CREAT. MATH. INFORM.
Volume **33** (2024), No. 2,
Pages 129 - 136

**Online version at https://creative-mathematics.cunbm.utcluj.ro/**
**Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X**
DOI: https://doi.org/10.37193/CMI.2024.02.01

# Closures of high submodules of $QTAG$-modules

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ABSTRACT. A right module $M$ over an associative ring $R$ with unity is a $QTAG$-module if every finitely generated submodule of any homomorphic image of $M$ is a direct sum of uniserial modules. We show the closures properties for certain high submodules by the $QTAG$-modules and vice versa. Important generalizations and certain related assertions of classical results in this direction are also established.

## 1. INTRODUCTION AND TERMINOLOGY

In 1976, Singh [17] introduced the concept of $TAG$-modules which was a generalization of torsion abelian groups. This encourages many researchers to investigate abelian group theory in $TAG$-modules. The notion of $TAG$-modules is one of the most important tools in module theory. Its importance lies behind the fact that this module can be applied in order to generalized torsion abelian group accurately. This kind of $TAG$-modules has been widely investigated. For details on the abelian groups behaving like modules, we refer to [1, 19].

Consider the following two conditions on a module $M$ over an arbitrary (associative, unitary) ring $R$.
"$(i)$ Every finitely generated submodule of any homomorphic image of $M$ is a direct sum of uniserial modules.
$(ii)$ Given any two uniserial submodules $U_1$ and $U_2$ of a homomorphic image of $M$, for any submodule $N$ of $U_1$, any non-zero homomorphism $\phi : N \to U_2$ can be extended to a homomorphism $\psi : U_1 \to U_2$, provided the composition length $d(U_1/N) \leq d(U_2/\phi(N))$."

When $M_R$ is a module and satisfies clauses $(i)$ and $(ii)$, it is called a $TAG$-module, and when $M_R$ has condition $(i)$ only, it is called a $QTAG$-module (see [18]). Significant work on this concept was produced by many authors, concentrating in the main in establishing when torsion abelian groups are actually $QTAG$-modules. They have also investigated some of their interesting properties and characterizations of these modules. It is worthwhile noticing that many of the developments in this direction are analogous to the earlier development of torsion abelian groups (see, for instance, [8, 14] ). The present work is a natural extension of the torsion abelian groups over to the area of $QTAG$-modules and certainly contributes to the overall knowledge of the structure of $QTAG$-modules.

Some of the fundamental concepts used in this paper have already appeared in one of the co-authors' previous works from [6] which is necessary for our successful presentation.

Throughout our discussion all rings below are assumed to be associative and with non-zero identity element; all modules are assumed to be unital $QTAG$-modules. A uniserial module $M$ is a module over a ring $R$, whose submodules are totally ordered by inclusion. This means simply that for any two submodules $S_1$ and $S_2$ of $M$, either $S_1 \subseteq S_2$

Received: 17.06.2023. In revised form: 01.02.2024. Accepted: 08.02.2024
2000 *Mathematics Subject Classification.* 16K20, 13C12, 13C13.
Key words and phrases. *QTAG-modules, closures, high submodules.*
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or $S_2 \subseteq S_1$. An element $u \in M$ is uniform, if $uR$ is a non-zero uniform (hence uniserial) module and for any module $M$ with a unique decomposition series, the symbol $d(M)$ will denote its decomposition length. If $u$ is an uniform element of $M$ (i.e., $u \in M$), then $e(u)$ is called the exponent of $u$, and $e(u) = d(uR)$. As usual, for such a module $M$, we set the height of $u$ in $M$ as $H_M(u) = \sup\{d(vR/uR) : v \in M,\ u \in vR \text{ and } v \text{ uniform}\}$. For every non-negative integer $t$, $H_t(M) = \{u \in M \mid H_M(u) \geq t\}$ denotes the $t$-th copies of $M$ which can be viewed as a submodule of $M$ consisting of all elements of height at least $t$. The topology of $M$ which admits as a base of neighbourhood of zero is known as $h$-topology. This topology has the submodules $H_t(M)$ with $t = 0, 1, \ldots, \infty$. In this way, a submodule $S$ of $M$ is called the closure in $M$ if $\overline{S} = \cap_{t=0}^{\infty}(S + H_t(M))$ and $S$ is closed with respect to this topology provided that $\overline{S} = S$. For a module $M$, the letter $M^1$ will always denote in the sequel the submodule of $M$, containing elements of infinite height. The sum of all simple submodules of $M$ is called the socle of $M$, denoted by $Soc(M)$ and a submodule $S$ of $Soc(M)$ is called a subsocle of $M$. For all $t \geq 0$, $f_M(t)$ is the $t$-th $Ulm$ invariant of $M$ and it is equal to $Soc(H_t(M))/Soc(H_{t+1}(M))$.

