CREAT. MATH. INFORM.
Volume **33** (2024), No. 1,
Pages 33 - 46

Online version at https://semnul.com/creative-mathematics/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2024.01.04

Fusion frame and its alternative dual in tensor product of Hilbert spaces

PRASENJIT GHOSH[1] and T. K. SAMANTA[2]

ABSTRACT. We study fusion frame in tensor product of Hilbert spaces and discuss some of its properties. The resolution of the identity operator on a tensor product of Hilbert spaces is being discussed. An alternative dual of a fusion frame in tensor product of Hilbert spaces is also presented.

1. INTRODUCTION

Fusion frame was investigated by P. Casazza and G. Kutyniok [2]. They define frames for closed subspaces of a given Hilbert spaces with respect to the orthogonal projections. In frame theory, fusion frame is a one kind of its generalization and it has been used in coding theory, data processing, signal processing and many other fields. Infact, the fusion frame theory is more elegent due to complicated relations between the structure of the sequence of weighted subspace and the local frames in the subspace. In fusion frame, the atomic resolution of the identity operator on Hilbert space was studied by M. S Asgari and Amil Khosraki [1]. They present a reconstruction formula and establish some useful results about resolution of the identity operator.

The basic concepts of tensor product of Hilbert spaces were described by S. Rabinson [11]. In Tensor Product of Hilbert spaces, the ideas of frames and Bases were studied by A. Khosravi and M. S. Asgari [9]. Reddy et al. [12] also studied the frame in tensor product of Hilbert spaces and presented the frame operator on tensor product of Hilbert spaces. The concepts of fusion frames and g-frames in tensor product of Hilbert spaces were introduced by Amir Khosravi and M. Mirzaee Azandaryani [10].

In this paper, fusion frame in tensor product of Hilbert spaces is developed and discuss a relationship among fusion frames in Hilbert spaces and their tensor products. We shall verify that in tensor product of Hilbert spaces, an image of a fusion frame under a bounded linear operator will be a fusion frame if the operator is invertible and unitary. The resolution of the identity operator on a tensor product of Hilbert spaces is presented. In tensor product of Hilbert spaces, we study an alternative dual of a fusion frame and see that the canonical dual of a fusion frame is also an alternative dual. Finally, we establish that an alternative dual of a fusion frame is a fusion frame in tensor product of Hilbert spaces.

Throughout this paper, we consider $(H, \langle \cdot, \cdot \rangle_1)$ and $(K, \langle \cdot, \cdot \rangle_2)$ be two separable Hilbert spaces. The identity operators on H and K are denoted by I_H and I_K respectively. $\mathcal{B}(H, K)$ is the collection of all bounded linear operators from H to K. Particularly, the space of all bounded linear operators on H is denoted by $\mathcal{B}(H)$. P_V is considered to be the orthogonal projection onto the closed subspace $V \subset H$. $\{V_i\}_{i \in I}$ and $\{W_j\}_{j \in J}$

Received: 01.11.2022. In revised form: 12.06.2023. Accepted: 19.06.2023
2020 *Mathematics Subject Classification.* 42C15, 46C07.
Key words and phrases. *Fusion frame, Resolution of identity operator, Canonical dual frame, Tensor product of Hilbert spaces, Tensor product of frames.*
Corresponding author: Prasenjit Ghosh; prasenjitpuremath@gmail.com

are the collections of closed subspaces of H and K, where $I,\, J$ are index sets. Consider the space

$$l^{2}\left(\{V_i\}_{i\in I}\right) = \left\{\{f_i\}_{i\in I} : f_i \in V_i\,,\, \sum_{i\in I} \|f_i\|_1^2 < \infty\right\}$$

with inner product defined by

$$\langle \{f_i\}_{i\in I}\,,\, \{g_i\}_{i\in I}\rangle = \sum_{i\in I} \langle f_i\,,\, g_i\rangle_1\,.$$

It is easy to verify that $l^{2}\left(\{V_i\}_{i\in I}\right)$ is a Hilbert space with respect to the above inner product [1]. In the similar way, we can define the space $l^{2}\left(\{W_j\}_{j\in J}\right)$.

2. Preliminaries

Let T be a bounded linear operators on H. Then T^{*} denotes the adjoint of T and for a closed subspace $V \subset H$, $\overline{V}$ denotes the closure of V.

Theorem 2.1. [6] *Let $T \in \mathcal{B}(H)$ and V be a closed subspace of H. Then $P_V\, T^{*} = P_V\, T^{*}\, P_{\overline{T\,V}}$. If in particular, T is an unitary operator (i.e $T^{*}T = I_H$), then $P_{\overline{T\,V}}\, T = T\, P_V$.*

Theorem 2.2. [7] *Let $\mathcal{S}(H)$ be the set of all self-adjoint operators on H and let $T,\, S \in \mathcal{S}(H)$. We say that $T \leq S$ if*

$$\langle T(f)\,,\, f\rangle_1 \leq \langle S(f)\,,\, f\rangle_1 \;\;\forall\, f \in H.$$

Definition 2.1. [2] Let $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. Then the family $V = \{(V_i,\, v_i) : i \in I\}$ is said to be a fusion frame for H if there exist constants $0 < A \leq B < \infty$ such that

$$A\,\|f\|_1^2 \leq \sum_{i\in I} v_i^2\,\|P_{V_i}(f)\|_1^2 \leq B\,\|f\|_1^2 \;\;\forall\, f \in H. \tag{2.1}$$

We call $A,\, B$ the fusion frame bounds. If the family V satisfies only right inequality of (2.1), then it is called a fusion Bessel sequence in H with bound B.

Definition 2.2. [2] Let $V = \{(V_i,\, v_i)\}_{i\in I}$ be a fusion Bessel sequence in H with bound B. The operator $T_V : l^{2}\left(\{V_i\}_{i\in I}\right) \to H$ defined as

$$T_V\left(\{f_i\}_{i\in I}\right) = \sum_{i\in I} v_i\, f_i \;\;\forall\, \{f_i\}_{i\in I} \in l^{2}\left(\{V_i\}_{i\in I}\right)$$

is called the synthesis operator and the operator given by

$$T_V^{*} : H \to l^{2}\left(\{V_i\}_{i\in I}\right),\; T_V^{*}(f) = \{v_i\, P_{V_i}(f)\}_{i\in I} \;\;\forall\, f \in H,$$

is called analysis operator. The operator $S_V : H \to H$ defined as

$$S_V(f) = T_V\, T_V^{*}(f) = \sum_{i\in I} v_i^2\, P_{V_i}(f) \;\;\forall\, f \in H,$$

is called fusion frame operator.

