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Alternating quadratic and cubic series with the tail of $\ln 2$

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ABSTRACT. In this paper we calculate several alternating quadratic series and a cubic series involving the tail of $\ln 2$.

1. INTRODUCTION

In this paper we calculate several alternating quadratic series and a cubic series involving the expression $\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots$. This term is connected to the tail of $\ln 2$ as follows

$$(-1)^{n-1}\left[\ln 2 - \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-2}}{n-1}\right)\right] = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots.$$

The goal of this paper is to extend the results recorded in [3], in problems 3.28, 3.29 and 3.45, concerning the calculation of some quadratic series involving the tail of $\ln 2$. Our results, which deal with the calculation of alternating series, are new in the literature and they are obtained based on a combination of techniques involving Abel's summation formula and shifting the index of summation. The last one allows us to reduce the calculation of an alternating quadratic series to a linear series.

Before stating the main results of the paper, we briefly mention two special functions that are used in our calculations.

The famous Riemann zeta function ζ is defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $\Re(s) > 1$. For $|z| \leq 1$, the Dilogarithm function $\operatorname{Li}_2(z)$ is introduced in the mathematical literature by $\operatorname{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\ln(1-t)}{t}\mathrm{d}t$. When $z = -1$ or $z = 1$, we obtain the values $\operatorname{Li}_2(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$ and $\operatorname{Li}_2(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}$. For properties of these and other special functions, the reader is referred to [5].

2. MAIN RESULTS

The main results of this paper are the following theorems.

Theorem 2.1. ***Quadratic series with the tail of*** $\ln 2$

The following identities hold:

(a) $$\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)^2 = -\frac{\zeta(2)}{4};$$

(b) $$\sum_{n=1}^{\infty}(-1)^n n\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)^2 = \frac{\ln 2}{2} - \frac{1}{2} - \frac{\zeta(2)}{8};$$

(c) $$\sum_{n=1}^{\infty}(-1)^n(2n-1)\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)^2 = \ln 2 - 1;$$

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(d) $\sum_{n=1}^{\infty}(-1)^n\left[n^2\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2-\frac{1}{4}\right]=\frac{\ln 2-1}{2}$;

(e) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2=-\frac{\ln^3 2}{3}+\frac{\ln 2}{2}\zeta(2)-\frac{3}{4}\zeta(3)$.

Theorem 2.2. ***A cubic series with the tail of*** $\ln 2$
The following equality is valid

$$\sum_{n=1}^{\infty}(-1)^n n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3=\frac{\ln 2}{4}-\frac{3\ln^2 2}{4}-\frac{\zeta(2)}{16}.$$

We collect, in the next lemma, some results that we need in proving Theorems 2.1 and 2.2.

Lemma 2.1. ***Linear series with the tail of*** $\ln 2$
The following equalities are true:

(i) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)=-\frac{1}{2}$;

(ii) $\sum_{n=1}^{\infty}(-1)^n\left[2n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)-1\right]=-\frac{1}{4}$;

(iii) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)=\frac{\ln^2 2}{2}-\frac{\zeta(2)}{2}$;

(iv) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)=\frac{\ln 2}{2}\zeta(2)-\zeta(3)$.

Before we prove the lemma, we observe that

$$\begin{aligned}\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots&=\int_0^1\left(x^{n-1}-x^n+x^{n+1}-\cdots\right)\mathrm{d}x\\&=\int_0^1\frac{x^{n-1}}{1+x}\mathrm{d}x.\end{aligned}\tag{2.1}$$

This implies

$$\lim_{n\to\infty}n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)=\frac{1}{2}\tag{2.2}$$

and it follows that

$$\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\sim\frac{1}{2n}.$$

This shows that the series in parts (a), (b), (c), (e) of Theorem 2.1, Theorem 2.2 and parts (i), (iii) and (iv) of Lemma 2.1 are all convergent.

Now we prove Lemma 2.1.

