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Uniqueness of Q-difference of meromorphic functions sharing a small function with finite weight

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ABSTRACT. In this paper, we study the value distribution and uniqueness of linear q-difference polynomial $L_k(f, E_q)$ and its q-difference operator $L_k(f, \Delta)$, for a transcendental meromorphic function f having zero order. Our results extends, improves and generalizes some of the earlier results due to Thin [Thin, N. V. Uniqueness of meromorphic functions and Q-difference polynomials sharing small functions. *Bull. Iranian Math. Soc.* **43** (2017), no. 3, 629–647]; Dyavanal and Desai [Dyavanal, R. S.; Desai, R. V. Uniqueness of q-shift difference polynomials of meromorphic functions sharing a small function. *Math. Bohem.* **145** (2020), no. 3, 241–253].

1. INTRODUCTION AND MAIN RESULTS

The Nevanlinna theory, or the value distribution theory, is a branch of complex analysis that has seen extensive research work. It mainly deals with studying the distribution of the zeros of the equation $f(z) = a$ in a disc $|z| \leq r$, where f is an entire or meromorphic function in the Gaussian complex plane $\mathbb{C}$, $z \in \mathbb{C}$ and $a \in \mathbb{C} \cup \{\infty\}$. The proofs in this paper uses the Nevanlinna theory and one can refer (Hayman [9], Yi and Yang [22], Yang [23]) for the standard definitions and notations.

Let f and g be two non-constant meromorphic functions defined in the open complex plane. For $a \in \mathbb{C} \cup \{\infty\}$ and $k \in \mathbb{Z}^+ \cup \{\infty\}$, the set $E(a, f) = \{z : f(z) - a = 0\}$ denotes all those a-points of f, where each a-point of f with multiplicity k is counted k times in the set and the set $\overline{E}(a, f) = \{z : f(z) - a = 0\}$, denotes all those a-points of f, where the multiplicities are ignored. If $f(z) - a$ and $g(z) - a$ assumes the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicity) and we have $E(a, f) = E(a, g)$; Suppose, if $f(z) - a$ and $g(z) - a$ assumes the same zeros ignoring the multiplicities, then we say that $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicity) and we will have $\overline{E}(a, f) = \overline{E}(a, g)$. If $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, then f and g share ∞ CM.

In general, for a meromorphic function $f(z)$, the quantity $m(r, f)$ denotes the proximity function of $f(z)$, while $N(r, f)$ denotes the counting function of poles of $f(z)$ whose multiplicities are taken into account (respectively $\overline{N}(r, f)$ denotes the reduced counting function when multiplicities are ignored). The quantity $N(r, a; f)$ $\Big($notation interchangable with $N\left(r, \frac{1}{f-a}\right)\Big)$ denotes the counting function of a points of $f(z)$ whose multiplicities are taken into account (respectively $\overline{N}(r, a; f)$ denotes the reduced counting function when multiplicities are ignored).

Suppose $f(z)$ and $g(z)$ share 1 IM and z_0 is a zero of $f(z) - 1$ of order p and also a zero of $g(z) - 1$ of order q, then $\overline{N}_L(r, 1; f)$ counts those 1-points of $f(z)$ and $g(z)$, where $p > q$.

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$\overline{N}_L(r, 1; g)$ is similarly defined. It is to be noted that each point in these counting functions are counted only once.

The Nevanlinna characteristic function of a meromorphic function f plays a very important role in the value distribution theory and it is denoted by $T(r, f)$. We have $T(r, f) = m(r, f) + N(r, f)$, which clearly shows that $T(r, f)$ is non-negative. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, if $T(r, a) = S(r, f)$, where $S(r, f)$ denotes any quantity, which satisfies $S(r, f) = o(T(r, f))$ as $r \to +\infty$ possibly outside a set I with finite linear measure $\lim\limits_{r\to\infty} \int\limits_{(1,r]\cap I} \frac{dt}{t} < \infty$.

Let us recall the following standard definitions of Nevanlinna theory.

Definition 1.1. The order $\rho(f)$ of a meromorphic function $f(z)$ is defined as,

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{log\, T(r, f)}{log\, r}.$$

Definition 1.2. [11] Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, then we say that f and g share the value a with the weight k.

The definition implies that if f and g share a value a with the weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if z_0 is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if z_0 is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n. We write f, g share (a, k) to mean that, f, g share the value a with the weight k. Clearly, if f, g share (a, k), then f, g share (a, p) for any integer p, such that, $0 \leq p < k$. Also we note that, f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.3. [11] Let f, g share the value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly, $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Around 2006, Halburd and Korhonen established the difference analogue of the Logarithmic Derivative Lemma [6] and also they developed the Nevanlinna theory for the difference operator [7] wherein they extended the classical Nevanlinna theory based on certain estimates involving the derivative $f \mapsto f'$ of a meromorphic function to a theory for the exact difference $f \mapsto \Delta f = f(z + c) - f(z)$. Their results gave the difference analogue of the second main theorem of Nevanlinna theory, as well as the difference analogues of the Nevanlinna defect relation, Picard's theorem and Nevanlinna's five value theorem. Then in 2007, Barnett. et. al [2] gave the q-shift equivalents of the Clunie and Mohon'ko lemmas which can be used to study value distribution of zero-order meromorphic solutions of large classes of q-difference equations and they proved the q-shift equivalent of the second main theorem of Nevanlinna theory and Picard-type theorem for the q-shift operator. Since that time, many mathematicians have developed a keen interest in the study of the q-difference analogues of Nevanlinna theory. We can observe many outstanding studies on the the uniqueness of difference analogues of Nevanlinna theory in ([3], [4], [10], [12], [14], [15], [17], [20]). One Such work can be seen in Zhao and Zhang [25] who in 2015, proved the following results.

Theorem A. [25] *Let $f(z)$ be a transcendental meromorphic function with zero order, and let n, k be positive integers. If $n > k + 5$, then $[f^n(z)f(qz + c)]^{(k)} - 1$ has infinitely many zeros.*

Theorem B. [25] *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, and let n, k be positive integers. If $n > 2k + 5$, and $[f^n(z)f(qz + c)]^{(k)}$ and $[g^n(z)g(qz + c)]^{(k)}$ share z or 1 CM, then $f \equiv tg$ for a constant t with $t^{n+1} = 1$.*

Theorem C. [25] *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, and let n, k be positive integers. If $n > 5k + 11$, and $[f^n(z)f(qz+c)]^{(k)}$ and $[g^n(z)g(qz+c)]^{(k)}$ share z or 1 IM, then $f \equiv tg$ for a constant t with $t^{n+1} = 1$.*

In 2017, Thin [19], by considering a general polynomial of degree n and having m distinct zeros proved the following results.

