M_1 \times (M_2 \varpi^1 M_3) \cong M_2 \varpi^2 (M_1 \times M_3)$*

*Proof.* Define $\alpha : Q_1 \times (Q_2 \times Q_3) \to (Q_1 \times Q_2) \times Q_3$ by $\alpha((q_1(q_2, q_3))) = (q_2, (q_1, q_3))$ and take $\beta$ as identity function on $X_1 \times X_3$. Also set $\varpi^2(x_1, x_3) = \varpi^1(x_3)$. Then,

$$\begin{aligned}&(\mu_1 \times (\mu_2 \varpi^1 \mu_3))((q_1, (q_2, q_3)), (x_1, x_3), (p_1, (p_2, p_3))) =\\ &\quad= \mu_1(q_1, x_1, p_1) \wedge (\mu_2 \varpi^1 \mu_3)((q_2, q_3), x_3, (p_2, p_3))\\ &\quad= \mu_1(q_1, x_1, p_1) \wedge \mu_2(q_2, \varpi^1(x_3), p_3) \wedge \mu_3(q_3, x_3, p_3)\\ &\quad= \mu_2(q_2, \varpi^1(x_3), p_3) \wedge \mu_1(q_1, x_1, p_1) \wedge \mu_3(q_3, x_3, p_3)\\ &\quad= \mu_2(q_2, \varpi^1(x_3), p_3) \wedge (\mu_1 \times \mu_3)((q_1, q_3), (x_1, x_3), (p_1, p_3))\\ &\quad= \mu_2(q_2, \varpi^2(x_1, x_3), p_3) \wedge (\mu_1 \times \mu_3)((q_1, q_3), (x_1, x_3), (p_1, p_3))\\ &\quad= (\mu_2 \varpi^2 (\mu_1 \times \mu_3))((q_2, (q_1, q_3)), (x_1, x_3), (p_2, (p_1, p_3)))\\ &\quad= (\mu_2 \varpi^2 (\mu_1 \times \mu_3))(\alpha((q_1, (q_2, q_3))), \beta(x_1, x_3), \alpha((p_1, (p_2, p_3))))\end{aligned}$$

□

**Theorem 3.4.** *Let $M_i = (Q_i, X_i, \mu_i)$ be FFSMs, $i = 1, 2, 3$. Then*

*(1)* $M_1\varpi^1(M_2 + M_3) \cong (M_1\varpi^2 M_2) + (M_1\varpi^3 M_3)$
*(2)* $M_1\varpi^1(M_2 \oplus M_3) \cong (M_1\varpi^2 M_2) \oplus (M_1\varpi^3 M_3)$
*(3)* $M_1\varpi^1(M_2 \bullet M_3) \cong (M_1\varpi^2 M_2) \bullet (M_1\varpi^3 M_3)$

*Proof.* Let $M_1, M_2, M_3$ be FFSMs.

(1) Define $\alpha : (Q_1 \times (Q_2 \cup Q_3) \to (Q_1 \times Q_2) \cup (Q_1 \times Q_3)$ by $\alpha((q_1, q)) = (q_1, q)$ and take $\beta$ as identity function on $X_2 \cup X_3$. Given $\varpi^1 : X_2 \cup X_3 \to X_1$, $\varpi^2 : X_2 \to X_1$ and $\varpi^3 : X_3 \to X_1$ denote $\varpi^2(x) = \varpi^1(x)$ and $\varpi^3(x) = \varpi^1(x)$. Then, $(\mu_1\varpi^1(\mu_2 + \mu_3))((q_1, q), (x_1, x), (p_1, p)) =$