We add some additional background material as well from [13]. The module $M$ is termed $h$-divisible if $M = M^1 = \cap_{t=0}^{\infty} H_t(M)$, or equivalently, if $H_1(M) = M$. With this in hand, we say that the module $M$ is $h$-reduced if it does not contain any $h$-divisible submodule. Moreover, the module $M$ is defined to be bounded if $\exists\, t \geq 0$ such that $H_M(u) \leq t$ for some $u \in M$. A submodule $S$ of $M$ is named $h$-pure in $M$ if for every non-negative integer $t$ the equality $S \cap H_t(M) = H_t(S)$ holds. A submodule $S$ of $M$ is termed a basic submodule of $M$, if $S$ is an $h$-pure submodule of $M$, $S$ is a direct sum of uniserial modules and $M/S$ is $h$-divisible.

It is well to note that various results for $TAG$-modules are also valid for $QTAG$-modules [12]. Our present work is motivated by the many significant results from the reference [10]. For a better understanding of the topic mentioned here, one must go through papers [5, 15, 16]. In what follows, all notations and notions are standard and will be in agreement with those used in [2, 3]; for the specific ones, we refer the readers to [20].

## 2. Chief results

The concept of a high submodule was introduced into the structure theory of $QTAG$-modules by Khan in [11]. Since then many papers have been written investigating the various properties of high submodules (see, for instance, [7]). Here, we continue the study of these high submodules of $QTAG$-module $M$ and consequently investigate the relation between the closure of a summand of $M$ and the closure of a summand of high submodules of $M$. For facilitating the exposition and for the convenience of the readers, we recall the definition of high submodules.

**Definition 2.1.** A submodule $S$ is called a high submodule of the $QTAG$-module $M$ if it is maximal with respect to having zero intersection with $H_\omega(M)$.

The following elementary, but useful lemma, possess a central position.

**Lemma 2.1.** *Let $M$ be a $QTAG$-module, and $S$ be a high submodule of $M$. Then $S/\overline{S}$ is a summand of $M/\overline{S}$.*

*Proof.* By hypothesis, $\overline{S}$ is a high submodule of $\overline{M}$ and hence $\overline{M}/\overline{S}$ is $h$-divisible. Since an $h$-divisible submodule is an absolute direct summand, we infer that

$$M/\overline{S} = (\overline{M}/\overline{S}) \oplus (S_1/\overline{S})$$

for some submodules $S_1$ of $M$ containing $S$. But

$$S/\overline{S} = (S_2/\overline{S}) \oplus (S_3/\overline{S}),$$

where $S_2/\overline{S}$ is $h$-divisible and $S_3/\overline{S}$ is $h$-reduced. Now we may write

$$S_1/\overline{S} = (S_4/\overline{S}) \oplus (S_2/\overline{S}) \oplus (S_5/\overline{S}),$$

where $(S_4/\overline{S}) \oplus (S_2/\overline{S})$ is $h$-divisible, and $S_5/\overline{S}$ is $h$-reduced with the property that $S_5 \supseteq S_3$. Observing that $S_2 + S_3 = S \subseteq S_2 + S_5$. Suppose $x \in S_2$, $y \in S_5$; if $(x + y) \in M^1$. Then

$$(x + y) + \overline{S} \in ((S_2 + S_5)/\overline{S})^1 = (S_2/\overline{S})^1 \oplus (S_5/\overline{S})^1 = S_2/\overline{S}.$$

It follows that $(x + y) \in S_2 \cap M^1 \subseteq S \cap M^1 = 0$. From the maximality of $S$, we have $S = S_2 + S_5$. This, in tern, implies that $S_5 = S_3$. Henceforth,

$$S/\overline{S} = (S_2/\overline{S}) \oplus (S_5/\overline{S}),$$

and consequently,

$$M/\overline{S} = (\overline{M}/\overline{S}) \oplus (S_4/\overline{S}) \oplus (S/\overline{S}).$$

We are finished. □

And so, we will verify the validity of the following theorem.