Theorem D. [19] *Let $f(z)$ be a transcendental meromorphic (resp. entire) function of zero order, $q(\neq 0)$ and c be complex constants, and k be a positive integer, and let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a non-constant polynomial with constant co-efficients $a_0, a_1, \ldots, a_{n-1}, a_n \neq 0$, and m be the number of distinct zeros of $P(z)$. Then for $n \geq m(k+1)+5$ (resp. $n \geq m(k+1)+3$), then $[P(f(z))f(qz + c)]^{(k)} - a(z)$ has infinitely many zeros, where $a(z) \not\equiv 0$ is a small function of f.*

Theorem E. [19] *Let $f(z)$ and $g(z)$ be two transcendental meromorphic (resp. entire) functions of zero order, $q(\neq 0)$ and c be complex constants, k be a positive integer, $a(z) \not\equiv 0$ be a meromorphic (resp. entire) small function and let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a non-constant polynomial with constant co-efficients $a_0, a_1, \ldots, a_{n-1}, a_n(\neq 0)$, and m be the number of distinct zeros of $P(z)$. If $n \geq 2m(k + 1) + 2k + 6$ (resp. $n \geq 2m(k + 1) + 4$) and $[P(f(z))f(qz + c)]^{(k)}$ and $[P(g(z))g(qz + c)]^{(k)}$ share $a(z)$, ∞ CM, then one of the following two results holds:*

(1) $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = LCM\{\lambda_j : j = 0, 1, \ldots, n\}$ denotes the lowest common multiple of $\lambda_j (j = 0, 1, \ldots, n)$ and

$$\lambda_j = \begin{cases} j+1, & if\, a_j \neq 0, \\ n+1, & if\, a_j = 0. \end{cases}$$

(2) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where

$$R(\omega_1, \omega_2) = P(\omega_1)\omega_1(qz + c) - P(\omega_2)\omega_2(qz + c).$$

Theorem F. [19] *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order, $q(\neq 0)$ and c be complex constants, k be a positive integer, $a(z) \not\equiv 0$ be a meromorphic small function and let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a non-constant polynomial with constant co-efficients $a_0, a_1, \ldots, a_{n-1}, a_n \neq 0$, and m be the number of distinct zeros of $P(z)$. If $n \geq 2m(k + 2) + 3m(k + 1) + 8k + 21$ and $[P(f(z))f(qz + c)]^{(k)}$ and $[P(g(z))g(qz + c)]^{(k)}$ share $a(z)$ IM, then either $[P(f(z))f(qz + c)]^{(k)} \cdot [P(g(z))g(qz + c)]^{(k)} \equiv a^2(z)$ or one of the conclusions of Theorem E holds.*

In 2019, Dyavanal and Desai [5] extended Theorems E and F to the q-difference operator $\Delta_q f(z) = f(qz + c) - f(z)$ and proved the following theorems.

Theorem G. [5] *Let $f(z)$ and $g(z)$ be two transcendental meromorphic (resp. entire) functions of zero order, such that $f(qz + c) - f(z) \not\equiv 0$ and $g(qz + c) - g(z) \not\equiv 0$, where q and c are non-zero complex constants, k, n, m are positive integers. Let $a(z)(\not\equiv 0)$ be a small function of $f(z)$ and $g(z)$. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a non-constant polynomial with constant co-efficients $a_0, a_1, \ldots, a_{n-1}, a_n(\neq 0)$ and m be the distinct zeros of $P(z)$. If $n > 2mk + 2m + 2k + 7$ (resp. $n > 2mk + 2m + 5$) and $[P(f(z))(f(qz+c) - f(z))]^{(k)}$ and $[P(g(z))(g(qz+c) - g(z))]^{(k)}$ share $a(z)$, ∞ CM, then one of the following two cases hold:*

(1) $f(z) \equiv tg(z)$ *for a constant* t *with* $t^d = 1$*, where* $d = LCM\{\lambda_j : j = 0, 1, ..., n\}$ *and*

$$\lambda_j = \begin{cases} j+1, & if\, a_j \neq 0, \\ n+1, & if\, a_j = 0. \end{cases}$$

(2) $f(z)$ *and* $g(z)$ *satisfy the algebraic equation* $R(f(z), g(z)) = 0$*, where*

$$R(\omega_1, \omega_2) = P(\omega_1)(\omega_1(qz+c) - \omega_1(z)) - P(\omega_2)(\omega_2(qz+c) - \omega_2(z)).$$

Theorem H. [5] *Let* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions of zero order, such that* $f(qz+c) - f(z) \not\equiv 0$ *and* $g(qz+c) - g(z) \not\equiv 0$*, where* q *and* c *are non-zero complex constants,* k*,* n*,* m *are positive integers. Let* $a(z)(\not\equiv 0)$ *be a small function of* $f(z)$ *and* $g(z)$*. Let* $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ *be a non-constant polynomial with constant co-efficients* $a_0, a_1, ..., a_{n-1}, a_n (\neq 0)$ *and* m *be the distinct zeros of* $P(z)$*. If* $n > 5mk + 7m + 8k + 25$ *and* $[P(f(z))(f(qz+c) - f(z))]^{(k)}$ *and* $[P(g(z))(g(qz+c) - g(z))]^{(k)}$ *share* $a(z)$ *IM, then either* $[P(f(z))(f(qz+c) - f(z))]^{(k)} \cdot [P(g(z))(g(qz+c) - g(z))]^{(k)} \equiv a^2$ *or one of the conclusions of Theorem G holds.*

Now, let us recall the definitions of q-shift and q-difference operator of a meromorphic function f.

Definition 1.4. For a meromorphic function f and c, $q(\neq 0) \in \mathbb{C}$, let us denote its q-shift, $E_{q,c}f$ and q-difference operator, $\Delta_{q,c}f$ respectively by $E_{q,c}f(z) = f(qz+c)$ and $\Delta_{q,c}f(z) = f(qz+c) - f(z)$, $\Delta^k_{q,c}f(z) = \Delta^{k-1}_{q,c}(\Delta_{q,c}f(z))$, for all $k \in \mathbb{N} - \{1\}$.

In 2021, Haldar [8] defined the linear q-shift and linear q-difference operators denoted respectively by $L_k(f, E_q)$ and $L_k(f, \Delta)$, for a non-constant meromorphic function f in a generalized way as follows

Definition 1.5. [8] *Let us define,*

$$L_k(f, E_q) = a_k f(q_k z + c_k) + a_{k-1} f(q_{k-1} z + c_{k-1}) + ... + a_0 f(q_0 z + c_0)$$

and

$$L_k(f, \Delta) = a_k \Delta_{q_k, c_k} f(z) + a_{k-1} \Delta_{q_{k-1}, c_{k-1}} f(z) + ... + a_0 \Delta_{q_0, c_0} f(z),$$

where $a_0, a_1, ..., a_k; q_0, q_1, ..., q_k; c_0, c_1, ..., c_k$ *are complex constants.*

From above definition one can easily observe that

$$L_k(f, \Delta) = L_k(f, E_q) - \sum_{j=0}^{k} a_j f(z).$$

If we choose $q_j = q^j$, $c_j = c$ and $a_j = (-1)^{k-j}\binom{k}{j}$ for $0 \leq j \leq k$, then $L_k(f, \Delta)$ reduces $\Delta^k_{q,c}f(z)$.

Now, it would be interesting to ask, *(i)* if the q-shift $f(qz+c)$ in Theorems D, E, F and the q-difference operator $\Delta_q f$=$f(qz+c) - f(z)$ in Theorems G, H can be extended any further and *(ii)* what happens if we consider an intermediate sharing, between CM and IM sharing? In this paper, we try to solve these interesting questions by considering linear q-difference polynomial $L_k(f, E_q)$ and its q-difference operator $L_k(f, \Delta)$, as defined in Definition 1.5, and we obtain Theorems 1.1, 1.2 and 1.3, which extends and generalizes the Theorems D, E and F respectively. Also, by considering the concept of weighted sharing introduced by Lahiri [11], we obtain Theorem 1.4 which answers the question *(ii)* and is an improvement of Theorem H.

The following are our main results.

