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Double Roman Domination in Cartesian Product

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ABSTRACT. Given a graph $G = (V, E)$, a function $f : V \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then there exist $v_1, v_2 \in N(v)$ such that $f(v_1) = 2 = f(v_2)$ or there exists $w \in N(v)$ such that $f(w) = 3$, and if $f(v) = 1$, then there exists $w \in N(v)$ such that $f(w) \geq 2$ is called a double Roman dominating function (DRDF). The weight of a DRDF f is the sum $f(V) = \sum_{v \in V} f(v)$, and the minimum among the weights of DRDFs on G is the double Roman domination number, $\gamma_{dR}(G)$, of G. In this paper, we study the impact of cartesian product on the double Roman domination number.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. If there is no ambiguity in the choice of G, then we write $V(G)$ and $E(G)$ as V and E respectively. Let $f : V \to \{0, 1, 2, 3\}$ be a function defined on V. Let $V_i^f = \{v \in V : f(v) = i\}$. (If there is no ambiguity, V_i^f is written as V_i.) Then f is a double Roman dominating function (DRDF) on a graph G if it satisfies the following conditions.

(i) If $v \in V_0$, then vertex v must have at least two neighbors in V_2 or at least one neighbor in V_3.
(ii) If $v \in V_1$, then vertex v must have at least one neighbor in $V_2 \cup V_3$.

The weight of a DRDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The double Roman domination number, $\gamma_{dR}(G)$, is the minimum among the weights of DRDFs on G, and a DRDF on G with weight $\gamma_{dR}(G)$ is called a γ_{dR}-function of G [8].

Let (V_0, V_1, V_2, V_3) be the ordered partition of V induced by f. Note that there exists a $1 - 1$ correspondence between the functions f and the ordered partitions (V_0, V_1, V_2, V_3) of V. Thus we will write $f = (V_0, V_1, V_2, V_3)$.

R. A. Beeler, T. W. Haynes and S. T. Hedetniemi pioneered the study of double Roman domination in [8]. The relationship between double Roman domination and Roman domination and the bounds on the double Roman domination number of a graph G in terms of its domination number were discussed by them. They also determined a sharp upper bound on $\gamma_{dR}(G)$ in terms of the order of G and characterized the graphs attaining this bound. In [1], it was verified that the decision problem associated with $\gamma_{dR}(G)$ is NP-complete for bipartite and chordal graphs. A characterization of graphs with small double Roman domination number was also provided by them. In [9], G. Hao et al. introduced the study of the double Roman domination of digraphs and L. Volkmann proposed a sharp lower bound on $\gamma_{dR}(G)$ in [10]. In [4], it was proved that $\gamma_{dR}(G) + 2 \leqslant \gamma_{dR}(M(G)) \leqslant \gamma_{dR}(G) + 3$, where $M(G)$ is the Mycielskian graph of G. A

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construction which confirms that there is no relation between the double Roman domination number of a graph and its induced subgraphs was also given in [4]. The impact of some graph operations on double Roman domination number was studied in [5] and [6]. In [3], J. Amjadi et al. improved an upper bound on $\gamma_{dR}(G)$ given in [8] by showing that for any connected graph G of order n with minimum degree at least two, $\gamma_{dR}(G) \leqslant \frac{8n}{7}$.

In [5], it is proved that for any graphs G and H the double Roman domination nummber of $G\Box H$ is at least $\frac{\gamma(G)\gamma_{dR}(H)}{2}$. In this paper we improve this lowerbound as $\gamma(G)\gamma_{dR}(H)$ for all graphs G having an efficient dominating set. The exact value of double Roman domination number of $P_2\Box P_n$ was found in [5]. In this paper, we extend this study to cartesian product of two graphs where factor graphs are complete graphs and cycles.

1.1. **Basic Definitions and Preliminaries.** The open neighborhood of a vertex $v \in V$ is the set $N(v) = \{u : uv \in E\}$, and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. The vertices in $N(v)$ are called the neighbors of v. For a set $D \subseteq V$, the open neighborhood is $N(D) = \cup_{v\in D} N(v)$ and the closed neighborhood is $N[D] = N(D) \cup D$. A set D is a dominating set if $N[D] = V$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set $S = \{u_1, u_2, \ldots, u_{\gamma(G)}\}$ such that $N[u_i]\cap N[u_j] = \emptyset$, for every $i, j \in \{1, 2, \ldots, \gamma(G)\}$, $i \neq j$, is called an efficient dominating set [11]. We say that a graph G is a double Roman graph if $\gamma_{dR}(G) = 3\gamma(G)$.