$$
\begin{aligned}
&= \mu_1(q_1, \varpi^1(x), p_1) \wedge (\mu_2 + \mu_3)(q, x, p) \\
&= \begin{cases} \mu_1(q_1, \varpi^1(x), p_1) \wedge \mu_2(q, x, p) & \text{if } q, p, \in Q_2 \text{ and } x \in X_2 \\ \mu_1(q_1, \varpi^1(x), p_1) \wedge \mu_3(q, x, p) & \text{if } q, p, \in Q_3 \text{ and } x \in X_3 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \mu_1(q_1, \varpi^2(x), p_1) \wedge \mu_2(q, x, p) & \text{if } q, p, \in Q_2 \text{ and } x \in X_2 \\ \mu_1(q_1, \varpi^3(x), p_1) \wedge \mu_3(q, x, p) & \text{if } q, p, \in Q_3 \text{ and } x \in X_3 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} (\mu_1\varpi^2\mu_2)((q_1, q), (x_1, x), (p_1, p)) & \text{if } q, p, \in Q_2 \text{ and } x \in X_2 \\ (\mu_1\varpi^3\mu_3)((q_1, q), (x_1, x), (p_1, p)) & \text{if } q, p, \in Q_3 \text{ and } x \in X_3 \\ 0 & \text{otherwise} \end{cases} \\
&= ((\mu_1\varpi^2\mu_2) + (\mu_1\varpi^3\mu_3))((q_1, q), x, (p_1, p)) \\
&= ((\mu_1\varpi^2\mu_2) + (\mu_1\varpi^3\mu_3))(\alpha((q_1, q)), \beta(x), \alpha((p_1, p)))
\end{aligned}
$$

(2) Similar to (1)

(3) Define $\alpha : (Q_1 \times (Q_2 \times Q_3) \to (Q_1 \times Q_2) \times (Q_1 \times Q_3)$ by $\alpha((q_1, (q_2, q_3))) = ((q_1, q_3), (q_2, q_3))$ and take $\beta$ as identity function on $X_2 \cup X_3$. Given $\varpi^1 : X_2 \cup X_3 \to X_1$, $\varpi^2 : X_2 \to X_1$ and $\varpi^3 : X_3 \to X_1$ denote $\varpi^2(x) = \varpi^1(x)$ and $\varpi^3(x) = \varpi^1(x)$. Then, $(\mu_1\varpi^1(\mu_2 \bullet \mu_3))((q_1, (q_2, q_3)), (x_1, x), (p_1, (p_2, p_3))) =$

$$
\begin{aligned}
&= \mu_1(q_1, \varpi^1(x), p_1) \wedge (\mu_2 \bullet \mu_3)((q_2, q_3), x, (p_2, p_3)) \\
&= \begin{cases} \mu_1(q_1, \varpi^1(x), p_1) \wedge \mu_2(q_2, x, p_2) & \text{if } x \in X_2 \text{ and } q_3 = p_3, \\ \mu_1(q_1, \varpi^1(x), p_1) \wedge \mu_3(q_3, x, p_3) & \text{if } x \in X_3 \text{ and } q_2 = p_2, \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \mu_1(q_1, \varpi^2(x), p_1) \wedge \mu_2(q_2, x, p_2) & \text{if } x \in X_2 \text{ and } q_3 = p_3, \\ \mu_1(q_1, \varpi^3(x), p_1) \wedge \mu_3(q_3, x, p_3) & \text{if } x \in X_3 \text{ and } q_2 = p_2, \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} (\mu_1\varpi^2\mu_2)((q_1, q_2), (x_1, x), (p_1, p_2)) & \text{if } x \in X_2 \text{ and } q_3 = p_3 \\ (\mu_1\varpi^3\mu_3)((q_1, q_3), (x_1, x), (p_1, p_3)) & \text{if } x \in X_3 \text{ and } q_2 = p_2, \\ 0 & \text{otherwise} \end{cases} \\
&= ((\mu_1\varpi^2\mu_2) \bullet (\mu_1\varpi^3\mu_3))(((q_1, q_2), (q_1, q_3)), (x_1, x), ((p_1, p_2), (p_1, p_3))) \\
&= ((\mu_1\varpi^2\mu_2) + (\mu_1\varpi^3\mu_3))(\alpha((q_1, (q_2, q_3))), \beta(x), \alpha((p_1, (p_2, p_3))))
\end{aligned}
$$

□

## 4. COVERING PROPERTIES OF THE RESTRICTED CASCADE PRODUCT

In this section we first recall the concept of covering given in [9, 21] and proved relations between restricted cascade product and all other products.