Theorem 1.1. *Let $f(z)$ be a zero order transcendental meromorphic (resp. entire) function. Let q_i and c_i $(i = 0, 1, ..., k)$ be complex constants, k, n, m, l be positive integers and let $P(z) = b_n z^n + b_{n-1} z^{n-1} + ... + b_1 z + b_0$ be a non-constant polynomial with constant co-efficients $b_0, b_1, ..., b_{n-1}, b_n (\neq 0)$, and m be the number of distinct zeros of $P(z)$. Then for $n \geq 3k + m(l+1) + 5$ (resp. $n \geq 2k + m(l+1) + 3$), $[P(f(z))L_k(f, E_q)]^{(l)} - a(z)$ has infinitely many zeros, where $a(z)(\not\equiv 0)$ is a small function of f.*

Remark 1.1. In Theorem 1.1, if we take $k = 0$ then $L_k(f, E_q) = (q_0 f + c_0)$ and, we get $n \geq m(l+1) + 5$ (resp. $n \geq m(l+1) + 3$, for entire function) and hence Theorem 1.1 reduces to Theorem D.

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic (resp. entire) functions of zero order. Let q_i and c_i $(i = 0, 1, ..., k)$ be complex constants, k, n, m, l be positive integers, $a(z)(\not\equiv 0)$ be a meromorphic (resp. entire) small function of $f(z)$ and $g(z)$. Let $P(z) = b_n z^n + b_{n-1} z^{n-1} + ... + b_1 z + b_0$ be a non-constant polynomial with constant co-efficients $b_0, b_1, ..., b_{n-1}, b_n (\neq 0)$, and m be the number of distinct zeros of $P(z)$. Suppose $[P(f(z)) L_k(f, \Delta)]^{(l)}$ and $[P(g(z))L_k(f, \Delta)]^{(l)}$ share $a(z), \infty$ CM, and $n > 2ml + 2m + (k+2)(l+4) - 1$ (resp. $n > 2ml + 2m + 3k + 5$) then one of the following two cases hold:*

(1) $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = LCM\{\lambda_j : j = 0, 1, ..., n\}$ denotes the lowest common multiple of $\lambda_j (j = 0, 1, ..., n)$ and

$$\lambda_j = \begin{cases} j+1, & if\, a_j \neq 0, \\ n+1, & if\, a_j = 0. \end{cases}$$

(2) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where

$$R(\omega_1, \omega_2) = P(\omega_1)L_k(\omega_1, \Delta) - P(\omega_2)L_k(\omega_2, \Delta).$$

Remark 1.2. In Theorem 1.2, if we take $k = 0$ then $L_k(f, \Delta) = (q_0 f + c_0) - f(z)$ and, we get $n > 2ml + 2m + 2l + 7$ (resp. $n > 2ml + 2m + 5$, for entire functions) and hence Theorem 1.2 reduces to Theorem G.

Theorem 1.3. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic (resp. entire) functions of zero order. Let q_i and c_i $(i = 0, 1, ..., k)$ be complex constants, k, n, m, l be positive integers, $a(z)(\not\equiv 0)$ be a meromorphic (resp. entire) small function of $f(z)$ and $g(z)$. Let $P(z) = b_n z^n + b_{n-1} z^{n-1} + ... + b_1 z + b_0$ be a non-constant polynomial with constant co-efficients $b_0, b_1, ..., b_{n-1}, b_n (\neq 0)$, and m be the number of distinct zeros of $P(z)$. Suppose $[P(f(z)) L_k(f, \Delta)]^{(l)}$ and $[P(g(z))L_k(f, \Delta)]^{(l)}$ share $a(z)$ IM and $n > (5l + 7)m + (4l + 13)(k + 2) - 1$ (resp. $n > 5ml + 6k + 7m + 11$) then either $[P(f(z))L_k(f, \Delta)]^{(l)} \cdot [P(g(z))L_k(g, \Delta)]^{(l)} \equiv a^2$ or one of the conclusions of Theorem 1.2 holds.*

Remark 1.3. In Theorem 1.3, if we take $k = 0$ then $L_k(f, \Delta) = (q_0 f + c_0) - f(z)$ and, we get $n > 5ml + 7m + 8l + 25$ (resp. $n > 5ml + 7m + 11$, for entire functions) and hence Theorem 1.3 reduces to Theorem H.

As a particular case of the above Theorems 1.2 and 1.3, we deduce the following corollaries.

Corollary 1.1. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order. Let q_i and c_i $(i = 0, 1, ..., k)$ be complex constants, k, n, m, l be positive integers, $a(z)(\not\equiv 0)$ be a small function of $f(z)$ and $g(z)$, and α a complex constant. If $n > 2l + (k+2)(l+4) + 1$ and $[(f-\alpha)^n L_k(f, \Delta)]^{(l)}$ and $[(g-\alpha)^n L_k(g, \Delta)]^{(l)}$ share $a(z), \infty$ CM, then one of the following two cases holds:*

(1) $f \equiv tg$ for a constant t with $t^{n+1} = 1$,

(2) f and g satisfy the algebraic equation $R(f,g)=0$, where

$$R(\omega_1,\omega_2)=(\omega_1-\alpha)^n L_k(\omega_1,\Delta)-(\omega_2-\alpha)^n L_k(\omega_2,\Delta).$$

Corollary 1.2. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order. Let q_i and c_i $(i=0,1,...,k)$ be complex constants, k,n,m,l be positive integers, $a(z)(\not\equiv 0)$ be a small function of $f(z)$ and $g(z)$, and α be a complex constant. If $n>(4l+13)(k+2)+(5l+7)-1$ and $[(f-\alpha)^n L_k(f,\Delta)]^{(l)}$ and $[(g-\alpha)^n L_k(g,\Delta)]^{(l)}$ share $a(z)$ IM, then one of the following three cases holds:*

(1) $[(f-\alpha)^n L_k(f,\Delta)]^{(l)}\cdot[(g-\alpha)^n L_k(g,\Delta)]^{(l)}\equiv a^2$
(2) $f\equiv tg$ for a constant t with $t^{n+1}=1$,
(3) f and g satisfy the algebraic equation $R(f,g)=0$, where

$$R(\omega_1,\omega_2)=(\omega_1-\alpha)^n L_k(\omega_1,\Delta)-(\omega_2-\alpha)^n L_k(\omega_2,\Delta).$$

Theorem 1.4. *Let $f(z)$ and $g(z)$ be two non-constant zero order meromorphic functions, $a(z)$ be a small function with respect to both f and g and n be a positive integer. Let $P(z)=b_nz^n+b_{n-1}z^{n-1}+...+b_1z+b_0$ be a non-constant polynomial with constant co-efficients $b_0,b_1,...,b_{n-1}$, $b_n(\neq 0)$, and m be the number of distinct zeros of $P(z)$. Suppose if $[P(f(z))L_k(f,\Delta)]^{(l)}$ and $[P(g(z))L_k(f,\Delta)]^{(l)}$ share $(a(z),w)$, where w is a non-negative integer; f and g share (∞,v), where $0\le v\le\infty$ and if n satisfies one of the following conditions:*

(i) $n>2m(l+2)+(l+6)(k+2)-1$, when $w\ge 2$ and $0\le v\le\infty$,
(ii) $n>\frac{5ml}{2}+\frac{9m}{2}+(3l+14)\left(1+\frac{k}{2}\right)-1$, when $w=1$ and $v=0$,
(iii) $n>5ml+7m+4kl+12k+8l+23$, when $w=0$ and $v=0$,

then either $[P(f(z))L_k(f,\Delta)]^{(l)}\cdot[P(g(z))L_k(g,\Delta)]^{(l)}\equiv a^2$ or one of the conclusions of Theorem 1.2 holds.

Remark 1.4. In Theorem 1.4, if we take $k=0$ then $L_k(f,\Delta)=(q_0f+c_0)-f(z)$ and, we get

(i) $n>2ml+4m+2l+11$, when $w\ge 2$ and $0\le v\le\infty$
(ii) $n>\frac{5ml}{2}+\frac{9m}{2}+3l+13$, when $w=1$ and $v=0$
(iii) $n>5ml+7m+8l+23$, when $w=0$ and $v=0$.

