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Fuzzy Noetherian Module

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ABSTRACT. P. S. Das [Sivaramakrishna Das, P. Fuzzy groups and level subgroups. *J. Math. Anal. Appl.* **84** (1981), no. 1, 264–269.] proved that the level subgroups of any fuzzy subgroup of a finite group form a chain. In this paper we extend it to modules and showed that the level submodules of a fuzzy module also form a chain. With the help of these results we introduced fuzzy noetherian module for noetherian modules. Some results are proved for level submodules of any fuzzy module of modules with composition series.

1. INTRODUCTION

In 1965, Zadeh [15] introduced the idea of fuzzy set on a nonempty set. He defined the fuzzy subset of a nonempty set X as a membership function $\lambda : X \to [0,1]$. Different types of generalizations of abstract mathematical structure into fuzzy context happen after the introduction of fuzzy subset of a set. In 1971, a milestone in the development of fuzzy group was laid by Rosenfeld [11]. The level set or a-cut [7] of a fuzzy set λ for $a \in [0,1]$ is defined as $\lambda_a = \{x/x \in X, \lambda(x) \geq a\}$. Das[13] showed that for a finite group, a chain is formed by the level subgroups of a fuzzy subgroup. In 1975, Negoita and Ralescu [9] came up with the concept of fuzzy module. After the introduction of the fuzzy module, many authors have studied fuzzy modules and its properties in their papers like [4], [5], [8]. Here we have showed that the level submodules of a fuzzy module also form a chain.

The properties of finitely generated submodules were first studied by Hilbert. An R-module M is Noetherian [3] if the ascending chain of submodules, partially ordered by inclusion terminates. It is named after the German mathematician Emmy Noether who was the first one to find the importance of this property. An R-module has a chain of submodules partially ordered by inclusion of finite length, so that no submodules are there in between consecutive submodules in the chain. Then that R-module has a composition series. we have studied the composition series using the level submodules of fuzzy module of R-module. Also we fuzzified the algebraic concept of noetherian modules to define the fuzzy noetherian module and some results relating to it.

2. PRELIMINARIES

Definition 2.1. [2] Let R be a ring. A left R-module is a set M together with
(1) a binary operation $+$ on M under which M is an abelian group, and
(2) an action of R on M (that is, a map $R \times M \to M$) denoted by ax, for all $a \in R$ and for all $x \in M$ which satisfies
  (a) $(a+b)x = ax + bx$, for all $a, b \in R, x \in M$
  (b) $(ab)x = a(bx)$, for all $a, b \in R, x \in M$ and

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(c) $a(x+y) = ax + ay$, for all $a \in R, x, y \in M$
If the ring R has a unity $'1'$ we impose the additional axiom:
(d) $1.x = x$, for all $x \in M$

Remark 2.1. Similar to this, a Right R-module can also be defined as the action of R on M on the right, indicated by xa, for all a in R and all x in M. In this paper, the R-module is interpreted as the left R-module.

Definition 2.2. [2] Let R be a ring and let M be an R-Module. An R-submodule of M is a subgroup N of M which is closed under the action of ring elements i.e $rn \in N, \ \forall r \in R, \ n \in N$.

Definition 2.3. [3] Let N be a submodule of an R-module M. Let the quotient abelian group M/N with addition defined by $(x+N)+(y+N) = (x+y)+N$ and multiplication given by $a.(x+N) = ax+N$, $a \in R, x, y \in M$ is an R-module. This R-module M/N known as the quotient(factor) module, under the action of R on M/N (i.e, a map $R \times M/N \to M/N$) gets the structure of an R-module.

Example 2.1.

(1) All additive abelian groups are $\mathbb{Z}$-modules and all subgroups of the group are the submodules of the $\mathbb{Z}$-module.
(2) If R is any ring then R is an R-module and all ideals of the ring R are the submodules of the R-module R.
(3) Let $R = F$, a field then every vectorspace over F are F-modules and the subspaces of the vectorspace are the submodules of F-module.
(4) $\mathbb{Z}$ is an $\mathbb{Z}$-module and $n\mathbb{Z}$ is a submodule of $\mathbb{Z}$ for any $n \in \mathbb{Z}$ then $\mathbb{Z}/n\mathbb{Z}$ is a factor $\mathbb{Z}$-module.