Remark 2.1. [2] Let V be a fusion frame with bounds $A,\, B$. Then from (2.1),

$$\langle A\,f\,,\, f\rangle_1 \leq \langle S_V(f)\,,\, f\rangle_1 \leq \langle B\,f\,,\, f\rangle_1 \;\;\forall\, f \in H.$$

The operator S_V is bounded, self-adjoint, positive and invertible. Now, according to the Theorem (2.2), we can write, $A\,I_H \leq S_V \leq B\,I_H$ and this gives $B^{-1}\,I_H \leq S_V^{-1} \leq A^{-1}\,I_H$.

Definition 2.3. [2] Let $V = \{(V_i, v_i)\}_{i \in I}$ be a fusion frame for H. Then $\{ (S_V^{-1} V_i, v_i) \}_{i \in I}$ is called the canonical dual fusion frame of V.

Theorem 2.3. [6] *Let $V = \{ (V_i, v_i) \}_{i \in I}$ be a fusion frame for H with bounds A, B and S_V be the corresponding frame operator. Then the canonical dual fusion frame of V is a fusion frame with bounds* $\dfrac{A}{\| S_V \|^2 \, \| S_V^{-1} \|^2}$, $B \, \| S_V \|^2 \, \| S_V^{-1} \|^2$.

Remark 2.2. [6] A reconstruction formula on H with the help of canonical dual fusion frame is given by

$$f = \sum_{i \in I} v_i^2 \, P_{S_V^{-1} V_i} \, S_V^{-1} \, P_{V_i} (f) \quad \forall \, f \in H.$$

Definition 2.4. [6] Let $V = \{ (V_i, v_i) \}_{i \in I}$ be a fusion frame for H and S_V be the corresponding frame operator. Then a fusion Bessel sequence $V' = \{ (V_i', v_i') \}_{i \in I}$ is said to be an alternative dual of V if

$$f = \sum_{i \in I} v_i \, v_i' \, P_{V_i'} \, S_V^{-1} \, P_{V_i} (f) \quad \forall \, f \in H.$$

Definition 2.5. [2] A family of bounded operators $\{ T_i \}_{i \in I}$ on H is called a resolution of identity operator on H if for all $f \in H$, we have $f = \sum\limits_{i \in I} T_i (f)$, provided the series converges unconditionally for all $f \in H$.

There are several ways to introduced the tensor product of Hilbert spaces. The tensor product of Hilbert spaces H and K is a certain linear space of operators which was represented by Folland in [5], Kadison and Ringrose in [8].

Definition 2.6. [12] The tensor product of Hilbert spaces H and K is denoted by $H \otimes K$ and it is defined to be an inner product space with respect to the inner product

$$\langle f \otimes g , f' \otimes g' \rangle = \langle f , f' \rangle_1 \, \langle g , g' \rangle_2 \quad \forall \, f, f' \in H \;\; \& \;\; g, g' \in K.$$

The norm on $H \otimes K$ is defined by

$$\| f \otimes g \| = \| f \|_1 \, \| g \|_2 \quad \forall \, f \in H \;\; \& \;\; g \in K. \tag{2.2}$$

The space $H \otimes K$ is complete with respect to the above inner product. Therefore, the space $H \otimes K$ is an Hilbert space.

For $Q \in \mathcal{B}(H)$ and $T \in \mathcal{B}(K)$, the tensor product of operators Q and T is denoted by $Q \otimes T$ and is defined as

$$(Q \otimes T) \, A = Q \, A \, T^* \quad \forall \; A \in H \otimes K.$$

Theorem 2.4. [5, 13] *Let $Q, Q' \in \mathcal{B}(H)$ and $T, T' \in \mathcal{B}(K)$. Then*

- (I) $Q \otimes T \in \mathcal{B}(H \otimes K)$ *and* $\| Q \otimes T \| = \| Q \| \, \| T \|$.
- (II) $(Q \otimes T) (f \otimes g) = Q (f) \otimes T (g)$ *for all* $f \in H, g \in K$.
- (III) $(Q \otimes T) (Q' \otimes T') = (Q Q') \otimes (T T')$.
- (IV) $Q \otimes T$ *is invertible if and only if Q and T are invertible, in which case* $(Q \otimes T)^{-1} = \left(Q^{-1} \otimes T^{-1} \right)$.
- (V) $(Q \otimes T)^* = (Q^* \otimes T^*)$.
- (VI) *Let $f, f' \in H \setminus \{0\}$ and $g, g' \in K \setminus \{0\}$. If $f \otimes g = f' \otimes g'$, then there exist constants a and b with $a \, b = 1$ such that $f = a \, f'$ and $g = b \, g'$.*

3. Fusion frame in tensor product of Hilbert spaces

In this section, fusion frame in the Hilbert space $H \otimes K$ have been discussed and some results associated to fusion frame in $H \otimes K$ are likely to be established. At the end, we discuss the relationship among the resolution of the identity operator on $H \otimes K$ and resolutions of the identity operators on H and K, respectively.

Definition 3.7. Let $\{ v_i \}_{i \in I}$ and $\{ w_j \}_{j \in J}$ be two families of positive weights i.e., $v_i > 0 \ \forall \, i \in I$ and $w_j > 0 \ \forall \, j \in J$ and $\{ V_i \otimes W_j : (i, j) \in I \times J \}$ be a family of closed subspaces of $H \otimes K$. Then the family $V \otimes W = \{ (V_i \otimes W_j, v_i w_j) \}_{i, j}$ is called a fusion frame for $H \otimes K$, if there exist $0 < A \leq B < \infty$ such that

$$A \, \| f \otimes g \|^2 \leq \sum_{i, j} v_i^2 w_j^2 \left\| P_{V_i \otimes W_j} (f \otimes g) \right\|^2 \leq B \, \| f \otimes g \|^2 \ \ \forall \, f \otimes g \in H \otimes K,$$

where $P_{V_i \otimes W_j}$ is the orthogonal projection of $H \otimes K$ onto $V_i \otimes W_j$. The constants A and B are called the frame bounds. If $A = B$ then it is called a tight fusion frame for $H \otimes K$. If the family $V \otimes W$ satisfies the inequality

$$\sum_{i, j} v_i^2 w_j^2 \left\| P_{V_i \otimes W_j} (f \otimes g) \right\|^2 \leq B \, \| f \otimes g \|^2 \ \ \forall \, f \otimes g \in H \otimes K,$$

then it is called a fusion Bessel sequence in $H \otimes K$ with bound B.