A simple graph G is said to be complete if every pair of distinct vertices of G are adjacent in G and a complete graph on n vertices is denoted by K_n. A path on n vertices P_n is the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$. If in addition, v_n is adjacent to v_1 and $n \geq 3$, it is called a cycle of length n, denoted by C_n.

The cartesian product of two graphs G and H, denoted by $G\Box H$, is the graph with vertex set $V(G) \times V(H)$ and any two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G\Box H$ if (i) $u_1 = u_2$ and $v_1v_2 \in E(H)$, or (ii) $u_1u_2 \in E(G)$ and $v_1 = v_2$. If $G = P_m$ and $H = P_n$, then the cartesian product $G\Box H$ is called the $m \times n$ grid graph and is denoted by $G_{m,n}$.

For any graph theoretic terminology and notations not mentioned here, the readers may refer to [7]. The following result is useful in this paper.

Proposition 1.1. [8] *In a double Roman dominating function of weight $\gamma_{dR}(G)$, no vertex needs to be assigned the value 1.*

Hence, without loss of generality, in determining the value $\gamma_{dR}(G)$ we can assume that $V_1 = \emptyset$ for all double Roman dominating functions under consideration.

2. Cartesian product

Theorem 2.1. *Let $G \in \Im$, where $\Im$ is the class of all graphs having an efficient dominating set and H be any graph. Then $\gamma_{dR}(G\Box H) \geqslant \gamma(G)\gamma_{dR}(H)$.*

Proof. Let $V(G)$ and $V(H)$ be the vertex sets of G and H respectively. Let $S=\{u_1, u_2, \ldots, u_{\gamma(G)}\}$ be an efficient dominating set for G. i.e., $\{N[u_1], N[u_2], \ldots, N[u_{\gamma(G)}]\}$ is a vertex partition of $V(G)$. Let $\{\Pi_1, \Pi_2, \ldots, \Pi_{\gamma(G)}\}$ is a vertex partition of $V(G\Box H)$, where $\Pi_i = N[u_i] \times V(H)$ for every $i \in \{1, 2, \ldots, \gamma(G)\}$. Consider a γ_{dR}-function $f = (V_0, V_2, V_3)$ of $G\Box H$. For every $i \in \{1, 2, \ldots, \gamma(G)\}$ we define the function $f_i : V(H) \to \{0, 2, 3\}$ as $f_i(v) = max\{f(u, v) : u \in N[u_i]\}$. In addition, for every $j \in \{0, 2, 3\}$, we define $X_j^{(i)} = \{v \in V(H) : f_i(v) = j\}$.

Now, if $v \in X_0^i$, then for every $u \in N[u_i]$, we have $(u,v) \in V_0$. Then there exists either $(u_i, v_j) \in V_3$ with $v_j \in N(v)$ or $(u_i, v_k), (u_i, v_l) \in V_2$ with $v_k, v_l \in N(v)$. Thus, every $v \in X_0^{(i)}$ has either a neighbor $v_j \in X_3^{(i)}$ or two neighbors $v_k, v_l \in X_2^{(i)}$ and hence, $f_i = (X_0^{(i)}, X_2^{(i)}, X_3^{(i)})$ is a double Roman dominating function on H for every $i \in \{1, 2, \ldots, \gamma(G)\}$. Therefore,

$$\begin{aligned}
\gamma_{dR}(G\Box H) &= 3|V_3| + 2|V_2| \\
&= \sum_{i=1}^{\gamma(G)} [3|V_3 \cap \Pi_i| + 2|V_2 \cap \Pi_i|] \\
&\geqslant \sum_{i=1}^{\gamma(G)} [3|X_3^{(i)}| + 2|X_2^{(i)}|] \\
&\geqslant \sum_{i=1}^{\gamma(G)} \gamma_{dR}(H) \\
&= \gamma(G)\gamma_{dR}(H).
\end{aligned}$$

□

The following result is an interesting consequence of the above theorem.

Corollary 2.1. *Let G and H be two graphs. If $G \in \Im$ and H is a double Roman graph, then $\gamma_{dR}(G\Box H) \geqslant 3\gamma(G)\gamma(H)$.*

Theorem 2.2. *For any graphs G and H of orders n_1 and n_2 respectively, $\gamma_{dR}(G\Box H) \leqslant 3\gamma(G)\gamma(H) + 2(n_1 - \gamma(G))(n_2 - \gamma(H))$.*

Proof. Let D_1 and D_2 be minimum dominating sets on G and H respectively. Let $V_3 = D_1 \times D_2, V_2 = (V(G) - D_1) \times (V(H) - D_2)$ and $V_0 = D_1 \times (V(H) - D_2) \cup (V(G) - D_1) \times D_2$. Since V_3 dominates V_0, $f = (V_0, V_2, V_3)$ is a DRDF on $G\Box H$. Therefore,

$$\begin{aligned}
\gamma_{dR}(G\Box H) &\leqslant 3|V_3| + 2|V_2| \\
&= 3|D_1||D_2| + 2|V(G) - D_1||V(H) - D_2| \\
&= 3\gamma(G)\gamma(H) + 2(n_1 - \gamma(G))(n_2 - \gamma(H)).
\end{aligned}$$