Definition 2.4. [1] The set of proper submodules of M, ordered by inclusion, has a maximal element called a maximal submodule N of M. A submodule N of M is hence maximal if and only if (i) $N \neq M$ and (ii) for each submodule N' such that $N \subset N' \subset M$, either $N = N'$ or $N' = M$.

Definition 2.5. [3] An R-module M is cyclic if $M = Rx$ for some $x \in M$, where $Rx = \{ax/ \ a \in R\}$.

Definition 2.6. [3] M is called a finitely generated R-module if $M = M_1 + M_2 + \cdots + M_n$, where each M_i is cyclic i.e $M_i = Rx_i$ for some $x_i \in M$. If $M_i = Rx_i$ for $i = 1, 2, \ldots$, then the set $\{x_1, x_2, \ldots, x_n\}$ is called a generating set for M.

Proposition 2.1. [3] *Let M be an R-module. The following conditions are equivalent*

(1) Any non-empty collection of submodules of M has a maximal element.
(2) For any increasing sequence of submodules of M, $M_1 \subset M_2 \subset M_3 \subset ...M_n \subset ...$, there exists some integer m such that $M_k = M_m$ for all $k \geq m$.
(3) Every submodule of M is finitely generated.

Definition 2.7. [3] An R-module M is noetherian if it satisfies any one of the above equivalent conditions.

Example 2.2.

(1) All finite dimensional vectorspaces over a field K are Noetherian $K-$modules.
(2) Any principal ideal ring is a noetherian R-module.
(3) Every finite abelian groups are noetherian $\mathbb{Z}$-modules.
(4) Let K be a field and x be an indeterminate, then $K[x]$ is a infinite dimensional vector space over K with basis $\{1, x, x^2, x^3, \ldots\}$. Since $K[x]$ is not finitely generated it is not a noetherian K-module.

Definition 2.8. [1] A left R-module $M \neq < 0 >$ is called irreducible if M contains no non-trivial submodules, whereas a module which contains a non-trivial submodule is called reducible.

Definition 2.9. [14] Let M be an R-module. A chain of submodules of M, $< 0 >= M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_r = M$ is called a composition series if for each $i \geq 1$, the factor (quotient) module ${}^{M_i}/_{M_{i-1}}$ is an irreducible R-module.

Definition 2.10. [1] The factor modules ${}^{M_i}/_{M_{i-1}}$ are called the factors of the composition series. The number of factors is called the length of the composition series.

Example 2.3.

(1) The $\mathbb{Z}$-module $\mathbb{Z}_6$ has a composition series of length 2, $< 0 > \subset < 2 > \subset < 1 >$.
(2) Consider the set of all complex numbers $\mathbb{C}$ as an $\mathbb{R}$-module the it has a composition series of length 2, $< 0 > \subset \mathbb{R} \subset \mathbb{C}$.
(3) The n-dimensional vectorspace over a field F is an F-module having composition series of length n.
(4) Let $R = K[x]$ where K is a field and $M = {}^{K[x]}/_{< x^2 >}$. Then M has a composition series $< 0 > \subset {}^{< x >}/_{< x^2 >} \subset M$.

Theorem 2.1. [1] *If M is an R-module which has a composition series, then any two composition series of M have the same length.*

3. Fuzzy Module

Definition 3.11. [10] Let R be a ring and let M be an R-module, then a fuzzy module on M is a map $\lambda : M \to [0, 1]$ satisfying the following conditions

(1) $\lambda(m_1 + m_2) \geq \min\{\lambda(m_1), \lambda(m_2)\}, \ \forall m_1, m_2 \in M$
(2) $\lambda(-m_1) = \lambda(m_1) \ \forall m_1 \in M$
(3) $\lambda(rm_1) \geq \lambda(m_1) \ \forall m_1 \in M, \ r \in R$
(4) $\lambda(0) = 1$

Example 3.4.