Definition 3.8. For $i \in I$ and $j \in J$, define the space $l^2 (\{ V_i \otimes W_j \})$

$$= \left\{ \{ f_i \otimes g_j \} : f_i \otimes g_j \in V_i \otimes W_j, \text{ and } \sum_{i, j} \| f_i \otimes g_j \|^2 < \infty \right\}$$

with inner product

$$\left\langle \{ f_i \otimes g_j \}, \{ f_i' \otimes g_j' \} \right\rangle_{l^2} = \sum_{i, j} \left\langle f_i \otimes g_j , f_i' \otimes g_j' \right\rangle$$

$$= \sum_{i, j} \langle f_i , f_i' \rangle_1 \, \langle g_j , g_j' \rangle_2 = \left(\sum_{i \in I} \langle f_i , f_i' \rangle_1 \right) \left(\sum_{j \in J} \langle g_j , g_j' \rangle_2 \right)$$

$$= \left\langle \{ f_i \}_{i \in I} , \{ f_i' \}_{i \in I} \right\rangle_{l^2 (\{ V_i \}_{i \in I})} \left\langle \{ g_j \}_{j \in J} , \{ g_j' \}_{j \in J} \right\rangle_{l^2 (\{ W_j \}_{j \in J})} .$$

It is easy to verify that the space $l^2 (\{ V_i \otimes W_j \})$ is an Hilbert space with respect to the above inner product.

Remark 3.3. Since $\{ V_i \}_{i \in I}$, $\{ W_j \}_{j \in J}$ and $\{ V_i \otimes W_j \}_{i, j}$ are the families of closed subspaces of H, K and $H \otimes K$ respectively, it is easy to verify that $P_{V_i \otimes W_j} = P_{V_i} \otimes P_{W_j}$.

For the remaining part of this paper, we denote the collections $\{ (V_i, v_i) \}_{i \in I}$, $\{ (W_j, w_j) \}_{j \in J}$, $\{ (V_i \otimes W_j, v_i w_j) \}_{i, j}$ and $\left\{ \left(V_i' \otimes W_j', v_i' w_j' \right) \right\}_{i, j}$ by V, W, $V \otimes W$ and $V' \otimes W'$, respectively.

Theorem 3.5. *Let V and W be the families of weighted closed subspaces in H and K, respectively. Then V and W are fusion frames for H and K if and only if $V \otimes W$ is a fusion frame for $H \otimes K$.*

Proof. First we suppose that V and W are fusion frames for H and K. Then there exist positive constants A, B and C, D such that

$$A \, \| f \|_1^2 \leq \sum_{i \in I} v_i^2 \, \| P_{V_i}(f) \|_1^2 \leq B \, \| f \|_1^2 \quad \forall f \in H \tag{3.3}$$

$$C \, \| g \|_2^2 \leq \sum_{j \in J} w_j^2 \, \left\| P_{W_j}(g) \right\|_2^2 \leq D \, \| g \|_2^2 \quad \forall g \in K. \tag{3.4}$$

Multiplying (3.3) and (3.4), and using the definition of norm on $H \otimes K$, we get

$$A C \| f \|_1^2 \, \| g \|_2^2 \leq \left(\sum_{i \in I} v_i^2 \| P_{V_i}(f) \|_1^2 \right) \left(\sum_{j \in J} w_j^2 \left\| P_{W_j}(g) \right\|_2^2 \right) \leq B D \| f \|_1^2 \, \| g \|_2^2$$

$$\Rightarrow A C \, \| f \otimes g \|^2 \leq \sum_{i, j} v_i^2 w_j^2 \, \| P_{V_i}(f) \|_1^2 \, \left\| P_{W_j}(g) \right\|_2^2 \leq B D \, \| f \otimes g \|^2$$

$$\Rightarrow A C \, \| f \otimes g \|^2 \leq \sum_{i, j} v_i^2 w_j^2 \, \left\| P_{V_i}(f) \otimes P_{W_j}(g) \right\|^2 \leq B D \, \| f \otimes g \|^2.$$

Therefore, for all $f \otimes g \in H \otimes K$, we have

$$A C \, \| f \otimes g \|^2 \leq \sum_{i, j} v_i^2 w_j^2 \, \left\| \left(P_{V_i} \otimes P_{W_j} \right) (f \otimes g) \right\|^2 \leq B D \, \| f \otimes g \|^2$$

$$\Rightarrow A C \, \| f \otimes g \|^2 \leq \sum_{i, j} v_i^2 w_j^2 \, \left\| P_{V_i \otimes W_j}(f \otimes g) \right\|^2 \leq B D \, \| f \otimes g \|^2.$$

This shows that $V \otimes W$ is a fusion frame for $H \otimes K$ with bounds $A C$ and $B D$.

Conversely, suppose that $V \otimes W$ is a fusion frame for $H \otimes K$ with bounds A and B. Then, for each $f \otimes g \in H \otimes K - \{\theta \otimes \theta\}$, we have

$$A \, \| f \otimes g \|^2 \leq \sum_{i, j} v_i^2 w_j^2 \, \left\| P_{V_i \otimes W_j}(f \otimes g) \right\|^2 \leq B \, \| f \otimes g \|^2$$

$$\Rightarrow A \, \| f \|_1^2 \, \| g \|_2^2 \leq \sum_{i, j} v_i^2 w_j^2 \, \left\| P_{V_i}(f) \otimes P_{W_j}(g) \right\|^2 \leq B \, \| f \|_1^2 \, \| g \|_2^2$$

$$\Rightarrow A \| f \|_1^2 \, \| g \|_2^2 \leq \left(\sum_{i \in I} v_i^2 \| P_{V_i}(f) \|_1^2 \right) \left(\sum_{j \in J} w_j^2 \left\| P_{W_j}(g) \right\|_2^2 \right) \leq B \| f \|_1^2 \, \| g \|_2^2.$$

Since $f \otimes g$ is non-zero vector, f and g are also non-zero vectors and therefore $\sum_{i \in I} v_i^2 \, \| P_{V_i}(f) \|_1^2$ and $\sum_{j \in J} w_j^2 \, \left\| P_{W_j}(g) \right\|_2^2$ are non-zero.

$$\Rightarrow \frac{A \| g \|_2^2}{\sum_{j \in J} w_j^2 \, \left\| P_{W_j}(g) \right\|_2^2} \, \| f \|_1^2 \leq \sum_{i \in I} v_i^2 \, \| P_{V_i}(f) \|_1^2 \leq \frac{B \| g \|_2^2}{\sum_{j \in J} w_j^2 \, \left\| P_{W_j}(g) \right\|_2^2} \, \| f \|_1^2$$

$$\Rightarrow A_1 \, \| f \|_1^2 \leq \sum_{i \in I} v_i^2 \, \| P_{V_i}(f) \|_1^2 \leq B_1 \, \| f \|_1^2 \quad \forall f \in H,$$

where $A_1 = \dfrac{A \| g \|_2^2}{\sum_{j \in J} w_j^2 \, \left\| P_{W_j}(g) \right\|_2^2}$ and $B_1 = \dfrac{B \| g \|_2^2}{\sum_{j \in J} w_j^2 \, \left\| P_{W_j}(g) \right\|_2^2}$. This shows that V is a fusion frame for H. Similarly, it can be shown that W is a fusion frame for K. □

Now, we validate this theorem by considering the following example.