If we consider the condition *(iii)*, it is definitely an improvement of Theorem H.

2. LEMMAS

This section provides all the necessary lemmas required to prove the theorems.
Let us define,

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right) \tag{2.1}$$

Lemma 2.1. [22] *Let f be a non-constant meromorphic function. Then*

$$T(r,P_n(f))=nT(r,f)+S(r,f).$$

Lemma 2.2. [21] *Let $f(z)$ be a non-constant meromorphic function of zero order and let q, η be two non-zero complex constants. Then on a set of lower logarithmic density 1, we have*

$$T(r,f(qz+\eta))=T(r,f)+S(r,f).$$

Lemma 2.3. [21] *Let $f(z)$ be a non-constant meromorphic function of zero order and let q, η be two non-zero complex constants. Then on a set of lower logarithmic density 1, we have*

$$N(r, f(qz+\eta)) = N(r, f) + S(r, f),$$

$$N\left(r, \frac{1}{f(qz+\eta)}\right) = N\left(r, \frac{1}{f}\right) + S(r, f).$$

Lemma 2.4. [16] *Let $f(z)$ be a non-constant zero order meromorphic function and let q, η be two non-zero complex numbers. Then on a set of logarithmic density 1, we have*

$$m\left(r, \frac{f(qz+\eta)}{f(z)}\right) = S(r, f).$$

Lemma 2.5. [24] *Let $f(z)$ and $g(z)$ be non-constant meromorphic functions, and let $a(z)$ ($\not\equiv 0, \infty$) be a small function of $f(z)$ and $g(z)$. If $f(z)$ and $g(z)$ share $a(z)$ IM, then one of the following three cases holds :*

(i)

$$\begin{aligned} T(r,f) \leq & N_2\left(r, \frac{1}{f}\right) + N_2(r,f) + N_2\left(r, \frac{1}{g}\right) + 2\left(\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r,f)\right) \\ & + N_2(r,g) + \left(\overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r,g)\right) + S(r,f) + S(r,g) \end{aligned}$$

and a similar inequality holds for $T(r,g)$,

(ii) $fg \equiv 1$,

(iii) $f \equiv g$.

Lemma 2.6. [13] *Let $f(z)$ be a non-constant meromorphic function and let p, k be positive integers. Then*

$$\begin{aligned} N_p\left(r, \frac{1}{f^{(k)}}\right) &\leq T(r, f^{(k)}) - T(r,f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r,f), \\ N_p\left(r, \frac{1}{f^{(k)}}\right) &\leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f). \end{aligned}$$

Lemma 2.7. [1] *Let F, G be two non-constant meromorphic function. If F, G share $(1,2)$ and (∞, k), where $0 \leq k \leq \infty$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,F) \leq & N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) \\ & + S(r,F) + S(r,G), \end{aligned}$$

where $\overline{N}_*(r,\infty;F,G)$ denotes the reduced counting function of those poles of F whose multiplicities differ from the multiplicities of the corresponding poles of G.

Lemma 2.8. [18] *Let F, G be two non-constant meromorphic functions sharing (1,1), $(\infty, 0)$ and $H \not\equiv 0$. Then*

$$\begin{aligned}
(i)\quad T(r,F) \le\ & N_2(r,0;F) + N_2(r,0;G) + \frac{3}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) \\
& + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G); \\
(ii)\quad T(r,G) \le\ & N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \frac{3}{2}\overline{N}(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;G) \\
& + \overline{N}_*(r,\infty;G,F) + S(r,F) + S(r,G).
\end{aligned}$$

Lemma 2.9. [18] *Let F, G be two non-constant meromorphic functions sharing (1,0), $(\infty, 0)$ and $H \not\equiv 0$. Then*

$$\begin{aligned}
(i)\quad T(r,F) \le\ & N_2(r,0;F) + N_2(r,0;G) + 3\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + 2\overline{N}(r,0;F) \\
& + \overline{N}(r,0;G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G); \\
(ii)\quad T(r,G) \le\ & N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,\infty;F) + 3\overline{N}(r,\infty;G) + \overline{N}(r,0;F) \\
& + 2\overline{N}(r,0;G) + \overline{N}_*(r,\infty;G,F) + S(r,F) + S(r,G).
\end{aligned}$$

Lemma 2.10. *Let $f(z)$ be a transcendental meromorphic function of zero order and $P(f) = a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0$. Let $F_1 = P(f)L_k(f,\Delta)$, where n is a positive integer. Then $(n-k-1)T(r,f) + S(r,f) \le T(r,F_1)$.*

Proof. From First Fundamental Theorem, Lemma 2.1 and 2.4, we obtain

$$\begin{aligned}
(n+1)T(r,f) &= T(r, f(z)P(f(z))) + S(r,f) \\
&\le T\left(r, \frac{f(z).F_1}{L_k(f,\Delta)}\right) + S(r,f) \\
&\le T(r,F_1) + T\left(r, \frac{L_k(f,\Delta)}{f(z)}\right) + S(r,f) \\
&\le T(r,F_1) + T\left(r, \frac{\sum_{j=0}^{k} a_j f(q_j z + c_j) - \sum_{j=0}^{k} a_j f(z))}{f(z)}\right) + S(r,f) \\
&\le T(r,F_1) + \sum_{j=0}^{k} T\left(r, \frac{a_j f(q_j z + c_j)}{f(z)}\right) + S(r,f) \\
&\le T(r,F_1) + \sum_{j=0}^{k} m\left(r, \frac{a_j f(q_j z + c_j)}{f(z)}\right) + \sum_{j=0}^{k} N\left(r, \frac{a_j f(q_j z + c_j)}{f(z)}\right) + S(r,f)
\end{aligned}$$

$(n+1)T(r,f) \le T(r,F_1) + (k+2)T(r,f) + S(r,f)$.

Thus, $(n-k-1)T(r,f) + S(r,f) \le T(r,F_1)$ on a set of logarithmic density 1.
This completes the proof of Lemma 2.10. □

3. Proof of Theorems

Let us define the following terms

$\mathcal{F}_1(z) = P(f(z))L_k(f,E_q)$ and $\mathcal{F}(z) = [P(f(z))L_k(f,E_q)]^{(l)}$,
$F_1(z) = P(f(z))L_k(f,\Delta)$ and $F(z) = [P(f(z))L_k(f,\Delta)]^{(l)}$,
$G_1(z) = P(g(z))L_k(g,\Delta)$ and $G(z) = [P(g(z))L_k(g,\Delta)]^{(l)}$.

3.1. **Proof of Theorem 1.1.**

Proof. We say that $\mathcal{F}(z) - a(z)$ has infinitely many zeros.