(1) Consider $R = M = \mathbb{Z}$. Define $\lambda : \mathbb{Z} \to [0, 1]$ by

$$\lambda(m) = \begin{cases} 1, & \text{if } m \in 2\mathbb{Z} \\ {}^1/_2 & \text{if } m \in \mathbb{Z} \setminus 2\mathbb{Z} \end{cases}$$

(2) Consider $R = M = \mathbb{Z}$. Define $\lambda : \mathbb{Z} \to [0, 1]$ by

$$\lambda(m) = \begin{cases} 1, & \text{if } m \in 4\mathbb{Z} \\ {}^1/_2 & \text{if } m \in 2\mathbb{Z} \setminus 4\mathbb{Z} \\ {}^1/_4 & \text{if } m \in \mathbb{Z} \setminus 2\mathbb{Z} \end{cases}$$

(3) Consider $R = \mathbb{Z}$, $M = \mathbb{Z}_6$ Define $\lambda_i : \mathbb{Z}_6 \to [0, 1], \ \ i = 1, 2, \ldots$ by

$$\lambda_1(m) = \begin{cases} 1, & \text{if } m \in \{0\} \\ {}^1/_2 & \text{if } m \in \{2, 4\} \\ {}^1/_3 & \text{if } m \in \{1, 3, 5\} \end{cases}$$

$$\lambda_2(m) = \begin{cases} 1, & \text{if } m \in \{0\} \\ {}^1/_2 & \text{if } m \in \{3\} \\ {}^1/_3 & \text{if } m \in \{1, 2, 4, 5\} \end{cases}$$

$$\lambda_3(m) = \begin{cases} 1, & \text{if } m \in \{0,2,4\} \\ 1/2 & \text{if } m \in \{1,3,5\} \end{cases}$$

$$\lambda_4(m) = \begin{cases} 1, & \text{if } m \in \{0,3\} \\ \frac{1}{2} & \text{if } m \in \{1,2,4,5\} \end{cases}$$

$$\lambda_5(m) = 1, \ \forall m \in \mathbb{Z}_6$$

Definition 3.12. [12], [5] Let μ and λ be two fuzzy modules of an R-module M, then λ is called a fuzzy submodule of μ if $\lambda \subseteq \mu$ (i.e $\lambda(m) \leq \mu(m) \ \forall m \in M$)

Example 3.5. In the preceding Example 3.4, we have $\lambda_2 \subseteq \lambda_4 \subseteq \lambda_5$ and $\lambda_1 \subseteq \lambda_3 \subseteq \lambda_5$.

Lemma 3.1. [12], [5] *Let λ be a fuzzy subset of an R-module M then the level subset λ_t is a submodule of M for all $t \in [0,1]$ if and only if λ is a fuzzy module of M.*

Definition 3.13. [12], [5] Let M be an R-module and λ be a fuzzy module of M, then the submodules λ_t, $t \in [0,1]$ are called the level submodules of λ.

Lemma 3.2. *Let M be an R-module and λ be a fuzzy R-module of M. If $t_1, t_2 \in [0,1]$, and $t_1 < t_2$ then $\lambda_{t_2} \subseteq \lambda_{t_1}$.*

Proof. Given λ is a fuzzy module on R-module M and $t_1 < t_2 \in [0,1]$ then clearly $\lambda_{t_2} = \{m \in M \ / \ \lambda(m) \geq t_2\} \subseteq \{m \in M \ / \ \lambda(m) \geq t_1\} = \lambda_{t_1}$ □

Lemma 3.3. *Let M be an R-module and λ be a fuzzy module of M. The level submodules λ_{t_1} and λ_{t_2} (with $t_1 < t_2$) of λ are equal if and only if there does not exists an $m \in M$ such that $t_1 \leq \lambda(m) < t_2$.*