3.1. **Example.** Let $H = \mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ be an orthonormal basis for H. Suppose that $V_1 = \overline{span}\{e_2, e_3\}$, $V_2 = \overline{span}\{e_1, e_3\}$ and $V_3 = \overline{span}\{e_1\}$ with $v_i = 1$, for $i = 1, 2, 3$. Now, for any $f = (f_1, f_2, f_3) \in H$, we have

$$\sum_{i=1}^{3} v_i^2 \| P_{V_i} f \|^2 = 2\left(f_1^2 + f_3^2\right) + f_2^2.$$

Thus,

$$\| f \|^2 \leq \sum_{i=1}^{3} v_i^2 \| P_{V_i} f \|^2 \leq 2\| f \|^2, \ \forall f \in H.$$

Hence, $\{(V_i, 1)\}_{i=1}^{3}$ is a fusion frame for H with bounds 1 and 2.

Next, we consider the Hilbert space $K = \mathbb{R}^2$ and $\{e_1, e_2\}$ be an orthonormal basis for K. Consider $W_1 = \overline{span}\{e_1, 2e_2\}$, $W_2 = \overline{span}\{e_2\}$ with $w_j = 1$, $j = 1, 2$. Thus, for any $g = (g_1, g_2) \in H$, we have

$$\sum_{j=1}^{2} w_j^2 \left\| P_{W_j} g \right\|^2 = g_1^2 + 5g_2^2.$$

Thus, $\{(W_j, 1)\}_{j=1}^{2}$ is a fusion frame for K with bounds 1 and 5. Therefore, by Theorem 3.5, $\{(V_i \otimes W_j, 1)\}_{i,j}$ is a fusion frame for $H \otimes K = \mathbb{R}^6$ with bounds 1 and 10.

Remark 3.4. Let $V \otimes W$ be a fusion frame for $H \otimes K$. According to the definition (2.2), the corresponding frame operator $S_{V \otimes W} : H \otimes K \to H \otimes K$ is given by

$$S_{V \otimes W}(f \otimes g) = \sum_{i,j} v_i^2 w_j^2 P_{V_i \otimes W_j}(f \otimes g) \ \forall f \otimes g \in H \otimes K.$$

Theorem 3.6. *Let S_V, S_W and $S_{V \otimes W}$ be the corresponding frame operators for the fusion frames V, W and $V \otimes W$, respectively. Then $S_{V \otimes W} = S_V \otimes S_W$ and $S_{V \otimes W}^{-1} = S_V^{-1} \otimes S_W^{-1}$.*

Proof. For each $f \otimes g \in H \otimes K$, we have

$$\begin{aligned} S_{V \otimes W}(f \otimes g) &= \sum_{i,j} v_i^2 w_j^2 P_{V_i \otimes W_j}(f \otimes g) \\ &= \sum_{i,j} v_i^2 w_j^2 \left(P_{V_i} \otimes P_{W_j}\right)(f \otimes g) \\ &= \sum_{i,j} v_i^2 w_j^2 \left(P_{V_i}(f) \otimes P_{W_j}(g)\right) \\ &= \left(\sum_{i \in I} v_i^2 P_{V_i}(f)\right) \otimes \left(\sum_{j \in J} w_j^2 P_{W_j}(g)\right) \\ &= S_V(f) \otimes S_W(g) = S_V \otimes S_W(f \otimes g). \end{aligned}$$

This implies that $S_{V \otimes W} = S_V \otimes S_W$. Since S_V and S_W are invertible, by (IV) of the Theorem (2.4), it follows that $S_{V \otimes W}^{-1} = S_V^{-1} \otimes S_W^{-1}$. □

Theorem 3.7. *Let V and W be fusion frames for H and K with frame bounds A, B and C, D having their corresponding fusion frame operators S_V and S_W, respectively. If T_1 and*

T_2 are invertible and unitary operators on H and K, respectively then the family given by $\Delta = \{(T_1 \otimes T_2)(V_i \otimes W_j), v_i w_j\}_{i,j}$ is a fusion frame for $H \otimes K$.

Proof. Since the operators T_1 and T_2 are invertible, by (IV) of the Theorem (2.4), $T_1 \otimes T_2$ is invertible and $(T_1 \otimes T_2)^{-1} = \left(T_1^{-1} \otimes T_2^{-1}\right)$. Also, by Theorem (2.1), for any $i \in I$ and $j \in J$, we get

$$\|P_{V_i} T_1^*(f)\|_1 \leq \|T_1^*\| \, \|P_{T_1 V_i}(f)\|_1 \quad \forall f \in H, \text{ and} \tag{3.5}$$

$$\left\|P_{W_j} T_2^*(g)\right\|_2 \leq \|T_2^*\| \, \left\|P_{T_2 W_j}(g)\right\|_2 \quad \forall g \in K. \tag{3.6}$$

Again, since T_1 and T_2 are invertible, for each $f \in H$ and $g \in K$, we obtain

$$\|f\|_1 \leq \left\|\left(T_1^{-1}\right)^*\right\| \, \|T_1^*(f)\|_1 \;\;\&\;\; \|g\|_2 \leq \left\|\left(T_2^{-1}\right)^*\right\| \, \|T_2^*(g)\|_2 . \tag{3.7}$$

Now, for each $f \otimes g \in H \otimes K$, using Theorem (2.4), we get

$$\sum_{i,j} v_i^2 w_j^2 \left\|P_{(T_1 \otimes T_2)(V_i \otimes W_j)}(f \otimes g)\right\|^2 = \sum_{i,j} v_i^2 w_j^2 \left\|P_{(T_1 V_i \otimes T_2 W_j)}(f \otimes g)\right\|^2$$

$$= \sum_{i,j} v_i^2 w_j^2 \left\|\left(P_{T_1 V_i} \otimes P_{T_2 W_j}\right)(f \otimes g)\right\|^2 \quad [\text{ by note (3.3) }]$$

$$= \left(\sum_{i \in I} v_i^2 \, \|P_{T_1 V_i}(f)\|_1^2\right) \left(\sum_{j \in J} w_j^2 \, \left\|P_{T_2 W_j}(g)\right\|_2^2\right) \quad [\text{ using (2.2) }] \tag{3.8}$$

$$\geq \frac{1}{\|T_1\|^2 \|T_2\|^2} \left(\sum_{i \in I} v_i^2 \|P_{V_i}(T_1^* f)\|_1^2\right) \left(\sum_{j \in J} w_j^2 \left\|P_{W_j}(T_2^* g)\right\|_2^2\right) \quad [\text{ by (3.5) \& (3.6) }]$$

$$\geq \frac{AC}{\|T_1\|^2 \, \|T_2\|^2} \|T_1^*(f)\|_1^2 \, \|T_2^*(g)\|_2^2 \quad [\text{ since } V,\ W \text{ are fusion frames }]$$

$$\geq \frac{AC}{\|T_1\|^2 \, \|T_2\|^2 \, \left\|T_1^{-1}\right\|^2 \, \left\|T_2^{-1}\right\|^2} \|f\|_1^2 \, \|g\|_2^2 \quad [\text{ by (3.7) }]$$

$$= \frac{AC}{\|T_1 \otimes T_2\|^2 \left\|(T_1 \otimes T_2)^{-1}\right\|^2} \, \|f \otimes g\|^2 .$$