**Definition 4.4.** [9, 21] $M_i = (Q_i, X_i, \mu_i)$ be a FFSMs, $i = 1, 2$. Let $\eta : Q_2 \to Q_1$ be surjective partial function and let $\xi : X_1 \to X_2$ be a function. Extend $\xi$ to a function $\xi^*$ of $X_1^*$ into $X_2^*$ by $\xi^*(\lambda) = \lambda$ and $\forall x \in X_1^*, \xi^*(x) = \xi(x_1)\xi(x_2)\dots\xi(x_n)$, where $x = x_1x_2\dots x_n$ and $x_i \in X_1, i = 1, 2, \dots, n$. Then the pair $(\eta, \xi)$ is called a covering of $M_1$ by $M_2$ , written as $M_1 \le M_2$, if and only if $\forall p_2, q_2$ belongs to the domain of $\eta$ and $x_1 \in X_1^*$, we have $\mu_1(\eta(q_2), x_1, \eta(p_2)) \le \mu_2(q_2, \xi(x_1), p_2)$.

Note that the properties (1) and (4) of the following theorem are established by Kavikumar et al. [4] for restricted cascade product of switchboard state machine in Theorem 3.19.

**Theorem 4.5.** *Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two FFSMs. Then*

*(1)* $M_1 \varpi M_2 \le M_1 \times M_2$
*(2)* $M_1 \varpi M_2 \le M_1 \wedge M_2$
*(3)* $M_1 \varpi M_2 \le M_1 \omega M_2$
*(4)* $M_1 \varpi M_2 \le M_1 \circ M_2$

*Proof.* Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two FFSMs.

(1) Let $\eta$ and $\xi$ be natural functions and set $\varpi(x_2) = x_2$. Then for any $(q_1, q_2), (p_1, p_2)$ belongs to the domain of $\eta$ and $(x_1, x_2) \in X_1 \times X_2$, we have

$$\begin{aligned}(\mu_1 \varpi \mu_2)(\eta((q_1, q_2)), x_2, \eta((p_1, p_2))) &= (\mu_1 \varpi \mu_2)((q_1, q_2), x_2, (p_1, p_2)) \\ &= \mu_1(q_1, \varpi(x_2), p_1) \wedge \mu_2(q_2, x_2, p_2) \\ &= \mu_1(q_1, x_1, p_1) \wedge \mu_2(q_2, x_2, p_2) \\ &= (\mu_1 \times \mu_2)((q_1, q_2), (x_1, x_2), (p_1, p_2)) \\ &= (\mu_1 \times \mu_2)((q_1, q_2), \xi((x_1, x_2)), (p_1, p_2))\end{aligned}$$

(2) Let $\eta$ be natural function and $\xi$ be identity function and set $\varpi(x) = x$. Then for any $(q_1, q_2), (p_1, p_2)$ belongs to the domain of $\eta$ and $x \in X$, we have

$$\begin{aligned}(\mu_1 \varpi \mu_2)(\eta((q_1, q_2)), x, \eta((p_1, p_2))) &= (\mu_1 \varpi \mu_2)((q_1, q_2), x, (p_1, p_2)) \\ &= \mu_1(q_1, \varpi(x), p_1) \wedge \mu_2(q_2, x, p_2) \\ &= \mu_1(q_1, x, p_1) \wedge \mu_2(q_2, x, p_2) \\ &= (\mu_1 \wedge \mu_2)((q_1, q_2), x, (p_1, p_2)) \\ &= (\mu_1 \wedge \mu_2)((q_1, q_2), \xi(x), (p_1, p_2))\end{aligned}$$

(3) We have Let $\eta$ be natural function and $\xi$ be identity function and set $\omega(q_2, x_2) = \varpi(x_2)$. Then for any $(q_1, q_2), (p_1, p_2)$ belongs to the domain of $\eta$ and $x_2 \in X_2$, we have

$$\begin{aligned}(\mu_1 \varpi \mu_2)(\eta((q_1, q_2)), x_2, \eta((p_1, p_2))) &= (\mu_1 \varpi \mu_2)((q_1, q_2), x_2, (p_1, p_2)) \\ &= \mu_1(q_1, \varpi(x_2), p_1) \wedge \mu_2(q_2, x_2, p_2) \\ &= \mu_1(q_1, \omega(q_2, x_2), p_1) \wedge \mu_2(q_2, x_2, p_2) \\ &= (\mu_1 \omega \mu_2)((q_1, q_2)), x_2, (p_1, p_2)) \\ &= (\mu_1 \omega \mu_2)((q_1, q_2), \xi(x_2), (p_1, p_2))\end{aligned}$$