Proof. First assume that $\lambda_{t_1} = \lambda_{t_2}$. Now suppose that there exists an $m \in M$ such that $t_1 \leq \lambda(m) < t_2$, then $\lambda_{t_2} \subsetneq \lambda_{t_1}$, as $m \in \lambda_{t_1}$ and $m \notin \lambda_{t_2}$ which is a contradiction to $\lambda_{t_1} = \lambda_{t_2}$.
Conversely suppose that there does not exists an $m \in M$ such that $t_1 \leq \lambda(m) < t_2$. As $t_1 < t_2$, $\lambda_{t_2} \subseteq \lambda_{t_1}$. Now

$$\begin{aligned} \lambda_{t_1} &= \{m \in M \ / \ \lambda(m) \geq t_1\} \\ &= \{m \in M \ / \ \lambda(m) \geq t_2\} \\ &= \lambda_{t_2} \end{aligned}$$

since there does not exists $m \in M$ such that $t_1 \leq \lambda(m) < t_2$. Thus $\lambda_{t_1} = \lambda_{t_2}$ □

Remark 3.2. If $t_1, t_2 \in Im(\lambda)$ and $t_1 < t_2$ then $\lambda_{t_2} \subsetneq \lambda_{t_1}$.

Lemma 3.4. *Let λ be a fuzzy module of an R-module M and $t, s \in Im(\lambda)$, then $\lambda_t = \lambda_s$ if and only if $t = s$.*

Proof. If $t = s$ then clearly $\lambda_t = \lambda_s$.
Now let $\lambda_t = \lambda_s$ since $t, s \in Im(\lambda)$, there exists $m_1, m_2 \in M$ such that $\lambda(m_1) = t$ and $\lambda(m_2) = s$. So $m_1 \in \lambda_t$ and $m_2 \in \lambda_s$ But since $\lambda_t = \lambda_s$, $m_1, m_2 \in \lambda_t$ and $\lambda_s \Rightarrow t = \lambda(m_1) \geq s$ and $s = \lambda(m_2) \geq t$
$\Rightarrow t = s$. □

Theorem 3.2. *Let M be an R-module and λ be a fuzzy module on M. If $Im(\lambda) = \{t_i \in [0,1] / \ \lambda(x) = t_i, \text{for some } x \in M\}$ then the only level submodules of λ are $\{\lambda_{t_i}\}, i = 1, 2, \ldots$.*

Proof. By Lemma 3.1 λ_t is a submodule of M $\forall t \in [0,1]$. Since λ is a fuzzy module of M, we have $\lambda(0) = 1$. So λ_1 is the smallest level submodule of λ and contains $\{0\}$.
If $t < 1$ then there are three cases.
Case (i) There doesnot exists a $t_i \in Im(\lambda)$ such that $t \leq t_i < 1$ then by Lemma 3.3 $\lambda_t = \lambda_1$.
Case(ii) $t_i < t < t_j$ where $t_i, t_j \in Im(\lambda)$, $t \notin Im(\lambda)$, then by the definition of level submodules and Lemma 3.3, we get $\lambda_{t_j} = \lambda_t \subset \lambda_{t_i}$.
Case (iii) $0 \leq t < t_r$ where t_r is the least element in $Im(\lambda)$ then $\lambda_{t_r} = M = \lambda_t$.
So if we consider any $t \in [0,1]$, then $\lambda_t = \lambda_{t_i}$ for some $t_i \in Im(\lambda)$. Hence all the level submodules of λ are of the form $\{\lambda_{t_i}\}, i = 1, 2, \ldots$. □

Lemma 3.5. *Let $M_1 \subset M_2 \subset \cdots \subset M_n \subset \ldots$ be an ascending chain of submodules of an R-module M then the fuzzy subset $\lambda : M \to [0,1]$ defined by*

$$\lambda(m) = \begin{cases} 1 & \text{if } m \in M_1 \\ 1/2 & \text{if } m \in M_2 \setminus M_1 \\ 1/3 & \text{if } m \in M_3 \setminus M_2 \\ \vdots & \\ 1/n & \text{if } m \in M_n \setminus M_{n-1} \\ \vdots & \end{cases} \tag{3.1}$$

is a fuzzy module on M.