Proof. (i) Formula (2.1) implies that

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right) &= \sum_{n=1}^{\infty}(-1)^n\int_0^1\frac{x^{n-1}}{1+x}\mathrm{d}x\\ &= -\int_0^1\frac{1}{1+x}\left(\sum_{n=1}^{\infty}(-x)^{n-1}\right)\mathrm{d}x\\ &= -\int_0^1\frac{1}{(1+x)^2}\mathrm{d}x\\ &= -\frac{1}{2}.\end{aligned}$$

(ii) It follows, based on (2.1), that

$$\begin{aligned}2n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)-1 &= 2n\left(\int_0^1\frac{x^{n-1}}{1+x}\mathrm{d}x-\int_0^1\frac{x^{n-1}}{2}\mathrm{d}x\right)\\ &= n\int_0^1 x^{n-1}\frac{1-x}{1+x}\mathrm{d}x.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n\left[2n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)-1\right] &= \sum_{n=1}^{\infty}(-1)^n n\int_0^1 x^{n-1}\frac{1-x}{1+x}\mathrm{d}x\\ &= -\int_0^1\frac{1-x}{1+x}\left[\sum_{n=1}^{\infty}n(-x)^{n-1}\right]\mathrm{d}x\\ &= \int_0^1\frac{x-1}{(x+1)^3}\mathrm{d}x\\ &= -\frac{1}{4}.\end{aligned}$$

(iii) We have, using (2.1), that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1\frac{x^{n-1}}{1+x}\mathrm{d}x\\ &= \int_0^1\frac{1}{x(1+x)}\left(\sum_{n=1}^{\infty}\frac{(-x)^n}{n}\right)\mathrm{d}x\\ &= -\int_0^1\frac{\ln(1+x)}{x(1+x)}\mathrm{d}x\\ &= \int_0^1\frac{\ln(1+x)}{1+x}\mathrm{d}x-\int_0^1\frac{\ln(1+x)}{x}\mathrm{d}x\\ &= \frac{\ln^2 2}{2}-\frac{\zeta(2)}{2}.\end{aligned}$$

(iv) According to (2.1), we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^1\frac{x^{n-1}}{1+x}\mathrm{d}x\\ &= \int_0^1\frac{1}{x(1+x)}\sum_{n=1}^{\infty}\frac{(-x)^n}{n^2}\mathrm{d}x\\ &= \int_0^1\frac{\mathrm{Li}_2(-x)}{x(1+x)}\mathrm{d}x\\ &= \int_0^1\frac{\mathrm{Li}_2(-x)}{x}\mathrm{d}x-\int_0^1\frac{\mathrm{Li}_2(-x)}{1+x}\mathrm{d}x.\end{aligned}\tag{2.3}$$

We calculate the first integral in (2.3) and we have

$$\int_0^1\frac{\mathrm{Li}_2(-x)}{x}\mathrm{d}x=\int_0^1\frac{1}{x}\sum_{n=1}^{\infty}\frac{(-x)^n}{n^2}\mathrm{d}x=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}=-\frac{3}{4}\zeta(3).\tag{2.4}$$

We calculate the second integral in (2.3). We integrate by parts, with $f(x)=\mathrm{Li}_2(-x)$, $f'(x)=-\frac{\ln(1+x)}{x}$, $g'(x)=\frac{1}{1+x}$, $g(x)=\ln(1+x)$, and we have that

$$\int_0^1\frac{\mathrm{Li}_2(-x)}{1+x}\mathrm{d}x=\ln(1+x)\mathrm{Li}_2(-x)\Big|_0^1+\int_0^1\frac{\ln^2(1+x)}{x}\mathrm{d}x=-\frac{\ln 2}{2}\zeta(2)+\frac{\zeta(3)}{4},\tag{2.5}$$

since $\int_0^1\frac{\ln^2(1+x)}{x}\mathrm{d}x=\frac{\zeta(3)}{4}$ (see [1, pp. 291–292]).

Combining (2.3), (2.4) and (2.5), the desired result holds and part (iv) of the lemma is proved. □

Now we are ready to prove Theorem 2.1.