(4) Let $\eta$ be natural function and $\xi : X_2 \to X_1^{Q_2}$ be function such that $\xi(x_2) = (f, x_2)$. Set $\varpi(x_2) = f(q_2)$. Then for any $(q_1, q_2), (p_1, p_2)$ belongs to the domain of $\eta$ and $x \in X$, we have

$$\begin{aligned}(\mu_1 \varpi \mu_2)(\eta((q_1, q_2)), x_2, \eta((p_1, p_2))) &= (\mu_1 \varpi \mu_2)((q_1, q_2), x_2, (p_1, p_2)) \\ &= \mu_1(q_1, \varpi(x_2), p_1) \wedge \mu_2(q_2, x_2, p_2) \\ &= \mu_1(q_1, f(q_2), p_1) \wedge \mu_2(q_2, x_2, p_2) \\ &= (\mu_1 \circ \mu_2)((q_1, q_2), (f, x_2), (p_1, p_2)) \\ &= (\mu_1 \circ \mu_2)((q_1, q_2), \xi(x_2), (p_1, p_2))\end{aligned}$$

□

**Theorem 4.6.** *Let $M_i = (Q_i, X_i, \mu_i)$ be FFSMs, $i = 1, 2, 3$ such that $M_1 \leq M_2$. Then $M_1 \varpi^1 M_3 \leq M_2 \varpi^2 M_3$*

*Proof.* Define $\eta' : (Q_2 \times Q_3) \to (Q_1 \times Q_3)$ by $\eta'((q_2, q_3)) = (\eta(q_2), q_3))$ and $\xi'$ as identity map on $X_3$. Set $\varpi^2(x_3) = \xi(\varpi^1(x_3))$ on $X_3$. Then for all $(q_2, q_3), (p_2, p_3)$ belongs to the domain of $\eta'$ and for all $x_3 \in X_3$, we have

$$\begin{aligned}(\mu_1 \varpi^1 \mu_3)(\eta'((q_2, q_3)), x_3, \eta'((p_2, p_3))) &= (\mu_1 \varpi^1 \mu_3)((\eta(q_2), q_3)), x_3, (\eta(p_2), p_3))) \\ &= \mu_1(\eta(q_2), \varpi^1(x_3), \eta(p_2)) \wedge \mu_3(q_3, x_3, p_3) \\ &\leq \mu_2(q_2, \xi(\varpi^1(x_3)), p_2) \wedge \mu_3(q_3, x_3, p_3), \text{ Since} M_1 \leq M_2 \\ &= \mu_2(q_2, \varpi^2(x_3), p_2) \wedge \mu_3(q_3, x_3, p_3) \\ &= (\mu_2 \varpi^2 \mu_3)((q_2, q_3), x_3, (p_2, p_3)) \\ &= (\mu_2 \varpi^2 \mu_3)((q_2, q_3), \xi'(x_3), (p_2, p_3))\end{aligned}$$

□

## 5. CONCLUSION

With the motivation of the concept of restricted product for fuzzy finite switchboard state machines [8], here in the paper the authors have introduced it for fuzzy finite state machines and pointed out that the condition of completeness is not necessary. The algebraic properties in terms of homomorphism and covering for restricted cascade product are also established. Similar to algebraic properties of other products of fuzzy finite state machines reported in Malik et al. [18, 19]; Mordeson and Malik [21]; Kim et al. [9] and Kumbhojkar and Chaudhari [11], the completeness does not hamper any result of present paper for the product of restricted cascade fuzzy finite state machines.

The concepts of homomorphism and covering plays vital role in establishing languages recognized by fuzzy finite automata [11, 21], the algebraic properties of restricted cascade product that we have established in this paper will help in unfolding more of these capabilities. In Theorem 3.2 it is established that restricted cascade product of two FFSMs is isomorphic to their other products, so structurally it can singly do the work of others, whereas Theorem 4.5 indicated that the direct, restricted direct, cascade and wreath products subsumes the restricted cascade products. The exchange property of cross product of two FFSMs with their restricted cascade product is discussed in the Theorem 3.3. In the Theorem 3.4 the distributive properties of restricted cascade product with another products like sum, direct sum and cartesian product are established and monotone property for covering is established in Theorem 3.5.

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