Proof. To prove λ is a fuzzy module on M, we need to prove that
(1) $\lambda(0) = 1$
(2) $\lambda(m_1 + m_2) \geq \min\{\lambda(m_1), \lambda(m_2)\}, \ \forall m_1, m_2 \in M$
(3) $\lambda(rm) \geq \lambda(m) \ \forall m \in M, \ r \in R$

From the definition of λ , $\lambda(m) = 1$ $\forall m \in M_1$ and M_1 is the smallest submodule of M in the ascending chain of submodules of M, So $\lambda(0) = 1$.
Consider $m_1, m_2 \in M \Rightarrow$ either (I) $m_1, m_2 \in M_i$, for some $i \in \mathbb{Z}^+$ or (II) $m_1 \in M_i$ and $m_2 \in M_j$ for $j < i \in \mathbb{Z}^+$ ($m_2 \in M_i$ and $m_1 \in M_j$ for $j < i \in \mathbb{Z}^+$)
When $m_1, m_2 \in M_i$, for some $i \in \mathbb{Z}^+$, there are the following cases
(1) $m_1, m_2 \in M_i \setminus M_{i-1}$, for $i \in \mathbb{Z}^+$
(2) $m_1 \in M_i \setminus M_{i-1}$ and $m_2 \in M_j$, for some $j < i \in \mathbb{Z}^+$
(3) $m_1, m_2 \in M_j \setminus M_{j-1}$, for $j < i \in \mathbb{Z}^+$
(4) $m_1 \in M_j \setminus M_{j-1}$ and $m_2 \in M_k$, for some $k < j < i \in \mathbb{Z}^+$

Case(1)
Consider $m_1, m_2 \in M_i \setminus M_{i-1}$, for $i \in \mathbb{Z}^+$ then $\lambda(m_1) = \lambda(m_2) = 1/i$.
So $min\{\lambda(m_1), \lambda(m_2)\} = 1/i$.
Now since $m_1, m_2 \in M_i \setminus M_{i-1}$ and M_i is a submodule of M, $m_1 + m_2 \in M_i \Rightarrow$ either $m_1 + m_2 \in M_i \setminus M_{i-1}$ or $m_1 + m_2 \in M_j$ for some $j < i \in \mathbb{Z}^+$
In both cases $\lambda(m_1 + m_2) \geq 1/i$ from the equation 3.1
So $\lambda(m_1 + m_2) \geq min\{\lambda(m_1), \lambda(m_2)\}$ $\forall m_1, m_2 \in M_i \setminus M_{i-1}$
Case(2)
Consider $m_1 \in M_i \setminus M_{i-1}$ and $m_2 \in M_j$, for some $j < i \in \mathbb{Z}^+$ then $\lambda(m_1) = 1/i$ and $\lambda(m_2) \geq 1/i$
$\Rightarrow min\{\lambda(m_1), \lambda(m_2)\} \geq 1/i$
As M_j and M_i are submodules of R-module M and $M_j \subset M_i$, we have $m_1 + m_2 \in M_i$, So $\lambda(m_1 + m_2) \geq 1/i$ from the equation 3.1
So we get $\lambda(m_1 + m_2) \geq min\{\lambda(m_1), \lambda(m_2)\}$ $\forall m_1 \in M_i \setminus M_{i-1}$ and $m_2 \in M_j$

Case(3)
Consider $m_1, m_2 \in M_j \setminus M_{j-1}$, for $j < i \in \mathbb{Z}^+$, then clearly it is similar to **Case(1)** and we get $\lambda(m_1 + m_2) \geq min\{\lambda(m_1), \lambda(m_2)\}\ \forall m_1, m_2 \in M_j \setminus M_{j-1}$
Case(4)
Consider $m_1 \in M_j \setminus M_{j-1}$ and $m_2 \in M_k$, for some $k < j < i \in \mathbb{Z}^+$, then clearly it is similar to **Case (2)** and we get $\lambda(m_1 + m_2) \geq min\{\lambda(m_1), \lambda(m_2)\}\ \forall m_1 \in M_j \setminus M_{j-1}$ and $m_2 \in M_k$
Now When $m_1 \in M_i$ and $m_2 \in M_j$ ($m_2 \in M_i$ and $m_1 \in M_j$) for some $j < i \in \mathbb{Z}^+$ then clearly $m_1, m_2 \in M_i$, Since $M_j \subset M_i$ as $j < i$. So from the above cases we get $\lambda(m_1 + m_2) \geq min\{\lambda(m_1), \lambda(m_2)\}$
$\therefore \lambda(m_1 + m_2) \geq min\{\lambda(m_1), \lambda(m_2)\}\ \forall m_1, m_2 \in M$