On the contrary, let us assume that $\mathcal{F}(z) - a(z)$ has either no or finitely many zeros. Then, from Lemma 2.4 and First Fundamental Theorem, we have,

$$\begin{aligned}
nT(r,f) &= T(r,P(f)) \\
&= T\left(r, \frac{\mathcal{F}_1}{L_k(f,E_q)}\right) \\
&\leq T(r,\mathcal{F}_1) + T\left(r, \frac{1}{L_k(f,E_q)}\right) + S(r,f) \\
&\leq T(r,\mathcal{F}_1) + (k+1)T(r,f) + S(r,f) \\
(n-k-1)T(r,f) + S(r,f) &\leq T(r,\mathcal{F}_1).
\end{aligned}$$

By Second Fundamental Theorem for small functions and Lemma 2.6, we have

$$\begin{aligned}
T(r,\mathcal{F}) &\leq \overline{N}(r,\mathcal{F}) + \overline{N}\left(r, \frac{1}{\mathcal{F}}\right) + \overline{N}\left(r, \frac{1}{\mathcal{F} - a(z)}\right) + S(r,f) \\
T(r,\mathcal{F}) &\leq \overline{N}(r,\mathcal{F}_1) + T(r,\mathcal{F}) - T(r,\mathcal{F}_1) + N_{l+1}\left(r, \frac{1}{\mathcal{F}_1}\right) + \overline{N}\left(r, \frac{1}{\mathcal{F} - a(z)}\right) + S(r,f) \\
T(r,\mathcal{F}_1) &\leq \overline{N}(r,P(f)) + \overline{N}(r,L_k(f,E_q)) + (l+1)\overline{N}\left(r, \frac{1}{P(f)}\right) + N\left(r, \frac{1}{L_k(f,E_q)}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{\mathcal{F} - a(z)}\right) + S(r,f)
\end{aligned}$$

$$(n-k-1)T(r,f) + S(r,f) \leq (k+2)T(r,f) + (l+1)mT(r,f) + (k+1)T(r,f) + S(r,f)$$

$$(n-3k-4-m(l+1))T(r,f) \leq S(r,f),$$

which contradicts $n \geq 3k + m(l+1) + 5$.

Similarly, we can prove the result for the entire functions using $N(r,f) = \overline{N}(r,f) = S(r,f)$, $N(r,g) = \overline{N}(r,g) = S(r,g)$ and we will get $n \geq 2k + m(l+1) + 3$.

This completes the proof of Theorem 1.1. □

3.2. **Proof of Theorem 1.2.**

Proof. Since F and G share $a(z)$, ∞ CM, there exists a non-zero constant β such that,

$$\frac{[P(f(z))L_k(f,\Delta)]^{(l)}/a(z) - 1}{[P(g(z))L_k(g,\Delta)]^{(l)}/a(z) - 1} = \beta \tag{3.1}$$

and we get,

$$F - a(z)(1-\beta) = \beta G.$$

Now, we will prove that $\beta = 1$

On the contrary, let us assume $\beta \neq 1$. Using the Second Fundamental Theorem and by Lemma 2.6, we get

$$\begin{aligned}
T(r,F) &\leq \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-a(z)(1-\beta)}\right) + S(r,f)\\
&\leq \overline{N}(r,F_1) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right) + S(r,f)\\
&\leq \overline{N}(r,F_1) + T(r,F) - T(r,F_1) + N_{l+1}\left(r,\frac{1}{F_1}\right) + l\overline{N}(r,G_1)\\
&\quad + N_{l+1}\left(r,\frac{1}{G_1}\right) + S(r,f) + S(r,g),
\end{aligned}$$

which implies,

$$\begin{aligned}
T(r,F_1) &\leq \overline{N}(r,F_1) + N_{l+1}\left(r,\frac{1}{F_1}\right) + l\overline{N}(r,G_1) + N_{l+1}\left(r,\frac{1}{G_1}\right) + S(r,f) + S(r,g)\\
&\leq T(r,f) + (k+1)T(r,f) + m(l+1)T(r,f) + (k+2)T(r,f) + lT(r,g)\\
&\quad + l(k+1)T(r,g) + m(l+1)T(r,g) + (k+2)T(r,g) + S(r,f) + S(r,g)
\end{aligned}$$

$$\begin{aligned}
(n-k-1)T(r,f) + S(r,f) &\leq (ml+m+2(k+2))T(r,f)\\
&\quad + (ml+m+(k+2)(l+1)T(r,g) + S(r,f) + S(r,g). \quad (3.2)
\end{aligned}$$

Similarly,

$$\begin{aligned}
(n-k-1)T(r,g) &\leq (ml+m+2(k+2))T(r,g) + (ml+m+(l+1)(k+2))T(r,f)\\
&\quad + S(r,f) + S(r,g). \quad (3.3)
\end{aligned}$$

From (3.2) and (3.3), we get

$$(n-k-1)\{T(r,f)+T(r,g)\} \leq (2ml+2m+(k+2)(l+3))\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)$$
$$(n-2ml-2m-(k+2)(l+4)+1)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g).$$

This is a contradicition to $n > 2ml+2m+(k+2)(l+4)-1$.

Thus, we get $\beta = 1$. Hence from (3.1), we have

$$[P(f(z))L_k(f,\Delta)]^{(l)} = [P(g(z))L_k(g,\Delta)]^{(l)}, \quad (3.4)$$

up on integrating the above equation for l times, we get

$$[P(f(z))L_k(f,\Delta)] = [P(g(z))L_k(g,\Delta)] + r(z), \quad (3.5)$$

where $r(z)$ is a polynomial of degree at most $(l-1)$. Suppose $r(z) \not\equiv 0$, then we get

$$\frac{P(f(z))L_k(f,\Delta)}{r(z)} = \frac{P(g(z))L_k(g,\Delta)}{r(z)} + 1. \quad (3.6)$$

Therefore, from Lemma 2.10 and the Second Fundamental Theorem, we have

$$\begin{aligned}
(n-k-1)T(r,f) &\leq T\left(r,\frac{P(f(z))L_k(f,\Delta)}{r(z)}\right) + S(r,f)\\
&\leq \overline{N}\left(r,\frac{P(f(z))L_k(f,\Delta)}{r(z)}\right) + \overline{N}\left(r,\frac{r(z)}{P(f(z))L_k(f,\Delta)}\right)\\
&\quad + \overline{N}\left(r,\frac{r(z)}{P(g(z))L_k(g,\Delta)}\right) + S(r,f)
\end{aligned}$$

$$(n-k-1)T(r,f) \leq T(r,f)+(k+1)T(r,f)+mT(r,f)+(k+2)T(r,f) \\ +mT(r,g)+(k+2)T(r,g)+S(r,f)+S(r,g)$$

$$(n-k-1)T(r,f) \leq (m+k+2)\{T(r,f)+T(r,g)\}+(k+2)T(r,f) \\ +S(r,f)+S(r,g). \tag{3.7}$$

Similarly, we have

$$(n-k-1)T(r,g) \leq (m+k+2)\{T(r,f)+T(r,g)\}+(k+2)T(r,g) \\ +S(r,f)+S(r,g). \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$(n-k-1)\{T(r,f)+T(r,g)\} \leq (2m+2k+4)\{T(r,f)+T(r,g)\} \\ +(k+2)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)$$

$$(n-2m-4k-7)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g).$$

This is a contradiction to $n > 2ml+2m+(k+2)(l+4)-1$. Therefore $r(z) \equiv 0$. Hence, (3.5) becomes

$$P(f(z))L_k(f,\Delta) = P(g(z))L_k(g,\Delta). \tag{3.9}$$

i.e.,

$$(b_nf^n+b_{n-1}f^{n-1}+...+b_1f+b_0)\{a_k[f(q_kz+c_k)-f(z)]+a_{k-1}[f(q_{k-1}z+c_k) \\ -f(z)]+...+a_1[f(q_1z+c_1)-f(z)]+a_0[f(q_0z+c_0)-f(z)]\}= \\ (b_ng^n+b_{n-1}g^{n-1}+...+g_1f+b_0)\{a_k[g(q_kz+c_k)-g(z)]+a_{k-1}[g(q_{k-1}z+c_k) \\ -g(z)]+...+a_1[g(q_1z+c_1)-g(z)]+a_0[g(q_0z+c_0)-g(z)]\}$$