Proof. (a) We calculate the series by shifting the index of summation. We have

$$\begin{aligned}&\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2\\ &=\ln^2 2+\sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2\\ &\overset{n-1=m}{=}\ln^2 2+\sum_{m=1}^{\infty}(-1)^{m}\left(\frac{1}{m+1}-\frac{1}{m+2}+\frac{1}{m+3}-\cdots\right)^2\\ &=\ln^2 2-\sum_{m=1}^{\infty}(-1)^{m-1}\left[\frac{1}{m}-\left(\frac{1}{m}-\frac{1}{m+1}+\frac{1}{m+2}-\cdots\right)\right]^2\\ &=\ln^2 2+\sum_{m=1}^{\infty}\frac{(-1)^m}{m^2}+2\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\frac{1}{m}-\frac{1}{m+1}+\frac{1}{m+2}-\cdots\right)\\ &\quad-\sum_{m=1}^{\infty}(-1)^{m-1}\left(\frac{1}{m}-\frac{1}{m+1}+\frac{1}{m+2}-\cdots\right)^2\end{aligned}$$

and it follows, having in view part (iii) of Lemma 2.1, that

$$\begin{aligned}
&2\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2\\
&=\ln^2 2+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}+2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)\\
&=\ln^2 2-\frac{\zeta(2)}{2}+2\left(\frac{\zeta(2)}{2}-\frac{\ln^2 2}{2}\right)\\
&=\frac{\zeta(2)}{2}.
\end{aligned}$$

We mention that this series was calculated by a different method, based on an integration technique, in [3, problem 3.45].

(b) We shift the index of summation and we obtain

$$\begin{aligned}
S&=\sum_{n=1}^{\infty}(-1)^n n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2\\
&=-\ln^2 2+\sum_{n=2}^{\infty}(-1)^n n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2\\
&\overset{n-1=i}{=}-\ln^2 2+\sum_{i=1}^{\infty}(-1)^{i+1}(i+1)\left(\frac{1}{i+1}-\frac{1}{i+2}+\frac{1}{i+3}-\cdots\right)^2\\
&=-\ln^2 2-\sum_{i=1}^{\infty}(-1)^i(i+1)\left[\frac{1}{i}-\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\frac{1}{i+3}+\cdots\right)\right]^2\\
&=-\ln^2 2-\sum_{i=1}^{\infty}(-1)^i(i+1)\left[\frac{1}{i^2}-\frac{2}{i}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)\right.\\
&\qquad\qquad\left.+\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)^2\right]\\
&=-\ln^2 2-\sum_{i=1}^{\infty}(-1)^i\frac{i+1}{i^2}+2\sum_{i=1}^{\infty}(-1)^i\frac{i+1}{i}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)\\
&\quad-\sum_{i=1}^{\infty}(-1)^i(i+1)\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)^2\\
&=-\ln^2 2-\sum_{i=1}^{\infty}\frac{(-1)^i}{i}-\sum_{i=1}^{\infty}\frac{(-1)^i}{i^2}+2\sum_{i=1}^{\infty}(-1)^i\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)\\
&\quad+2\sum_{i=1}^{\infty}\frac{(-1)^i}{i}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)-S-\sum_{i=1}^{\infty}(-1)^i\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)^2.
\end{aligned}\tag{2.6}$$

It follows, having in view part (a) of Theorem 2.1 and parts (i) and (iii) of Lemma 2.1, that

$$2S=-\ln^2 2+\ln 2+\frac{\zeta(2)}{2}-1+2\left(\frac{\ln^2 2}{2}-\frac{\zeta(2)}{2}\right)+\frac{\zeta(2)}{4}=\ln 2-1-\frac{\zeta(2)}{4}.$$

Now we give **another proof**, *a gem in the theory of quadratic series*, of parts (a) and (b) of Theorem 2.1.

Let

$$S=\sum_{n=1}^{\infty}(-1)^n n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2$$

and

$$T=\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2.$$

We apply Abel's summation formula, which states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of real numbers and $A_n=\sum_{k=1}^{n}a_k$, then $\sum_{k=1}^{n}a_kb_k=A_nb_{n+1}+\sum_{k=1}^{n}A_k(b_k-b_{k+1})$, or, the infinite version

$$\sum_{k=1}^{\infty}a_kb_k=\lim_{n\to\infty}A_nb_{n+1}+\sum_{k=1}^{\infty}A_k(b_k-b_{k+1}). \tag{2.7}$$