Now consider $m \in M$ and $r \in R$
$m \in M \Rightarrow m \in M_i$ for some $i \in \mathbb{Z}^+$ then either $m \in M_i \setminus M_{i-1}$ or $m \in M_j$ for some $j < i \in \mathbb{Z}^+$
If $m \in M_i - M_{i-1}$ then $\lambda(m) = 1/i$ and $rm \in M_i$ as M_i is a submodule of M, So we have

$$\begin{aligned} \lambda(rm) &\geq 1/i \\ &= \lambda(m) \end{aligned}$$

So $\lambda(rm) \geq \lambda(m)\ \forall m \in M_i \setminus M_{i-1}$ and $r \in R$
Now if $m \in M_j$ for some $j < i \in \mathbb{Z}^+$ then either $m \in M_j \setminus M_{j-1}$ or $m \in M_k$ for some $k < j < i \in \mathbb{Z}^+$
If $m \in M_j \setminus M_{j-1}$ then $\lambda(m) = 1/j$ and $rm \in M_j$ as M_j is a submodule of M, So we have

$$\begin{aligned} \lambda(rm) &\geq 1/j \\ &= \lambda(m) \end{aligned}$$

So $\lambda(rm) \geq \lambda(m)\ \forall m \in M_j \setminus M_{j-1}$ and $r \in R$
If $m \in M_k$ then again the process repeats and so we get $\lambda(rm) \geq \lambda(m)\ \forall m \in M$ and $r \in R$.

$\therefore \lambda$ is a fuzzy module of M. □

Theorem 3.3. *Every submodule N of an R-module M can be written as a level submodule of some fuzzy module λ of R-module M.*

Proof. Let N be a submodule of the R-module M. Define $\lambda : M \to [0, 1]$ by

$$\lambda(m) = \begin{cases} 1 & \text{if } m \in \{0\} \\ t & \text{if } m \in N \setminus \{0\} \\ 0 & \text{if } m \in M \setminus N \end{cases}$$

where $0 < t \leq 1$
Now consider the ascending chain $< 0 > \subseteq N \subseteq M$ of submodules of M. Then by lemma 3.5 λ is a fuzzy module on M. Also $\lambda_t = N$, So N is written as the level submodule of the fuzzy module λ on M. Hence the result. □

Theorem 3.4. *Let λ be a fuzzy module on R-module M and let $1 = t_0 > t_1 > t_2 > \cdots > t_n \geq 0$ be the images of λ with level cardinality $n + 1$, then the level submodules of λ form a chain of submodules of R-module M of length n.*

Proof. By Lemma 3.3, $t_i < t_j \Rightarrow \lambda_{t_j} \subsetneq \lambda_{t_i}$, for $t_i, t_j \in Im(\lambda)$. Also the level submodules of the fuzzy R-module λ are submodules of M by lemma 3.1. So for $1 = t_0 > t_1 > t_2 > \cdots > t_n \in Im(\lambda)$, $\lambda_{t_0} \subset \lambda_{t_1} \subset \lambda_{t_2} \subset \cdots \subset \lambda_{t_n} = M$ form a chain of submodules of R-module M of length n. □

Theorem 3.5. *Let λ be a fuzzy module on R-module M with level cardinality $n+1$, $1 = t_0 > t_1 > t_2 > \cdots > t_n \geq 0 \in Im(\lambda)$ and if the factor level submodules ${}^{\lambda_{t_i}}/_{\lambda_{t_{i-1}}}$ is irreducible for $i = 1, 2, \ldots, n$ then λ has a composition series of level submodules of length n.*