On the other hand, since T_1 and T_2 are unitary operators, again by Theorem (2.1), $P_{T_1 V_i} T_1 = T_1 P_{V_i}$ and $P_{T_2 W_j} T_2 = T_2 P_{W_j}$. Then, for all $f \otimes g \in H \otimes K$, we have

$$\begin{aligned}
&\sum_{i,\,j} v_i^2\, w_j^2\, \left\| P_{(T_1 \otimes T_2)(V_i \otimes W_j)}(f \otimes g) \right\|^2 \\
&= \left(\sum_{i \in I} v_i^2\, \| P_{T_1 V_i}(f) \|_1^2 \right) \left(\sum_{j \in J} w_j^2\, \| P_{T_2 W_j}(g) \|_2^2 \right) \quad [\text{ by (3.8) }] \\
&= \left(\sum_{i \in I} v_i^2\, \left\| T_1\, P_{V_i}\left(T_1^{-1} f\right) \right\|_1^2 \right) \left(\sum_{j \in J} w_j^2\, \left\| T_2\, P_{W_j}\left(T_2^{-1} g\right) \right\|_2^2 \right) \\
&\leq \| T_1 \|^2\, \| T_2 \|^2 \left(\sum_{i \in I} v_i^2\, \left\| P_{V_i}\left(T_1^{-1} f\right) \right\|_1^2 \right) \left(\sum_{j \in J} w_j^2\, \left\| P_{W_j}\left(T_2^{-1} g\right) \right\|_2^2 \right) \\
&\leq B\, D\, \| T_1 \|^2\, \| T_2 \|^2\, \left\| T_1^{-1}(f) \right\|_1^2\, \left\| T_2^{-1}(g) \right\|_2^2 \quad [\text{ since } V,\ W \text{ are fusion frames }] \\
&\leq B\, D\, \| T_1 \|^2\, \| T_2 \|^2\, \left\| T_1^{-1} \right\|^2\, \left\| T_2^{-1} \right\|^2\, \| f \|_1^2\, \| g \|_2^2 \\
&= B\, D\, \| T_1 \otimes T_2 \|^2\, \left\| (T_1 \otimes T_2)^{-1} \right\|^2\, \| f \otimes g \|^2.
\end{aligned}$$

Hence, Δ is a fusion frame for $H \otimes K$. □

Theorem 3.8. *The corresponding fusion frame operator for the fusion frame* Δ *is given by* $(T_1 \otimes T_2)\, S_{V \otimes W}\, (T_1 \otimes T_2)^{-1}$.

Proof. For each $f \otimes g \in H \otimes K$, we have

$$\begin{aligned}
&\sum_{i,\,j} v_i^2\, w_j^2\, P_{(T_1 \otimes T_2)(V_i \otimes W_j)}(f \otimes g) = \sum_{i,\,j} v_i^2\, w_j^2\, \left(P_{T_1 V_i} \otimes P_{T_2 W_j} \right)(f \otimes g) \\
&= \left(\sum_{i \in I} v_i^2\, P_{T_1 V_i}(f) \right) \otimes \left(\sum_{j \in J} w_j^2\, P_{T_2 W_j}(g) \right) \\
&= \left(\sum_{i \in I} v_i^2\, T_1\, P_{V_i}\left(T_1^{-1} f\right) \right) \otimes \left(\sum_{j \in J} w_j^2\, T_2\, P_{W_j}\left(T_2^{-1} g\right) \right) \quad [\text{ by Theorem (2.1) }] \\
&= T_1\, S_V\, \left(T_1^{-1}(f)\right) \otimes T_2\, S_W\, \left(T_2^{-1}(g)\right) \\
&= (T_1 \otimes T_2)\,(S_V \otimes S_W)\,\left(T_1^{-1} \otimes T_2^{-1}\right)(f \otimes g) \ [\text{ by Theorem (2.4) }] \\
&= (T_1 \otimes T_2)\, S_{V \otimes W}\, (T_1 \otimes T_2)^{-1}\, (f \otimes g).
\end{aligned}$$

This shows that $(T_1 \otimes T_2)\, S_{V \otimes W}\, (T_1 \otimes T_2)^{-1}$ is the corresponding fusion frame operator for Δ. □

Definition 3.9. A family of bounded operators $\{T_i \otimes U_j\}_{i,\,j}$ on a tensor product of Hilbert space $H \otimes K$ is called a resolution of the identity operator on $H \otimes K$, if for all $f \otimes g \in H \otimes K$, we have

$$f \otimes g = \sum_{i,\,j} (T_i \otimes U_j)(f \otimes g),$$

provided the series converges unconditionally for all $f \otimes g \in H \otimes K$.

Proposition 3.1. *If the families of bounded operators $\{T_i\}_{i \in I}$ and $\{U_j\}_{j \in J}$ on H and K are the resolutions of the identity operator on H and K, then $\{T_i \otimes U_j\}_{i, j}$ is a resolution of the identity operator on $H \otimes K$.*

Proof. Since $\{T_i\}_{i \in I}$ and $\{U_j\}_{j \in J}$ are the resolutions of the identity operator on H and K, respectively, we have

$$f = \sum_{i \in I} T_i(f) \quad \forall f \in H \text{ and } \quad g = \sum_{j \in J} U_j(g) \quad \forall g \in K.$$

Then, for all $f \otimes g \in H \otimes K$, we have

$$f \otimes g = \left(\sum_{i \in I} T_i(f)\right) \otimes \left(\sum_{j \in J} U_j(g)\right) = \sum_{i, j} (T_i \otimes U_j)(f \otimes g).$$

This completes the proof. □

Remark 3.5. Let V and W be fusion frames for H and K with their associated frame operators S_V and S_W, respectively. By reconstruction formula we can write

$$f = \sum_{i \in I} v_i^2 S_V^{-1} P_{V_i}(f) \quad \forall f \in H \text{ and } \quad g = \sum_{j \in J} w_j^2 S_W^{-1} P_{W_j}(g) \ \forall g \in K.$$

Then it is easy to verify that

$$f \otimes g = \sum_{i, j} v_i^2 w_j^2 S_{V \otimes W}^{-1} P_{V_i \otimes W_j}(f \otimes g) \ \forall f \otimes g \in H \otimes K.$$

This shows that the family of operators $\left\{v_i^2 w_j^2 S_{V \otimes W}^{-1} P_{V_i \otimes W_j}\right\}_{i, j}$ is resolution of the identity operator on $H \otimes K$.