Let $h=\frac{f}{g}$, we consider the following cases

Case 1. If $h(z)$ is a constant, then substituting $f=gh$ in the above equation, we have

$$(b_ng^nh^n+b_{n-1}g^{n-1}h^{n-1}+...+b_1gh+b_0)\{a_kh[g(q_kz+c_k)-g(z)]+a_{k-1}h[g(q_{k-1}z+c_k) \\ -g(z)]+...+a_1h[g(q_1z+c_1)-g(z)]+a_0h[g(q_0z+c_0)-g(z)]\}= \\ (b_ng^n+b_{n-1}g^{n-1}+...+g_1f+b_0)\{a_k[g(q_kz+c_k)-g(z)]+a_{k-1}[g(q_{k-1}z+c_k)-g(z)] \\ +...+a_1[g(q_1z+c_1)-g(z)]+a_0[g(q_0z+c_0)-g(z)]\}$$

which implies,

$$b_ng^n[L_k(g,\Delta)](h^{n+1}-1)+b_{n-1}g^{n-1}[L_k(g,\Delta)](h^n-1)+... \\ +b_1g[L_k(g,\Delta)](h^2-1)+b_0[L_k(g,\Delta)](h-1)=0.$$

This implies $h^d=1$, where $d=LCM\{\lambda_j : j=0,1,...,n\}$ and

$$\lambda_j = \begin{cases} j+1, & if\, a_j \neq 0, \\ n+1, & if\, a_j = 0. \end{cases}$$

Thus, $f \equiv tg$, where t is a constant with $t^d=1$, where $d=LCM\{\lambda_j : j=0,1,...,n\}$ and

$$\lambda_j = \begin{cases} j+1, & if\, a_j \neq 0, \\ n+1, & if\, a_j = 0. \end{cases}$$

Case 2. Suppose $h(z)$ is not a constant, then f and g satisfy the algebraic equation $R(f,g)=0$, where

$$R(\omega_1,\omega_2) = P(\omega_1)(L_k(\omega_1,\Delta)) - P(\omega_2)(L_k(\omega_2,\Delta)).$$

Similarly, we can prove the result for the entire functions using $N(r,f) = \overline{N}(r,f) = S(r,f)$, $N(r,g) = \overline{N}(r,g) = S(r,g)$.

This completes the proof of Theorem 1.2. □

3.3. **Proof of Theorem 1.3.**

Proof. We see that F and G share $a(z)$ IM. If *(i)* of Lemma 2.5 holds, then using Lemma 2.10, we obtain

$$\begin{aligned}
T(r,F) \le{}& N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G) + 2\left(\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F)\right)\\
&+\left(\overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G)\right) + S(r,f) + S(r,g)\\
T(r,F) \le{}& N_2(r,F) + T(r,F) - T(r,F_1) + N_{l+2}\left(r,\frac{1}{F_1}\right) + N_{l+2}\left(r,\frac{1}{G_1}\right) + l\overline{N}(r,G_1)\\
&+ N_2(r,G) + 2\left(N_{l+1}\left(r,\frac{1}{F_1}\right) + l\overline{N}(r,F_1) + \overline{N}(r,F)\right) + N_{l+1}\left(r,\frac{1}{G_1}\right)\\
&+ l\overline{N}(r,G_1) + \overline{N}(r,G) + S(r,f) + S(r,g),
\end{aligned}$$

which implies,

$$\begin{aligned}
T(r,F_1) \le{}& 2(k+2)T(r,f) + m(l+2)T(r,f) + (k+2)T(r,f) + m(l+2)T(r,g)\\
&+ (k+2)T(r,g) + l(k+2)T(r,g) + 2(k+2)T(r,g) + 2m(l+1)T(r,f)\\
&+ 2(k+2)T(r,f) + 2l(k+2)T(r,f) + 2(k+2)T(r,f) + m(l+1)T(r,g)\\
&+ (k+2)T(r,g) + l(k+2)T(r,g) + (k+2)T(r,g) + S(r,f) + S(r,g)\\
\le{}& \{(2l+7)(k+2) + (3l+4)m\}T(r,f) + \{(2l+5)(k+2) + (2l+3)m\}T(r,g)\\
&+ S(r,f) + S(r,g).
\end{aligned}$$

From Lemma 2.10, we have

$$\begin{aligned}
(n-k-1)T(r,f) \le{}& \{(2l+7)(k+2) + (3l+4)m\}T(r,f)\\
&+ \{(2l+5)(k+2) + (2l+3)m\}T(r,g) + S(r,f) + S(r,g). \qquad (3.10)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(n-k-1)T(r,g) \le{}& \{(2l+7)(k+2) + (3l+4)m\}T(r,g)\\
&+ \{(2l+5)(k+2) + (2l+3)m\}T(r,f) + S(r,f) + S(r,g). \qquad (3.11)
\end{aligned}$$

Combining the above two inequalities (3.10) and (3.11), we get

$$\begin{aligned}
(n-k-1)T(r,f) + T(r,g) \le{}& \{(4l+12)(k+2) + (5l+7)m\}\{T(r,f) + T(r,g)\}\\
&+ S(r,f) + S(r,g).
\end{aligned}$$

Thus, we get

$$\{n - (4l+13)(k+2) - (5l+7)m + 1\}\{T(r,f) + T(r,g)\} \le S(r,f) + S(r,g).$$

This is a contradiction to $n > (4l+13)(k+2) + (5l+7)m - 1$.

Thus by Lemma 2.5, we have either $FG \equiv a^2(z)$ or $F = G$.

Case 3. Suppose $FG \equiv a^2(z)$ then,

$$[P(f)L_k(f,\Delta)]^{(l)} \cdot [P(g)L_k(g,\Delta)]^{(l)} \equiv a^2(z).$$

This is one of the conclusions of Theorem 1.3.

Case 4. Suppose $F = G$, then by (3.9), using a similar argument as in Theorem 1.2, one of the conclusions of Theorem 1.2 holds.
Similarly, we can prove the result for the entire functions using $N(r,f) = \overline{N}(r,f) = S(r,f)$, $N(r,g) = \overline{N}(r,g) = S(r,g)$.

This completes the proof of Theorem 1.3. □

3.4. **Proof of Theorem 1.4.**

Proof. Let $F^* = \dfrac{F}{a(z)}, \quad G^* = \dfrac{G}{a(z)}$ and $H = \left(\dfrac{F^{*\prime\prime}}{F^{*\prime}} - \dfrac{2F^{*\prime}}{F^*-1}\right) - \left(\dfrac{G^{*\prime\prime}}{G^{*\prime}} - \dfrac{2G^{*\prime}}{G^*-1}\right)$.

From the hypothesis we have $F(z)$ and $G(z)$ share $(a(z), w)$ and f, g share (∞, v). It follows that F^* and G^* share $(1, w)$ and (∞, v). We now discuss the following two cases separately.

Case 5. We assume that $H \not\equiv 0$. Now we consider the following three subcases.