Proof. By Lemma 3.4 for $1 = t_0 > t_1 > t_2 > \cdots > t_n \in Im(\lambda)$, $\lambda_{t_0} \subset \lambda_{t_1} \subset \lambda_{t_2} \subset \cdots \subset \lambda_{t_n} = M$ form a chain of submodules of R-module M of length n. Given the factor modules ${}^{\lambda_{t_i}}/_{\lambda_{t_{i-1}}}$ of the ascending chain $\lambda_{t_0} \subset \lambda_{t_1} \subset \lambda_{t_2} \subset \cdots \subset \lambda_{t_n} = M$ of level submodules of the fuzzy R-module λ is irreducible for $i = 1, 2, \ldots, n$ then by the definition of composition series of an R-module, it forms a composition series of the R-module M. □

Remark 3.3. $|\lambda|$ denote the level cardinality of the fuzzy R-module λ on R-module M.

Theorem 3.6. *Let M has a composition series of length n and if λ is a fuzzy module on module M then $|\lambda| \leq n+1$ and if $|\lambda| = n+1$ then the chain of level submodules of fuzzy R-module λ of M form a composition series of M.*

Proof. We know that corresponding to each submodule of M there exists a level submodule of some fuzzy R-module λ of M by theorem 3.3. If possible λ is a fuzzy R-module of M with level cardinality $> n+1$, say $n+2$ then $t_0 = 1 > t_1 > \cdots > t_{n+1}$ be the $n+2$ distinct values of $Im(\lambda)$ then by lemma 3.3 , $\lambda_{t_0} \subsetneq \lambda_{t_1} \subsetneq \lambda_{t_2} \subsetneq \ldots \lambda_{t_n} \subsetneq \lambda_{t_{n+1}}$ is an ascending chain of level submodules of λ of length $n+1$. By lemma 3.1 each level submodule of λ is a submodule of M. $\therefore$ there exist an ascending chain of submodules of M of length $n+1$, which is a contradiction. So every fuzzy R-module on M has level cardinality $\leq n+1$.
Now let λ be a fuzzy R-module of M with level cardinality $n+1$, then as above the chain of level submodule is a chain of submodule of M of length n. By theorem 2.1 the chain of level submodules of λ on M of length n form a composition series of M. □

Theorem 3.7. *If every fuzzy module λ on an R-module M has finite level cardinality, then M is noetherian.*

Proof. Let λ be a fuzzy R-module of M with finite level cardinality. Corresponding to each fuzzy R-module λ on M, the level submodules form a chain of submodules of M by lemma 3.3.
If possible assume that M is not noetherian then there exists a chain of submodules of M which doesnot terminate, say $M_1 \subset M_2 \subset \cdots \subset M_n \subset \ldots$ then by lemma 3.5 we can form a fuzzy module on M with cardinality not finite, which is a contradiction. Hence the result. □

Remark 3.4. Let N be a submodule of an R-module M and let λ be a fuzzy module on M then $\nu = \lambda|N$ is a fuzzy module on N, where $\lambda|N$ is the restriction of the domain M to the submodule N.

Theorem 3.8. *Let M be a Noetherian R-module, then corresponding to each ascending chain of submodules $\{0\} = M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$ of M which terminates at M_r, for some $r \in \mathbb{Z}^+$, there exists fuzzy R-modules ν_i on M_i, $i = 1, 2, 3, \ldots$ and an ascending chain of fuzzy submodules $\lambda_1 \subseteq \lambda_2 \subseteq \ldots$ defined on M such that it terminates at λ_r, $r \in \mathbb{Z}^+$ and $\nu_i = \lambda_i|M_i$, $i = 1, 2, \ldots$.*

Proof. Suppose $\{0\} = M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$ be an ascending chain of submodules of M which terminates at M_r.