Let $x_n=\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots$. We calculate the series in part (a) of Theorem 2.1 by using Abel's summation formula, with $a_n=1$ and $b_n=(-1)^n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2=(-1)^nx_n^2$, and we have that

$$\begin{aligned}
T&=\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^2\\
&=\lim_{n\to\infty}n(-1)^{n+1}\left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots\right)^2\\
&\quad+\sum_{n=1}^{\infty}n\left[(-1)^nx_n^2-(-1)^{n+1}x_{n+1}^2\right]\\
&=\sum_{n=1}^{\infty}n(-1)^n\left(x_n^2+x_{n+1}^2\right)\\
&=\sum_{n=1}^{\infty}n(-1)^n\left[(x_n+x_{n+1})^2-2x_nx_{n+1}\right]\\
&\overset{x_n+x_{n+1}=\frac{1}{n}}{=}\sum_{n=1}^{\infty}n(-1)^n\left[\frac{1}{n^2}-2x_n\left(\frac{1}{n}-x_n\right)\right]\\
&=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}-2\sum_{n=1}^{\infty}(-1)^nx_n+2\sum_{n=1}^{\infty}(-1)^nnx_n^2\\
&=-\ln 2-2\left(-\frac{1}{2}\right)+2\sum_{n=1}^{\infty}(-1)^nnx_n^2\\
&=-\ln 2+1+2S.
\end{aligned}$$

We used that $\lim_{n\to\infty}n\left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots\right)^2=0$, which follows based on (2.2).

It follows that $T-2S=1-\ln 2$. On the other hand, equality (2.6) implies that $2S+T=\ln 2-1-\frac{\zeta(2)}{2}$. Solving the system of linear equations we obtain that $T=-\frac{\zeta(2)}{4}$ and $S=\frac{\ln 2}{2}-\frac{1}{2}-\frac{\zeta(2)}{8}$.

(c) Multiply the series in part (b) by 2 and subtract the series from part (a).

(d) Let $x_n = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots$ and observe that $x_n + x_{n+1} = \frac{1}{n}$. The following identity can be proved by mathematical induction

$$(-1)^1 1^2 + (-1)^2 2^2 + \cdots + (-1)^n n^2 = (-1)^n \frac{n(n+1)}{2}, \quad n \geq 1. \tag{2.8}$$

We calculate the series in part (d) by applying Abel's summation formula with $a_n = (-1)^n n^2$ and $b_n = x_n^2 - \frac{1}{4n^2}$, and we have, based on (2.8), that

$$\begin{aligned}
&\sum_{n=1}^{\infty} (-1)^n \left[n^2 \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right)^2 - \frac{1}{4} \right] \\
&= \lim_{n\to\infty} (-1)^n \frac{n(n+1)}{2} \left[x_{n+1}^2 - \frac{1}{4(n+1)^2} \right] \\
&\quad + \sum_{n=1}^{\infty} (-1)^n \frac{n(n+1)}{2} \left[x_n^2 - \frac{1}{4n^2} - x_{n+1}^2 + \frac{1}{4(n+1)^2} \right] \\
&= \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n n(n+1) \left[\frac{1}{n}(x_n - x_{n+1}) + \frac{1}{4(n+1)^2} - \frac{1}{4n^2} \right] \\
&= \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n (n+1) \left(x_n - x_{n+1} + \frac{n}{4(n+1)^2} - \frac{1}{4n} \right) \\
&= \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n (n+1) \left(2x_n - \frac{1}{n} + \frac{1}{4(n+1)} - \frac{1}{4n} - \frac{1}{4(n+1)^2} \right) \\
&= \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n (n+1) \left(2x_n - \frac{1}{n} - \frac{1}{4n(n+1)} - \frac{1}{4(n+1)^2} \right) \\
&= \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n (n+1) \left(2x_n - \frac{1}{n} \right) + \frac{1}{8} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + \frac{1}{8} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n (2nx_n - 1) + \sum_{n=1}^{\infty} (-1)^n x_n - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} + \frac{1}{8} \\
&\overset{\text{Lemma 2.1 } (i),(ii)}{=} -\frac{1}{8} - \frac{1}{2} + \frac{\ln 2}{2} + \frac{1}{8} \\
&= \frac{\ln 2 - 1}{2}.
\end{aligned}$$

We used in the preceding calculations that $\lim_{n\to\infty} n(n+1) \left[x_{n+1}^2 - \frac{1}{4(n+1)^2} \right] = 0$, which follows based on (2.2).

(e) We mention that the series in part (e) of Theorem 2.1 was calculated, by an integration technique, by Boyadzhiev in [2, formula 19].