Subcase 1. Suppose that $w \geq 2$ and $0 \leq v \leq \infty$, then using Lemma 2.7, we obtain

$$\begin{aligned} T(r,F) &\leq T(r,F^*) + S(r,F^*) \\ &\leq N_2(r,0;F^*) + N_2(r,0;G^*) + \overline{N}(r,\infty;F^*) + \overline{N}(r,\infty;G^*) + \overline{N}_*(r,\infty;F^*,G^*) \\ &\quad + S(r,F^*) + S(r,G^*) \\ &\leq N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) \\ &\quad + S(r,F) + S(r,G). \end{aligned} \tag{3.12}$$

Noting that,

$$\begin{aligned} \overline{N}_*(r,\infty;F^*,G^*) &= \overline{N}_L(r,\infty;F) + \overline{N}_L(r,\infty;G) \\ &\leq \overline{N}(r,\infty;F) = \overline{N}(r,\infty;G). \end{aligned} \tag{3.13}$$

We obtain from (3.12) and (3.13) that

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,F) + S(r,G).$$

By using Lemma 2.6 and 2.10, we have

$$\begin{aligned} T(r,F) &\leq T(r,F) - T(r,F_1) + N_{l+2}(r,0;F_1) + l\overline{N}(r,\infty;G_1) + N_{l+2}(r,0;G_1) \\ &\quad + 2\overline{N}(r,\infty;F_1) + \overline{N}(r,\infty;G_1) + S(r,f) + S(r,g) \\ T(r,F_1) &\leq (l+2)mT(r,f) + (k+2)T(r,f) + lT(r,g) + l(k+1)T(r,g) + (l+2)mT(r,g) \\ &\quad + (k+2)T(r,g) + 2T(r,f) + 2(k+1)T(r,f) + T(r,g) + (k+1)T(r,g) \\ &\quad + S(r,f) + S(r,g) \end{aligned}$$

$$(n-k-1)T(r,f) \leq \{ml+2m+3k+6\}T(r,f) + \{ml+lk+2m+2k+2l+4\}T(r,g) + S(r,f) + S(r,g).$$

Similarly, we have for $T(r,g)$

$$(n-k-1)T(r,g) \leq \{ml+2m+3k+6\}T(r,g) + \{ml+lk+2m+2k+2l+4\}T(r,f) + S(r,f) + S(r,g).$$

Combining the above two inequalities, we get

$$(n-k-1)\{T(r,f)+T(r,g)\} \leq \{2ml+4m+5k+lk+2l+10\}\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g),$$

which contradicts $n > 2m(l+2) + (l+6)(k+2) - 1$

Subcase 2. Let $w = 1$ and $v = 0$, then using (3.13) and Lemma 2.8 we obtain

$$\begin{aligned} T(r,F) &\le T(r,F^*) + S(r,F*) \\ &\le N_2(r,0;F^*) + N_2(r,0;G^*) + \frac{3}{2}\overline{N}(r,\infty;F^*) + \overline{N}(r,\infty;G^*) + \overline{N}_*(r,\infty;F^*,G^*) \\ &\quad + \frac{1}{2}\overline{N}(r,0;F^*) + S(r,F^*) + S(r,G^*) \\ &\le N_2(r,0;F) + N_2(r,0;G) + \frac{3}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) \\ &\quad + \frac{1}{2}\overline{N}(r,0;F) + S(r,F) + S(r,G) \\ &\le N_2(r,0;F) + N_2(r,0;G) + \frac{5}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \frac{1}{2}\overline{N}(r,0;F) \\ &\quad + S(r,F) + S(r,G), \end{aligned}$$

using Lemma 2.6 and 2.10, we have

$$\begin{aligned} T(r,F) \le T(r,F) - T(r,F_1) + N_{l+2}(r,0;F_1) + l\overline{N}(r,\infty;G_1) + N_{l+2}(r,0;G_1) + \frac{5}{2}\overline{N}(r,\infty;F_1) \\ + \overline{N}(r,\infty;G_1) + \frac{1}{2}\{l\overline{N}(r,\infty;F_1) + N_{l+1}(r,0;F_1)\} + S(r,f) + S(r,g) \end{aligned}$$

$$\begin{aligned} (n-k-1)T(r,f) \le (l+2)mT(r,f) + (k+2)T(r,f) + lT(r,g) + l(k+1)T(r,g) \\ + (l+2)mT(r,g) + (k+2)T(r,g) + \frac{5}{2}T(r,f) + \frac{5}{2}(k+1)T(r,f) \\ + \frac{1}{2}\{lT(r,f) + l(k+1)T(r,f) + (l+1)mT(r,f) + (k+2)T(r,f)\} \\ + T(r,g) + (k+1)T(r,g) + S(r,f) + S(r,g) \end{aligned}$$

$$\begin{aligned} (n-k-1)T(r,f) \le \left(\frac{3ml}{2} + \frac{5m}{2} + 4k + l + \frac{lk}{2} + 8\right) T(r,f) \\ + (ml + 2m + 2k + 2l + lk + 4)T(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{3.14}$$

Similarly, we have for $T(r,g)$

$$\begin{aligned} (n-k-1)T(r,g) \le \left(\frac{3ml}{2} + \frac{5m}{2} + 4k + l + \frac{lk}{2} + 8\right) T(r,g) \\ + (ml + 2m + 2k + 2l + lk + 4)T(r,f) + S(r,f) + S(r,g). \end{aligned} \tag{3.15}$$

Now, by combining the above two inequalities (3.14) and (3.15), we get

$$\begin{aligned} (n-k-1)\{T(r,f)+T(r,g)\} \le \left(\frac{5ml}{2} + \frac{9m}{2} + 6k + 3l + \frac{3lk}{2} + 12\right)\{T(r,f)+T(r,g)\} \\ + S(r,f) + S(r,g), \end{aligned}$$

which contradicts $n > \frac{5ml}{2} + \frac{9m}{2} + (3l+14)\left(1 + \frac{k}{2}\right) - 1$

Subcase 3. Let $w = 0$ and $v = 0$, then using (3.13) and Lemma 2.9, we obtain

$$\begin{aligned} T(r,F) &\leq T(r,F^*) + S(r,F) \\ &\leq N_2(r,0;F^*) + N_2(r,0;G^*) + 3\overline{N}(r,\infty;F^*) + 2\overline{N}(r,\infty;G^*) + 2\overline{N}(r,0;F^*) \\ &\quad + \overline{N}_*(r,\infty;F^*,G^*) + \overline{N}(r,0;G^*) + S(r,F^*) + S(r,G^*) \\ &\leq N_2(r,0;F) + N_2(r,0;G) + 3\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + 2\overline{N}(r,0;F) \\ &\quad + \overline{N}_*(r,\infty;F,G) + \overline{N}(r,0;G) + S(r,F) + S(r,G) \\ &\leq N_2(r,0;F) + N_2(r,0;G) + 4\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + 2\overline{N}(r,0;F) \\ &\quad + \overline{N}(r,0;G) + S(r,F) + S(r,G). \end{aligned}$$

Using Lemma 2.6 and 2.10 we get,

$$\begin{aligned} T(r,F) \leq{} & T(r,F) - T(r,F_1) + N_{l+2}(r,0;F_1) + l\overline{N}(r,\infty;G_1) + N_{l+2}(r,0;G_1) + 4\overline{N}(r,\infty;F_1) \\ & + 2\overline{N}(r,\infty;G_1) + 2\{l\overline{N}(r,\infty;F_1) + N_{l+1}(r,0;F_1)\} + l\overline{N}(r,\infty;G_1) + N_{l+1}(r,0;G_1) \\ & + S(r,F) + S(r,G) \end{aligned}$$

$$\begin{aligned} (n-k-1)T(r,f) \leq{} & (l+2)mT(r,f) + (k+2)T(r,f) + lT(r,g) + l(k+1)T(r,g) \\ & + (l+2)mT(r,g) + (k+2)T(r,g) + 4T(r,f) + 4(k+1)T(r,f) \\ & + 2T(r,g) + 2(k+1)T(r,g) + 2lT(r,f) + 2l(k+1)T(r,f) \\ & + 2(l+1)mT(r,f) + 2(k+2)T(r,f) + lT(r,g) + l(k+1)T(r,g) \\ & + (l+1)mT(r,g) + (k+2)T(r,g) + S(r,f) + S(r,g) \end{aligned}$$

$$\begin{aligned} (n-k-1)T(r,f) \leq{} & \{3ml + 4m + 2lk + 7k + 4l + 14\}T(r,f) \\ & + \{2ml + 3m + 2lk + 4k + 4l + 8\}T(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{3.16}$$

Similarly, we get for $T(r,g)$,

$$\begin{aligned} (n-k-1)T(r,g) \leq{} & \{3ml + 4m + 2lk + 7k + 4l + 14\}T(r,g) \\ & + \{2ml + 3m + 2lk + 4k + 4l + 8\}T(r,f) + S(r,f) + S(r,g). \end{aligned} \tag{3.17}$$

Now, by combining the above two inequalities (3.16) and (3.17), we get

$$(n-k-1)\{T(r,f)+T(r,g)\} \leq (5ml+7m+4lk+11k+8l+22)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

which contradicts, $n > 5ml + 7m + 4lk + 12k + 8l + 23$.