Define the fuzzy modules $\nu_k : M_k \to [0,1]$ by

$$\nu_k(m) = \begin{cases} 1 & \text{if } m \in M_1 \\ 1/2 & \text{if } m \in M_2 \setminus M_1 \\ 1/3 & \text{if } m \in M_3 \setminus M_2 \\ \vdots & \\ 1/k & \text{if } m \in M_k \setminus M_{k-1}, \text{ k=1,2, \dots, r} \end{cases}$$

and $\nu_k(m) = \nu_r(m) \ \forall k \geq r$, as the ascending chain of submodules of M terminates at M_r (i.e $M_r = M_{r+1} = M_{r+2} = \dots$)
Now define $\lambda_k : M \to [0,1]$ by

$$\lambda_k(m) = \begin{cases} \nu_k(m) & \forall m \in M_k \\ 0 & \forall m \in M \setminus M_k \end{cases}$$

$k = 1, 2, \dots, r$ then $\lambda_1 \subseteq \lambda_2 \subseteq \dots$ is an ascending chain of fuzzy submodules on M and terminates at λ_r, since $\lambda_r = \lambda_{r+1} = \lambda_{r+2} = \dots$ by definition. and $\nu_i = \lambda_i | M_i, \quad i = 1, 2, \dots, r$ and also $\nu_k = \nu_{k+i} | M_k$, $k = 1, 2, \dots r$, $i = 1, 2, \dots$
Hence the theorem. □

Definition 3.14. The fuzzy module λ_r of the noetherian R-module M, which terminates at r in the theorem 3.8, is called a fuzzy noetherian module of M with level cardinality r.

Example 3.6. Consider the four dimensional vectorspace $M = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$ with basis $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ as a $\mathbb{Q}$-module, and the ascending chain of submodules of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is $\{0\} = M_1 \subset \mathbb{Q} = M_2 \subset \mathbb{Q} \oplus \mathbb{Q}\sqrt{2} = M_3 \subset \mathbb{Q} \oplus \mathbb{Q}\sqrt{2} \oplus \mathbb{Q}\sqrt{3} = M_4 \subset \mathbb{Q}(\sqrt{2}, \sqrt{3}) = M$. As in theorem 3.8, the corresponding fuzzy modules on $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ are

(1)

$$\lambda_1(m) = \begin{cases} 1 & \text{if } m \in M_1 \\ 0 & \text{if } m \in M \setminus M_1 \end{cases}$$

(2)

$$\lambda_2(m) = \begin{cases} 1 & \text{if } m \in M_1 \\ 1/2 & \text{if } m \in M_2 \setminus M_1 \\ 0 & \text{if } m \in M \setminus M_2 \end{cases}$$

(3)

$$\lambda_3(m) = \begin{cases} 1 & \text{if } m \in M_1 \\ 1/2 & \text{if } m \in M_2 \setminus M_1 \\ 1/3 & \text{if } m \in M_3 \setminus M_2 \\ 0 & \text{if } m \in M \setminus M_3 \end{cases}$$

(4)

$$\lambda_4(m) = \begin{cases} 1 & \text{if } m \in M_1 \\ 1/2 & \text{if } m \in M_2 \setminus M_1 \\ 1/3 & \text{if } m \in M_3 \setminus M_2 \\ 1/4 & \text{if } m \in M_4 \setminus M_3 \\ 0 & \text{if } m \in M \setminus M_4 \end{cases}$$

(5)

$$\lambda_5(m) = \begin{cases} 1 & \text{if } m \in M_1 \\ 1/2 & \text{if } m \in M_2 \setminus M_1 \\ 1/3 & \text{if } m \in M_3 \setminus M_2 \\ 1/4 & \text{if } m \in M_4 \setminus M_3 \\ 1/5 & \text{if } m \in M \setminus M_4 \end{cases}$$

then clearly we have $\lambda_1 \subset \lambda_2 \subset \lambda_3 \subset \lambda_4 \subset \lambda_5$ is an ascending chain of fuzzy submodules of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ which terminates at λ_5 and so the level cardinality of the fuzzy noetherian module is 5. Similarly if we consider any other chain of submodules of M, then the proper submodules in that chain are any one of the submodules M_i, $i = 1, 2, 3, 4$ and the fuzzy submodules are defined accordingly as in the above cases.

4. Conclusion

In this paper, we fuzzified the algebraic concept of noetherian modules in order to define the fuzzy noetherian module and some related results. P. S. Das [13] proved that there is a one-one correspondence between the subgroups of a group and the equivalence classes of level subgroups of the collection of all fuzzy subgroups of the group. we are trying to extend this result in the case of an R-module in our future work.

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