Here, we calculate the series using a technique based on evaluating the alternating cubic series $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right)^3$ by shifting the index of summation. We have

$$\begin{aligned}
S &= \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3\\
&= -\ln^3 2+\sum_{n=2}^{\infty}(-1)^n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3\\
&\overset{n-1=i}{=} -\ln^3 2+\sum_{i=1}^{\infty}(-1)^{i+1}\left(\frac{1}{i+1}-\frac{1}{i+2}+\frac{1}{i+3}-\cdots\right)^3\\
&= -\ln^3 2-\sum_{i=1}^{\infty}(-1)^i\left[\frac{1}{i}-\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)\right]^3\\
&= -\ln^3 2-\sum_{i=1}^{\infty}(-1)^i\left[\frac{1}{i^3}-\frac{3}{i^2}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)\right.\\
&\qquad\left.+\frac{3}{i}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)^2-\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)^3\right]\\
&= -\ln^3 2-\sum_{i=1}^{\infty}\frac{(-1)^i}{i^3}+3\sum_{i=1}^{\infty}\frac{(-1)^i}{i^2}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)\\
&\quad -3\sum_{i=1}^{\infty}\frac{(-1)^i}{i}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)^2+S.
\end{aligned}$$

It follows, having in view part (iv) of Lemma 2.1, that

$$\begin{aligned}
&3\sum_{i=1}^{\infty}\frac{(-1)^i}{i}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)^2\\
&= -\ln^3 2-\sum_{i=1}^{\infty}\frac{(-1)^i}{i^3}+3\sum_{i=1}^{\infty}\frac{(-1)^i}{i^2}\left(\frac{1}{i}-\frac{1}{i+1}+\frac{1}{i+2}-\cdots\right)\\
&= -\ln^3 2+\frac{3}{4}\zeta(3)+3\left(\frac{\ln 2}{2}\zeta(2)-\zeta(3)\right)\\
&= -\ln^3 2+\frac{3}{2}\ln 2\cdot\zeta(2)-\frac{9}{4}\zeta(3).
\end{aligned}$$

Theorem 2.1 is proved. □

Now we give the proof of Theorem 2.2.

Proof. We apply Abel's summation formula, with $a_n = n$ and $b_n = (-1)^n x_n^3$, where

$$x_n = \frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots,$$

and we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^n n\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\cdots\right)^3\\
&=\lim_{n\to\infty}(-1)^{n+1}\frac{n(n+1)}{2}\left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots\right)^3\\
&\quad+\sum_{n=1}^{\infty}\frac{n(n+1)}{2}(-1)^n(x_n^3+x_{n+1}^3)\\
&\overset{x_n+x_{n+1}=\frac{1}{n}}{=}\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)(x_n^2-x_nx_{n+1}+x_{n+1}^2)\\
&\overset{x_n+x_{n+1}=\frac{1}{n}}{=}\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)\left(3x_n^2-\frac{3}{n}x_n+\frac{1}{n^2}\right)\\
&=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\left(3nx_n^2+3x_n^2-3x_n-\frac{3}{n}x_n+\frac{1}{n}+\frac{1}{n^2}\right)\\
&=\frac{3}{2}\sum_{n=1}^{\infty}(-1)^n nx_n^2+\frac{3}{2}\sum_{n=1}^{\infty}(-1)^n x_n^2-\frac{3}{2}\sum_{n=1}^{\infty}(-1)^n x_n\\
&\quad-\frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}x_n+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\\
&\overset{(*)}{=}\frac{3}{2}\left(\frac{\ln 2}{2}-\frac{1}{2}-\frac{\zeta(2)}{8}\right)-\frac{3}{8}\zeta(2)+\frac{3}{4}-\frac{3}{2}\left(\frac{\ln^2 2}{2}-\frac{\zeta(2)}{2}\right)-\frac{\ln 2}{2}-\frac{\zeta(2)}{4}\\
&=\frac{\ln 2}{4}-\frac{3\ln^2 2}{4}-\frac{\zeta(2)}{16}.
\end{aligned}$$

At step (*) we used parts (a) and (b) of Theorem 2.1 and parts (i) and (iii) of Lemma 2.1. We also used in the preceding calculations that $\lim_{n\to\infty} n(n+1)\left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots\right)^3=0$, which follows based on (2.2). □

We mention that other challenging alternating quadratic series involving the tail of various special functions, as well as open problems, can be found in [4].

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