**Theorem 2.1.** *Let $S$ be a high submodule of the QTAG-module $M$ such that $S = \overline{S} \oplus S_1$ for some submodules $S_1$ of $M$. Then $M = S_1 \oplus S_2$ such that $S_2/\overline{M}$ is $h$-divisible in $M/\overline{M}$.*

*Proof.* In accordance with Lemma 2.1, one may see that

$$M/\overline{S} = (\overline{M}/\overline{S}) \oplus (S_3/\overline{S}) \oplus (S/\overline{S})$$

with $(\overline{M}/\overline{S}) \oplus (S_3/\overline{S})$ is $h$-divisible. Let $S_2 = \overline{M} + S_3$. Then

$$M/\overline{S} = (S_2/\overline{S}) \oplus (\overline{S} \oplus S_1)/\overline{S},$$

and hence $M = S_1 \oplus S_2$. Furthermore, with Lemma 2.1 at hand, we subsequently deduce that

$$M/\overline{M} = (S_2/\overline{M}) \oplus (S_1 \oplus \overline{M})/\overline{M}.$$

Since $S_2/\overline{S} = (\overline{M}/\overline{S}) \oplus (S_3/\overline{S})$ is $h$-divisible, it follows that $S_2/\overline{M}$ is $h$-divisible. In fact, $(S_1 \oplus \overline{M})/\overline{M} \cong S_1$ being $h$-reduced forces that $S_2/\overline{M}$ is $h$-divisible in $M/\overline{M}$. Moreover, it is readily verified that $S_2$ is the only summand of $M$ complementary to $S_1$. Thus if $M = S_4 \oplus S_1 = S_2 \oplus S_1$, we deduce that $\overline{M} \subseteq S_4$, and

$$M/\overline{M} = (S_4/\overline{M}) \oplus (S_1 \oplus \overline{M})/\overline{M} = (S_2/\overline{M}) \oplus (S_1 \oplus \overline{M})/\overline{M}.$$

Observe that $S_4/\overline{M}$ is $h$-divisible in $M/\overline{M}$, since as mentioned before $S_2/\overline{M}$ is so. Consequently, $S_2 = S_4$, as required and this gives the result. □

Next, we concentrate on the following theorem.

**Theorem 2.2.** *Let $S$ be a high submodule of an $h$-reduced QTAG-module $M$. Then $\overline{M}$ is a summand of M if and only if $\overline{S}$ is a summand of S such that $M/\overline{M}$ is $h$-reduced.*

*Proof.* To treat the necessity, observe that $M = \overline{M} \oplus S_1$, for some submodules $S_1$ of $M$. Moreover, let $\overline{S}$ be a high submodule of $\overline{M}$. Thus $\overline{S} \oplus S_1$ has no element of infinite height in $M$. It is fairly to see that $\overline{S} \oplus S_1$ is high in $M$, and hence $\overline{S}$ is a summand of $S$. Since $M/\overline{M} \cong S_1$, it easily follows that $M/\overline{M}$ is $h$-reduced.

Concerning the sufficiency, suppose that $M/\overline{M}$ is $h$-reduced, and that $\overline{S}$ is a summand of $S$. Setting $S = \overline{S} \oplus S_1$, for some submodules $S_1$ of $M$. In virtue of Theorem 2.1, write $M = S_1 \oplus S_2$, where $S_2/\overline{M}$ is $h$-divisible in $M/\overline{M}$. Indeed, $M/\overline{M}$ is $h$-reduced, and hence $S_2 = \overline{M}$. Therefore $M = \overline{M} \oplus S_1$, and we are done. □

We come now to a significant characterization of high submodules of $h$-reduced $QTAG$-modules.