Theorem 3.9. *Let V and W be fusion frames for H and K with frame bounds A, B and C, D having their corresponding fusion frame operators S_V and S_W, respectively. Then the family $\left\{v_i^2 w_j^2 (T_i \otimes U_j)\right\}_{i, j}$ is a resolution of the identity operator on $H \otimes K$, where $T_i \otimes U_j = P_{V_i \otimes W_j} S_{V \otimes W}^{-1}$ for $i \in I$ and $j \in J$. Furthermore,*

$$\frac{AC}{B^2 D^2} a^2 b^2 \|f \otimes g\|^2 \leq \sum_{i, j} v_i^2 w_j^2 \|(T_i \otimes U_j)(f \otimes g)\|^2 \leq \frac{BD}{A^2 C^2} a^2 b^2 \|f \otimes g\|^2,$$

for all $f \otimes g \in H \otimes K$, where a and b are constants with $ab = 1$.

Proof. Since S_V and S_W are fusion frame operators for V and W, respectively, for all $f \in H$, $g \in K$, we have

$$f = \sum_{i \in I} v_i^2 P_{V_i}\left(S_V^{-1} f\right) \text{ and } \quad g = \sum_{j \in J} w_j^2 P_{W_j}\left(S_W^{-1} g\right).$$

Now, for all $f \otimes g \in H \otimes K$, we have

$$\begin{aligned} f \otimes g &= \left(\sum_{i \in I} v_i^2 P_{V_i} \left(S_V^{-1} f \right) \right) \otimes \left(\sum_{j \in J} w_j^2 P_{W_j} \left(S_W^{-1} g \right) \right) \\ &= \sum_{i, j} v_i^2 w_j^2 \left(P_{V_i} S_V^{-1} (f) \otimes P_{W_j} S_W^{-1} (g) \right) \\ &= \sum_{i, j} v_i^2 w_j^2 \left(P_{V_i} \otimes P_{W_j} \right) \left(S_V^{-1} \otimes S_W^{-1} \right) (f \otimes g) \\ &= \sum_{i, j} v_i^2 w_j^2 P_{V_i \otimes W_j} S_{V \otimes W}^{-1} (f \otimes g). \end{aligned}$$

This shows that $\left\{ v_i^2 w_j^2 (T_i \otimes U_j) \right\}_{i, j}$ is a resolution of the identity operator on $H \otimes K$, where $T_i \otimes U_j = P_{V_i \otimes W_j} S_{V \otimes W}^{-1} = P_{V_i} S_V^{-1} \otimes P_{W_j} S_W^{-1}$. Now, by (VI) of the Theorem (2.4), there exist constants a and b with $a b = 1$ such that

$$T_i (f) = a P_{V_i} S_V^{-1} (f) \ \forall f \in H, \quad \text{and} \quad U_j (g) = b P_{W_j} S_W^{-1} (g) \ \forall g \in K.$$

Then, for each $f \otimes g \in H \otimes K$, we have

$$\begin{aligned} &\sum_{i, j} v_i^2 w_j^2 \| (T_i \otimes U_j) (f \otimes g) \|^2 = \sum_{i, j} v_i^2 w_j^2 \| T_i (f) \otimes U_j (g) \|^2 \\ &= \sum_{i, j} v_i^2 w_j^2 \| T_i (f) \|_1^2 \| U_j (g) \|_2^2 \\ &= \left(\sum_{i \in I} v_i^2 \| T_i (f) \|_1^2 \right) \left(\sum_{j \in J} w_j^2 \| U_j (g) \|_2^2 \right) \\ &= \left(\sum_{i \in I} v_i^2 \left\| a P_{V_i} S_V^{-1} (f) \right\|_1^2 \right) \left(\sum_{j \in J} w_j^2 \left\| b P_{W_j} S_W^{-1} (g) \right\|_2^2 \right) \qquad (3.9) \\ &\leq B D a^2 b^2 \left\| S_V^{-1} (f) \right\|_1^2 \left\| S_W^{-1} (g) \right\|_2^2 \quad [\text{ since } V, W \text{ are fusion frames }] \\ &\leq \frac{B D}{A^2 C^2} a^2 b^2 \| f \|_1^2 \| g \|_2^2 = \frac{B D}{A^2 C^2} a^2 b^2 \| f \otimes g \|^2. \\ &\left[\text{since } B^{-1} I_H \leq S_V^{-1} \leq A^{-1} I_H \text{ and } D^{-1} I_K \leq S_W^{-1} \leq C^{-1} I_K \right]. \end{aligned}$$

On the other hand, using (3.9), we get

$$\begin{aligned} &\sum_{i, j} v_i^2 w_j^2 \| (T_i \otimes U_j) (f \otimes g) \|^2 \geq A C a^2 b^2 \left\| S_V^{-1} (f) \right\|_1^2 \left\| S_W^{-1} (g) \right\|_2^2 \\ &\geq \frac{A C}{B^2 D^2} a^2 b^2 \| f \|_1^2 \| g \|_2^2 = \frac{A C}{B^2 D^2} a^2 b^2 \| f \otimes g \|^2. \end{aligned}$$

This completes the proof. □

4. Alternative dual fusion frame in tensor product of Hilbert spaces

In this section, an alternative dual of a fusion frame in $H \otimes K$ is discussed.

Theorem 4.10. *Let V and W be fusion frames for H and K with frame bounds A, B and C, D having their corresponding fusion frame operators S_V and S_W, respectively. Then the family $\Lambda = \left\{ S_{V \otimes W}^{-1} (V_i \otimes W_j), v_i w_j \right\}_{i, j}$ is a fusion frame for $H \otimes K$.*

Proof. By Theorem 2.3, for all $f \in H$ and $g \in K$, we have

$$\frac{A \| f \|_1^2}{\| S_V \|^2 \left\| S_V^{-1} \right\|^2} \leq \sum_{i \in I} v_i^2 \left\| P_{S_V^{-1} V_i}(f) \right\|_1^2 \leq B \| S_V \|^2 \left\| S_V^{-1} \right\|^2 \| f \|_1^2, \tag{4.10}$$

$$\frac{C \| g \|_2^2}{\| S_W \|^2 \left\| S_W^{-1} \right\|^2} \leq \sum_{j \in J} w_j^2 \left\| P_{S_W^{-1} W_j}(g) \right\|_2^2 \leq D \| S_W \|^2 \left\| S_W^{-1} \right\|^2 \| g \|_2^2 \tag{4.11}$$