Case 6. We now assume that $H \equiv 0$. Then

$$\left(\frac{F^{*\prime\prime}}{F^{*\prime}} - \frac{2F^{*\prime}}{F^* - 1}\right) - \left(\frac{G^{*\prime\prime}}{G^{*\prime}} - \frac{2G^{*\prime}}{G^* - 1}\right) = 0$$

Integrating both sides of the above equality twice we get,

$$\frac{1}{F^* - 1} = \frac{A}{G^* - 1} + B, \tag{3.18}$$

where $A(\neq 0)$ and B are constants. From (3.18) it is obvious that F^*, G^* share the value 1 CM and hence they share the value 1 with weight $w = 2$, and therefore, $n > 2m(l+2) + (l+6)(k+2) - 1$.

We now discuss the following three subcases separately.

Subcase 4. Suppose that $B \neq 0$ and $A = B$, then from (3.18), we obtain

$$\frac{1}{F^* - 1} = \frac{BG^*}{G^* - 1} \tag{3.19}$$

If $B = -1$, then from (3.19), we obtain

$$F^* G^* = 1,$$

i.e.,

$$[P(f)L_k(f, \Delta)]^{(l)} \cdot [P(g)L_k(g, \Delta)]^{(l)} = a^2,$$

which is one of the conclusions of Theorem 1.4.

If $B \neq -1$, then from (3.19), we have

$$\frac{1}{F^*} = \frac{BG^*}{(1+B)G^* - 1} \quad \text{and so} \quad \overline{N}\left(r, \frac{1}{1+B}; G^*\right) = \overline{N}(r, 0; F^*).$$

Using the Second Fundamental Theorem and Lemma 2.6, we have

$$\begin{aligned} T(r, G) &\leq T(r, G^*) + S(r, G) \\ &\leq \overline{N}(r, 0; G^*) + \overline{N}\left(r, \frac{1}{1+B}; G^*\right) + \overline{N}(r, \infty; G^*) + S(r, G) \\ &\leq \overline{N}(r, 0; F^*) + \overline{N}(r, 0; G^*) + \overline{N}(r, \infty; G^*) + S(r, G) \\ &\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + S(r, G). \end{aligned} \tag{3.20}$$

Using (3.20), Lemma 2.6 and Lemma 2.10, we have

$$\begin{aligned} T(r, G) \leq\ & l\overline{N}(r, \infty; F_1) + N_{l+1}(r, 0; F_1) + T(r, G) - T(r, G_1) + N_{l+1}(r, 0; G_1) \\ & + \overline{N}(r, \infty; G_1) + S(r, g) \end{aligned}$$

$$\begin{aligned} (n-k-1)T(r, g) \leq\ & lT(r, f) + l(k+1)T(r, f) + (l+1)mT(r, f) + (k+2)T(r, f) \\ & + (l+1)mT(r, g) + (k+2)T(r, g) + S(r, f) + S(r, g) \end{aligned}$$

$$(n-k-1)T(r, g) \leq (lk+ml+2l+m+k+2)T(r, f) + (ml+m+k+2)T(r, g) + S(r, f) + S(r, g).$$

Similarly, we have for $T(r, f)$

$$(n-k-1)T(r, f) \leq (lk+ml+2l+m+k+2)T(r, g) + (ml+m+k+2)T(r, f) + S(r, f) + S(r, g).$$

Thus by combining the above two inequalities, we get

$$(n-k-1)\{T(r, f)+T(r, g)\} \leq (2ml+lk+2l+2m+2k+4)\{T(r, f)+T(r, g)\} + S(r, f) + S(r, g)$$

$$(n - 2ml - lk - 2l - 2m - 3k - 5)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

which contradicts $n > 2m(l+2) + (l+6)(k+2) - 1$

Subcase 5. Let $B \neq 0$ and $A \neq B$. Then from (3.18), we get

$$F^* = \frac{(B+1)G^* - (B - A + 1)}{BG^* + (A - B)} \quad \text{and so} \quad \overline{N}\left(r, \frac{B - A + 1}{B + 1}; G^*\right) = \overline{N}(r, 0; F^*).$$

Proceeding in a manner similar to subcase 1, we can arrive at a contradiction.

Subcase 6. Let $B = 0$ and $A \neq 0$. Then from (3.18), we get

$$F^* = \frac{G + A - 1}{A} \quad \text{and} \quad G = AF - (A - 1).$$

If $A \neq 1$, it follows that,

$$\overline{N}\left(r, \frac{A-1}{A}; F^*\right) = \overline{N}(r, 0; G^*) \quad \text{and} \quad \overline{N}(r, 1-A; G^*) = \overline{N}(r, 0; F^*).$$

Using the similar argument as in subcase 1, we obtain a contradiction. Thus $A = 1$, which implies $F^* = G^*$, and therefore,

$$[P(f)L_k(f, \Delta)]^{(l)} = [P(g)L_k(g, \Delta)]^{(l)}.$$

Integrating the above equation for l times, we get

$$P(f)L_k(f, \Delta) = P(g)L_k(g, \Delta) + R(z),$$

where $R(z)$ is a polyomial of degree atmost $l-1$, then by a similar argument as in Theorem 1.2, we will obtain the required result.

This completes the proof of Theorem 1.4. □

4. Conclusions

Using the Nevanlinna theory, we have studied the value distribution and uniqueness of linear q-difference polynomial $L_k(f, E_q)$ and its q-difference operator $L_k(f, \Delta)$, for a transcendental meromorphic function f having zero order. Since $L_k(f, E_q)$ and $L_k(f, \Delta)$ are generalised forms of the q-shift $f(qz+c)$ and the q-difference operator $\Delta_q f(z)$ respectively, our results extends and generalizes the Theorems D, E, & F due to Thin and Theorems G & H due to Dyavanal and Desai. Also, by considering the concept of weighted sharing introduced by Lahiri, we have obtained Theorem 1.4 which is an intermediate of Theorems G & H due to Dyavanal and Desai.

Also, we can pose the following open questions.

Open questions:

1. What happens to Theorems 1.2-1.4, if we replace the linear q-difference operator $L_k(f, \Delta)$ by the non-linear product of q-difference $\prod_{j=1}^{k}(\Delta_{q,c}^{n} f)^{\mu_j}$, where n, μ_j are positive integers and q, c are non-zero complex constants?
2. Can the condition for the lower bound n in Theorems 1.1-1.4 be reduced any further?
3. What happens to the Theorems 1.1-1.4, if we study them using the concepts of weakly weighted sharing, truncated sharing which are weaker than weighted sharing?

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