**Theorem 2.3.** *Let $M$ be an $h$-reduced $QTAG$-module such that $M = \overline{M} \oplus N$, for some submodules $N$ of $M$. Then there is a 1-1 correspondence between the set of all high submodules of $M$ and the set $\mathcal{F} = \{\cup_{\alpha \in I} Hom(N, \overline{M}/S_\alpha) : (S_\alpha)_{\alpha \in I}$ is the set of all high submodules of $\overline{M}\}$.*

*Proof.* Suppose $S_\alpha$ is a high submodule of $\overline{M}$, and let $\beta \in \mathrm{Hom}(N, \overline{M}/S_\alpha)$. Then $\beta$ induces an isomorphism

$$\gamma : N/\mathrm{ker}(\beta) \longrightarrow L/S_\alpha$$

such that $S_\alpha \subseteq L \subseteq \overline{M}$. Let

$$S = \{a + b : \gamma(a + \mathrm{ker}(\beta)) = a + S_\alpha\}.$$

Indeed, we first claim that $S$ is a high submodule of $M$. To prove this, we foremost see that $M$ is $h$-reduced, and hence $M^1 = (\overline{M})^1$. It is apparently seen that $S \cap \overline{M} = S_\alpha$ and thus immediately $S \cap M^1 = \{0\}$.

Let $u$ be an uniform element of $M$ such that $u \notin S$. Then $u = b + c$, $b \in N$, $c \in \overline{M}$ and $c \notin S$. Therefore, there exists $a \in L$ such that $(a + b) \in S$. Hence,

$$(b + c) - (a + b) = (c - a) \notin S.$$

If now $S_\alpha$ is a high submodule of $\overline{M}$. This insures at once that

$$\langle uR,\ SR\rangle \cap M^1 \supseteq \langle (c - a)R,\ S_\alpha R\rangle \neq \{0\},$$

and hence $S$ is a high submodule of $M$. In fact, it is elementary to verify that distinct $\beta$'s in $\cup_{\alpha \in I}\mathrm{Hom}(N, \overline{M}/S_\alpha)$ give rise to distinct high submodules.

Next, since $S$ is a high submodule of $M$ such that $M = \overline{M} \oplus N$. It easily follows that $\overline{S}$ is a high submodule of $\overline{M}$. We further observe that $b \in N$, $b \notin S$. Then

$$tb + x = y \in M^1 = (\overline{M})^1$$

for some integer $t$ and $x \in S$. Since $y \in M^1$, we write $y = tu$, where $u \in \overline{M}$. From the $h$-purity of $S$, we have $x = tz$ for some $z \in S$. Thus

$$b + z - u = c \in \overline{M}$$

and so

$$b + (u - c) = -z \in S.$$

Hence the $QTAG$-module of $N$ components of the element of $S$ in $N$. Let $L$ be the $QTAG$-module of $\overline{M}$ components of the element of $S$. Then $\overline{S} \subseteq L \subseteq \overline{M}$. It is plainly seen that $S$ is a subdirect sum of $L$ and $N$, and

$$N/(S \cap N) \cong L/(S \cap L) = L/\overline{S}.$$

The proof of the theorem is completed. □

The question whether all closures of high submodules are summands, has a significance in the theory of $QTAG$-modules. We conjecture that the problem has a negative answer in general, but nevertheless we shall inspect in the sequel its validity for $h$-reduced $QTAG$-module. However, we now have the following example.

**Example 2.1.** Let $M$ be a $QTAG$-module such that $\overline{M}$ is $h$-reduced with non-zero elements of infinite height. Consider any high submodule $S_1$ of $M$ such that $\overline{S_1}$ is high in $\overline{M}$. If $S_2$ is a bounded submodule of $M$, then $M = \overline{M} \oplus S_2$. Since $\overline{M}/\overline{S_1}$ is $h$-divisible, then there is a submodule $S_3$ of $\overline{M}$ such that $d(S_3/\overline{S_1}) = \infty$. Likewise, let $S_4$ be any submodule of $S_2$ such that $d(S_2/S_4) = \infty$. Let $\gamma : S_3/\overline{S_1} \longrightarrow S_2/S_4$ be an isomorphism such that

$S_1 = \{a + b : \gamma(a + \overline{S_1}) = b + S_4\}$. By virtue of Theorem 2.3, $S_1$ is a high submodule of $M$, and $\overline{S_1}$ is the closure of $S_1$.