Multiplying the inequalities (4.10) and (4.11) and using (2.2), we get

$$\frac{A C \| f \otimes g \|^2}{\| S_V \|^2 \| S_W \|^2 \left\| S_V^{-1} \right\|^2 \left\| S_W^{-1} \right\|^2} \leq \sum_{i, j} v_i^2 w_j^2 \left\| P_{S_V^{-1} V_i}(f) \otimes P_{S_W^{-1} W_j}(g) \right\|^2$$
$$\leq B D \| S_V \|^2 \| S_W \|^2 \left\| S_V^{-1} \right\|^2 \left\| S_W^{-1} \right\|^2 \| f \otimes g \|^2.$$

Therefore, for each $f \otimes g \in H \otimes K$, we get

$$\Rightarrow \frac{A C \| f \otimes g \|^2}{\| S_V \otimes S_W \|^2 \left\| S_V^{-1} \otimes S_W^{-1} \right\|^2} \leq \sum_{i, j} v_i^2 w_j^2 \left\| P_{S_V^{-1} V_i \otimes S_W^{-1} W_j}(f \otimes g) \right\|^2$$
$$\leq B D \| S_V \otimes S_W \|^2 \left\| S_V^{-1} \otimes S_W^{-1} \right\|^2 \| f \otimes g \|^2$$
$$\Rightarrow \frac{A C \| f \otimes g \|^2}{\| S_{V \otimes W} \|^2 \left\| S_{V \otimes W}^{-1} \right\|^2} \leq \sum_{i, j} v_i^2 w_j^2 \left\| P_{S_{V \otimes W}^{-1}(V_i \otimes W_j)}(f \otimes g) \right\|^2$$
$$\leq B D \| S_{V \otimes W} \|^2 \left\| S_{V \otimes W}^{-1} \right\|^2 \| f \otimes g \|^2.$$

This shows that Λ is a fusion frame for $H \otimes K$ with bounds $\dfrac{A C}{\| S_{V \otimes W} \|^2 \left\| S_{V \otimes W}^{-1} \right\|^2}$ and $B D \| S_{V \otimes W} \|^2 \left\| S_{V \otimes W}^{-1} \right\|^2$. □

Definition 4.10. Let $V \otimes W$ be a fusion frame for $H \otimes K$ and $S_{V \otimes W}$ be the corresponding fusion frame operator. Then a fusion Bessel sequence $V' \otimes W'$ in $H \otimes K$ is said to be an alternative dual of $V \otimes W$ if for all $f \otimes g \in H \otimes K$,

$$f \otimes g = \sum_{i, j} v_i w_j v_i' w_j' P_{V_i' \otimes W_j'} S_{V \otimes W}^{-1} P_{V_i \otimes W_j}(f \otimes g).$$

Remark 4.6. According to note (2.2), a reconstruction formula on K is also described by

$$g = \sum_{j \in J} w_j^2 P_{S_W^{-1} W_j} S_W^{-1} P_{W_j}(g) \quad \forall g \in K.$$

Thus, for each $f \otimes g \in H \otimes K$, we get

$$f \otimes g = \left(\sum_{i \in I} v_i^2 P_{S_V^{-1} V_i} S_V^{-1} P_{V_i}(f) \right) \otimes \left(\sum_{j \in J} w_j^2 P_{S_W^{-1} W_j} S_W^{-1} P_{W_j}(g) \right)$$
$$= \sum_{i, j} v_i^2 w_j^2 \left(P_{S_V^{-1} V_i} \otimes P_{S_W^{-1} W_j} \right) \left(S_V^{-1} \otimes S_W^{-1} \right) \left(P_{V_i} \otimes P_{W_j} \right) (f \otimes g)$$
$$= \sum_{i, j} v_i^2 w_j^2 P_{S_V^{-1} V_i \otimes S_W^{-1} W_j} S_{V \otimes W}^{-1} P_{V_i \otimes W_j}(f \otimes g)$$
$$= \sum_{i, j} v_i^2 w_j^2 P_{S_{V \otimes W}^{-1}(V_i \otimes W_j)} S_{V \otimes W}^{-1} P_{V_i \otimes W_j}(f \otimes g).$$

Thus, we see that the canonical dual frame $\left\{ S_{V \otimes W}^{-1} \left(V_i \otimes W_j \right), v_i w_j \right\}_{i, j}$ is an alternative dual fusion frame for $H \otimes K$.

Theorem 4.11. *Let V and W be fusion frames for H and K with their alternative dual $V' = \{ (V_i', v_i') \}_{i \in I}$ and $W' = \left\{ \left(W_j', w_j' \right) \right\}_{j \in J}$, respectively. Then $V' \otimes W'$ is an alternative dual of the fusion frame $V \otimes W$ for $H \otimes K$.*

Proof. By Theorem (3.5), $V \otimes W$ is a fusion frame for $H \otimes K$ and $V' \otimes W'$ is a fusion Bessel sequence in $H \otimes K$. Since V' and W' are alternative dual sequences of V and W, for each $f \in H$ and $g \in K$, we get

$$f = \sum_{i \in I} v_i v_i' P_{V_i'} S_V^{-1} P_{V_i} (f) \text{ and } g = \sum_{j \in J} w_j w_j' P_{W_j'} S_W^{-1} P_{W_j} (g).$$

Then, for all $f \otimes g \in H \otimes K$, using the Theorem (2.4), we get

$$\begin{aligned} f \otimes g &= \left(\sum_{i \in I} v_i v_i' P_{V_i'} S_V^{-1} P_{V_i} (f) \right) \otimes \left(\sum_{j \in J} w_j w_j' P_{W_j'} S_W^{-1} P_{W_j} (g) \right) \\ &= \sum_{i, j} v_i w_j v_i' w_j' \left(P_{V_i'} \otimes P_{W_j'} \right) \left(S_V^{-1} \otimes S_W^{-1} \right) \left(P_{V_i} \otimes P_{W_j} \right) (f \otimes g) \\ &= \sum_{i, j} v_i w_j v_i' w_j' P_{V_i' \otimes W_j'} S_{V \otimes W}^{-1} P_{V_i \otimes W_j} (f \otimes g). \end{aligned}$$

This completes the proof. □

Theorem 4.12. *Let V and W be fusion frames for H and K with bounds (B_1, D_1) and (B_2, D_2) having their alternative duals $V' = \{ (V_i', v_i') \}_{i \in I}$ and $W' = \left\{ \left(W_j', w_j' \right) \right\}_{j \in J}$, respectively. Then $V' \otimes W'$ is a fusion frame for $H \otimes K$.*