□

3. Particular cases

The double Roman domination number of $K_1 \Box P_m \cong P_m$ and $K_2 \Box P_m \cong G_{2,m}$ was studied in [2] and [5] respectively. It is clear that $\gamma_{dR}(K_n \Box P_1) = \gamma_{dR}(K_n) = 3$, for $n \geqslant 2$. Also, it is easy to verify that the value of $\gamma_{dR}(K_3 \Box P_m)$, for $m = 2, 3$, is $6, 8$, respectively. Hence we are excluding the values $m = 1, 2, 3$ in the following theorems.

Theorem 3.3. *For integers $m \geqslant 4$, $\gamma_{dR}(K_3 \Box P_m) \leqslant 2m + 3$.*

Proof. Let $V(K_3) = \{u_1, u_2, u_3\}$ and $V(P_m) = \{v_1, v_2, \ldots, v_m\}$. Let $A = \{(u_1, v_1)\} \cup \{(u_1, v_{3l}) : 1 \leqslant l \leqslant \lfloor \frac{m}{3} \rfloor - 1\} \cup \{(u_2, v_{3l+2}), (u_3, v_{3l+1}) : 0 \leqslant l \leqslant \lfloor \frac{m}{3} \rfloor - 1\}$.
Case 1: $m \equiv 0 \ (mod\ 3)$.
Let $f_1 = (V_0', V_2', V_3')$, where $V_3' = \{(u_1, v_m)\}$, $V_2' = A$ and $V_0' = V - (V_2' \cup V_3')$. Each vertex $(u_1, v_{3l+1}) \in V_0'$ has two neighbors $(u_1, v_{3l}), (u_3, v_{3l+1}) \in V_2'$. Also, each vertex $(u_1, v_{3l+2}) \in V_0'$ has two neighbors $(u_1, v_{3(l+1)}), (u_2, v_{3l+2}) \in V_2'$ except for (u_1, v_{m-1}). But (u_1, v_{m-1}) has a neighbor $(u_1, v_m) \in V_3'$. Similarly, we can prove that each vertex $(u_i, v_j) \in V_0', i = 2, 3$ and $j = 1, 2, \ldots, n$, is double Roman dominated and hence f_1 is a

DRDF. (See Figure 1.)

Case 2: $m \equiv 1\ (mod\ 3)$.

Let $f_2 = (V_0'', V_2'', V_3'')$, $V_3'' = \{(u_3, v_m)\}$, $V_2'' = A \cup \{(u_1, v_{m-1})\}$ and $V_0'' = V - (V_2'' \cup V_3'')$.

Case 3: $m \equiv 2\ (mod\ 3)$.

Let $f_3 = (V_0''', V_2''', V_3''')$, where $V_3''' = \{(u_2, v_m)\}$, $V_2''' = A \cup \{(u_1, v_{m-2}), (u_3, v_{m-1})\}$ and $V_0''' = V - (V_2''' \cup V_3''')$.

As in Case 1, we can prove that f_2 and f_3 are DRDFs. Hence the result. □

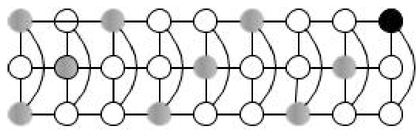

FIGURE 1. DRDF f_1 for $K_3 \square P_9$. Black circle denote vertex in V_3', grey circles denote vertices in V_2' and empty circles denote vertices in V_0'.

Theorem 3.4. *For integers $m \geqslant 4, n \geqslant 4$, $\gamma_{dR}(K_n \square P_m) = 3m$.*

Proof. Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and $V(P_m) = \{v_1, v_2, \ldots, v_m\}$. In any DRDF of $K_n \square P_m$, we have to use at least value 3 on each copy of K_n. Thus $\gamma_{dR}(K_n \square P_m) \geqslant 3m$. Now, $f = (V_0, \emptyset, V_3)$, where $V_3 = \{(u_1, v_l) : 1 \leqslant l \leqslant m\}$ and $V_0 = V - V_3$ is a DRDF with weight $3m$. Hence the result follows. □

By the definition, the number of vertices should be at least 3 in a cycle. The double Roman domination number of $K_1 \square C_m \cong C_m$ was studied in [2]. Also, it can be verified that $\gamma_{dR}(K_2 \square C_3) = 6 = \gamma_{dR}(K_3 \square C_3)$. Hence we are considering $m \geqslant 4$ in the following theorems.