Suppose $\overline{S_1}$ is a summand of $S_1$ such that $S_1 = \overline{S_1} \oplus S_5$. It is plainly seen that $S_5$ is a subdirect sum of $S_6$ and $S_2$ where $\overline{S_1} \subseteq S_6 \subseteq S_3$. However, since $S_6/(S_6 \cap S_5) \cong S_2/(S_2 \cap S_5)$, we obtain that $S_6 \cap S_5 = \{0\}$. Hence $S_6$ is a homomorphic image of $S_2$. But $S_6$ is $h$-reduced and the homomorphic image of $S_2$ is zero. In other words, we must have $S_6 = \{0\}$. Observe that $S_6 \supseteq \overline{S_1} = \{0\}$, and moreover it follows that no high submodule of $S_3$ is zero. This contradiction proves our assertion after all.

We are now ready to discuss the question of whether or not any two high submodules of an $h$-reduced $QTAG$-module are isomorphic. Let $N$ be a submodule of a $QTAG$-module $M$, and let $\widetilde{N}$ be the image under the natural homomorphism from $M$ onto $M/M^1$. It is easy to verify that $M$ is an $h$-reduced $QTAG$-module without elements of infinite height. Thus if $N$ is a high submodule of $M$, $N \cong \widetilde{N}$. This provides us a natural way to study the properties of high submodules without actually looking at these submodules themselves.

So, we turn next to the following observation, which is parallel to an assertion due to Irwin [9].

**Theorem 2.4.** *Suppose $\{S_1, S_2\}$ is a pair of high submodules of a $QTAG$-module $M$. Then, for some $t \in Z^+$*

$$Soc(H_t(\widetilde{S_1}))/Soc(H_{t+1}(\widetilde{S_1})) = Soc(H_t(\widetilde{S_2}))/Soc(H_{t+1}(\widetilde{S_2})).$$

*In particular, $S_1$ and $S_2$ have the same $Ulm$ invariants.*

*Proof.* We first notice that $Soc(\widetilde{S_1}) = Soc(\widetilde{S_2})$. Then, for all uniform elements $x \in S_1$, we observe that $e(x) = e(x + M^1)$. We next assume that $x \in Soc(S_1) \backslash S_1 \cap S_2$, there exists $y \in S_2$ such that $x - y = z \neq 0$ where $z \in M^1$. Furthermore, since $e(y) = 1$, it follows easily that $x = y + z$. This gives that $Soc(\widetilde{S_1}) \subset Soc(\widetilde{S_2})$, and hence by symmetry $Soc(\widetilde{S_1}) = Soc(\widetilde{S_2})$.

In the remaining case when $Soc(H_t(\widetilde{S_1})) = Soc(H_t(\widetilde{S_2}))$, we assume that an uniform element $a \in M$ with $(H_t(S_1) + b) \cap M^1 = 0$ where $d(aR/bR) = t$ for some $b \in M$ and $b \notin H_t(S_1)$. From the $h$-purity of $S_1$, we get that $b \notin H_t(S_1)$ where $d(aR/bR) = t$. This in tern, implies that $b \notin S_1$ where $d(aR/bR) = t$. Therefore, there exists $c \in S_1$ such that $c + kb = d$ for some $k \geqslant 0$, $d \in M^1$ and $d(aR/bR) = t$. This follows that $c \in H_t(S_1)$, contrary to assumption. Hence in virtue of hypothesis, we have

$$Soc(H_t(\widetilde{S_1}))/Soc(H_{t+1}(\widetilde{S_1})) = Soc(H_t(\widetilde{S_2}))/Soc(H_{t+1}(\widetilde{S_2})),$$

since the numerators are equal and the denominators are equal, and hence the $Ulm$ invariants of $\widetilde{S_1}$ and $\widetilde{S_2}$ are equal. Finally the fact that $S_1 \cong \widetilde{S_1}$ gives us that $\widetilde{S_1}$ and $\widetilde{S_2}$ have the same $t$-th $Ulm$ invariants. □

This brings us to another technical observation.