Proof. Since V' and W' are fusion Bessel sequences in H and K, respectively, by Theorem (3.5), $V' \otimes W'$ is a fusion Bessel sequence in $H \otimes K$. Also, since V' and W' are alternative dual sequences of V and W, respectively, by Theorem (4.11), $V' \otimes W'$ is an alternative dual of the fusion frame $V \otimes W$ for $H \otimes K$. Now, for each $f \otimes g \in H \otimes K$, we have

$$\begin{aligned} &\| f \otimes g \|^2 = \langle f \otimes g, f \otimes g \rangle \\ &= \left\langle \sum_{i, j} v_i w_j v_i' w_j' P_{V_i' \otimes W_j'} S_{V \otimes W}^{-1} P_{V_i \otimes W_j} (f \otimes g), f \otimes g \right\rangle \\ &= \sum_{i, j} v_i w_j v_i' w_j' \left\langle S_{V \otimes W}^{-1} P_{V_i \otimes W_j} (f \otimes g), P_{V_i' \otimes W_j'} (f \otimes g) \right\rangle \\ &= \sum_{i, j} v_i w_j v_i' w_j' \left\langle S_V^{-1} P_{V_i} (f) \otimes S_W^{-1} P_{W_j} (g), P_{V_i'} (f) \otimes P_{W_j'} (g) \right\rangle \\ &= \left(\sum_{i \in I} v_i v_i' \left\langle S_V^{-1} P_{V_i} (f), P_{V_i'} (f) \right\rangle_1 \right) \left(\sum_{j \in J} w_j w_j' \left\langle S_W^{-1} P_{W_j} (g), P_{W_j'} (g) \right\rangle_2 \right) \\ &\leq \left(\sum_{i \in I} v_i v_i' \left\| S_V^{-1} P_{V_i} (f) \right\|_1 \left\| P_{V_i'} (f) \right\|_1 \right) \left(\sum_{j \in J} w_j w_j' \left\| S_W^{-1} P_{W_j} (g) \right\|_2 \left\| P_{W_j'} (g) \right\|_2 \right) \end{aligned}$$

$$\leq \left(\sum_{i \in I} v_i^2 \left\| S_V^{-1} P_{V_i}(f) \right\|_1^2 \right)^{1/2} \left(\sum_{i \in I} (v_i')^2 \left\| P_{V_i'}(f) \right\|_1^2 \right)^{1/2} \times$$

$$\left(\sum_{j \in J} w_j^2 \left\| S_W^{-1} P_{W_j}(g) \right\|_2^2 \right)^{1/2} \left(\sum_{j \in J} (w_j')^2 \left\| P_{W_j'}(g) \right\|_2^2 \right)^{1/2} \quad [\text{ by C-S inequality }]$$

$$\leq \sqrt{D_1 D_2} \left\| S_V^{-1} \right\| \left\| S_W^{-1} \right\| \|f\|_1 \|g\|_2 \left(\sum_{i \in I} (v_i')^2 \left\| P_{V_i'}(f) \right\|_1^2 \right)^{1/2} \times$$

$$\left(\sum_{j \in J} (w_j')^2 \left\| P_{W_j'}(g) \right\|_2^2 \right)^{1/2} \quad [\text{ since } V,\ W \text{ are fusion frames }]$$

$$= \sqrt{D_1 D_2} \left\| S_{V \otimes W}^{-1} \right\| \|f \otimes g\| \left(\sum_{i,j} (v_i')^2 (w_j')^2 \left\| P_{V_i' \otimes W_j'}(f \otimes g) \right\|^2 \right)^{1/2}$$

$$\Rightarrow \frac{1}{D_1 D_2 \left\| S_{V \otimes W}^{-1} \right\|^2} \|f \otimes g\|^2 \leq \sum_{i,j} (v_i')^2 (w_j')^2 \left\| P_{V_i' \otimes W_j'}(f \otimes g) \right\|^2 .$$

This completes the proof. □

5. CONCLUSION

In this paper, in the setting of tensor product of Hilbert spaces, we give the ideas of fusion frame and alternative dual fusion frame and then establish some characterizations of them. Yet it remains to establish another few important concepts of fusion frame theory like, perturbation, stability etc. in the setting of tensor product of Hilbert spaces.

REFERENCES

[1] Asgari, M. S.; Khosravi, A. Frames and bases of subspaces in Hilbert spaces. *J. Math. Anal. Appl.* ***308*** *(2005), 541-553.*

[2] Casazza, P.; Kutyniok, G. Frames of subspaces. *Cotemporary Math, AMS* **345** (2004), 87–114.

[3] Daubechies, I.; Grossmann, A.; Mayer, Y. Painless nonorthogonal expansions, *J. Math. Phys.* ***27*** *(1986), no. 5, 1271–1283.*

[4] Duffin, R. J.; Schaeffer, A. C. A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.* ***72*** *(1952), 341–366.*

[5] Folland, G. B. A Course in abstract harmonic analysis, *CRC Press BOCA Raton, Florida.*

[6] Gavruta, P. On the duality of fusion frames. *J. Math. Anal. Appl.* **333** (2007) 871-879.

[7] Jain, P. K.; Ahuja, O. P. Functional Analysis, *New Age International Publisher.* 1995.

[8] Kadison, R. V.; Ringrose, J. R. Fundamentals of the theory of operator algebras, Vol. I, *Academic Press. New York* 1983.

[9] Khosravi, A.; Asgari, M. S. Frames and Bases in Tensor Product of Hilbert spaces, *Intern. Math. Journal.* ***4*** *2003, no. 6, 527–537.*

[10] Khosravi, A.; Azandaryani; Mirzaee, M. Fusion frames and g-frames in tensor product and direct sum of Hilbert spaces. *Appl. Anal. Discrete Math.* ***6*** *(2012), 287–303.*

[11] Rabinson, S. Hilbert space and tensor products. *Lecture notes September 8, 1997.*

[12] Reddy, G. U.; Reddy, N. G.; Reddy, B. K. Frame operator and Hilbert-Schmidt Operator in Tensor Product of Hilbert Spaces, *Journal of Dynamical Systems and Geometric Theories.* ***7*** *(2009), 61–70.*

[13] Wang Y. H.; Li, Y. Z. Tensor product dual frames. *J. Inequal. Appl., doi.org/10.1186/s13660-019-2034-6.*

[14] Xiao, X. C.; Zhu, Y. C.; Shu, Z. B.; Ding, M. L., G-frames with bounded linear operators, *Rocky Mountain, Journal of Mathematics.* ***45*** *(2015), no. 2, 675–693.*

[1]DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF CALCUTTA
35, BALLYGUNGE CIRCULAR ROAD, KOLKATA, 700019, WEST BENGAL, INDIA
Email address: prasenjitpuremath@gmail.com

[2]DEPARTMENT OF MATHEMATICS
ULUBERIA COLLEGE
ULUBERIA, HOWRAH, 711315, WEST BENGAL, INDIA
Email address: mumpu_tapas5@yahoo.co.in