Theorem 3.5. *For integers $m \geqslant 4$,*

$$\gamma_{dR}(K_n \square C_m) \leqslant \begin{cases} 2m, & if\ \ n = 2, \\ 2m + 2, & if\ \ n = 3. \end{cases}$$

Proof. ***Case 1:*** $n = 2$.

Let $V(K_2) = \{u_1, u_2\}$ and $V(C_m) = \{v_1, v_2, \ldots, v_m\}$. Let $A = \{(u_1, v_j) : j\ is\ an\ odd\ number \leqslant m\} \cup \{(u_2, v_j) : j\ is\ an\ even\ number \leqslant m\}$. Let $f = (V_0, V_2, \emptyset)$, where $V_2 = A$ and $V_0 = V - V_2$. It is easy to prove that f is a DRDF.

Case 2: $n = 3$.

Let $V(K_3) = \{u_1, u_2, u_3\}$ and $V(C_m) = \{v_1, v_2, \ldots, v_m\}$. Let $A = \{(u_1, v_{3l+1}), (u_2, v_{3l+2}) : 0 \leqslant l \leqslant \lfloor \frac{m}{3} \rfloor\} \cup \{(u_3, v_{3l}) : 1 \leqslant l \leqslant \lfloor \frac{m}{3} \rfloor\}$.

Subcase (i): $m \equiv 0\ (mod\ 3)$.

Let $f_1 = (V_0', V_2', \emptyset)$, where $V_2' = A \cup \{(u_1, v_m)\} - \{(u_1, v_{3\lfloor \frac{m}{3} \rfloor+1}), (u_2, v_{3\lfloor \frac{m}{3} \rfloor+2})\}$ and $V_0' = V - V_2'$. Each vertex $(u_1, v_{3l}) \in V_0'$ has two neighbors $(u_1, v_{3l+1}), (u_3, v_{3l}) \in V_2'$. Also, each vertex $(u_1, v_{3l+2}) \in V_0'$ has two neighbors $(u_1, v_{3l+1}), (u_2, v_{3l+2}) \in V_2'$. Similarly, we can prove that each vertex $(u_i, v_j) \in V_0'$, $i = 2, 3$ and $j = 1, 2, \ldots, n$, is double Roman dominated and hence f_1 is a DRDF.

Subcase (ii): $m \equiv 1\ (mod\ 3)$.

Let $f_2 = (V_0'', V_2'', \emptyset)$, where $V_2'' = A \cup \{(u_3, v_m)\} - \{(u_2, v_{3\lfloor \frac{m}{3} \rfloor+2})\}$ and $V_0'' = V - V_2''$.

Subcase (iii): $m \equiv 2\ (mod\ 3)$.

Let $f_3 = (V_0''', V_2''', \emptyset)$, where $V_2''' = A \cup \{(u_3, v_m)\}$ and $V_0''' = V - V_2'''$.

As in Subcase (i), we can prove that f_2 and f_3 are DRDFs.
Hence the result is true. □

Theorem 3.6. *For integers $m \geqslant 4, n \geqslant 4$, $\gamma_{dR}(K_n \square C_m) = 3m$.*

Proof. Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and $V(C_m) = \{v_1, v_2, \ldots, v_m\}$. In any DRDF of $K_n \square C_m$, we have to use at least value 3 on each copy of K_n. Thus $\gamma_{dR}(K_n \square C_m) \geqslant 3m$. Now, $f = (V_0, \emptyset, V_3)$, where $V_3 = \{(u_1, v_l) : 1 \leqslant l \leqslant m\}$ and $V_0 = V - V_3$ is a DRDF with weight $3m$. Hence the result follows. (See Figure 2.) □

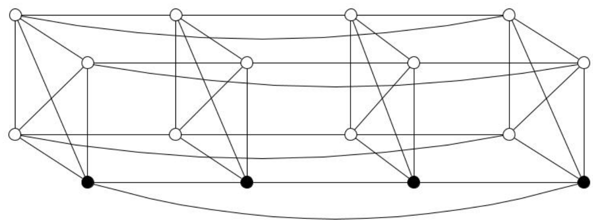

FIGURE 2. DRDF for $K_4 \square P_4$. Black circles denote vertices in V_3 and empty circles denote vertices in V_0.

Conclusion

In this paper we have improved an existing lowerbound of $\gamma_{dR}(G \square H)$ for graphs $G \in \Im$, where $\Im$ is the class of all graphs having an efficient dominating set. Proving this lower boung for all graphs is an open problem. We have also found the double Roman domination number of $K_n \square P_m$ and $K_n \square C_m$. Finding similar results for some other classes of graphs is also open.

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