**Theorem 2.5.** *Suppose $\{S_1, S_2\}$ is a pair of high submodules of a $QTAG$-module $M$. Then*
*$(i)$ $M/S_1 \cong M/S_2$;*
*$(ii)$ $S_1/\overline{S_1} \cong S_2/\overline{S_2}$;*
*$(iii)$ $M/\overline{S_1} \cong M/\overline{S_2}$.*

*Proof.* $(i)$ This follows straightforward, since $S_1$ and $S_2$ are high submodules of $M$.
$(ii)$ Let $S_3$ is a submodule of $M$, and let $\widetilde{S_3} = (S_3 + \overline{M})/\overline{M}$. Then $\widetilde{S_1}$ is maximal disjoint from $\widetilde{M^1}$ in $\widetilde{M}$, that is, $\widetilde{S_1} \cap \widetilde{M^1} = 0$.

To show this, suppose that $x + \overline{M} = y + \overline{M}$ with $x \in S_1$, $y \in M^1$. Then $x - y \in \overline{M}$. It follows that for some integer $t$, $tx = ty = 0$, so that $x + \overline{M} = 0$.

Next, suppose $z + \overline{M} \notin \widetilde{S_1}$, and that $\langle z + \overline{M},\ \widetilde{S_1} \cap \widetilde{M^1} \rangle = 0$. Since $S_1$ is a high submodule of $M$, there exists $x \in S_1$ and an integer $t$ such that $0 \neq x = tz = y \in M^1$. For each $a \in S_1$ choose $b \in M$ such that $ta + tz = tb$. Basically, one may choose $y \in \overline{M}$, and hence $b \in \overline{M}$. Thus

$$a + z - b = y \in \overline{M},$$

and so

$$z + \overline{M} = -a + (b + y) + \overline{M} = -a + \overline{M} \in \widetilde{S_1}.$$

But $z + \overline{M} \notin \widetilde{S_1}$. We conclude that $\widetilde{S_1} \cap \widetilde{M^1} = 0$. Since $\widetilde{M^1} \subseteq (\widetilde{M})^1$ is $h$-divisible, $\widetilde{M}$ contains a minimal $h$-divisible submodule $\widetilde{S_4}$ of $S_4$ which contains $\widetilde{M^1}$. Consequently, $\widetilde{S_1} \cap \widetilde{S_4} = \{0\}$. Note that $\widetilde{S_4}$ is an absolute summand, so that $\widetilde{M} = \widetilde{S_1} \oplus \widetilde{S_4}$, for any high submodule $S_1$ of $M$. Thus $\widetilde{S_1} \cong \widetilde{S_2}$. And finally, since

$$S_1/\overline{S_1} = S_1/(S_1 \cap \overline{M}) \cong (S_1 + \overline{M})/\overline{M} = \widetilde{S_1} \cong \widetilde{S_2} = S_2/\overline{S_2},$$

the claim $(ii)$ follows.
$(iii)$ First we note that
$M/\overline{S_1} = (\overline{M}/\overline{S_1}) \oplus (S_5/\overline{S_1})$ and $M/S_2 \cong \overline{M} \oplus (S_6/\overline{S_1})$, for some submodules $S_5$ and $S_6$ of $M$. Since $\overline{S_1}$ and $\overline{S_2}$ are high submodules of $\overline{M}$, we have $\overline{M}/\overline{S_1} \cong \overline{M}/\overline{S_2}$. But $S_5/\overline{S_1} \cong M/\overline{M} \cong S_6/\overline{S_2}$. Hence $M/\overline{S_1} \cong M/\overline{S_2}$, as needed. □

The concept of $\Sigma$-uniserial modules, play a prominant role to the study of $QTAG$-modules. We shall say that a $QTAG$-module $M$ is a $\Sigma$-uniserial [4] if it is isomorphic to a direct sum of uniserial modules. Notice that $\Sigma$-uniserial modules are separable (i.e., $M^1 = 0$). It is apparent to see that these modules are necessarily $\omega$-bounded, that is they have zero first $Ulm$ submodule (i.e., $M^1 = \cap_{t=1}^{\infty} H_t(M)$).

Likewise, a $QTAG$-module $M$ is called a $\Sigma$-module (see [11]) if its high submodules are the direct sum of uniserial modules. It is well known that if $M$ is a $\Sigma$-module, then all its high submodules are $\Sigma$-uniserial, and that the separable $\Sigma$-modules are precisely the $\Sigma$-uniserial modules.

We now proceed by proving the following statement.

**Corollary 2.1.** *Suppose $\{S_1, S_2\}$ is a pair of high submodules of a $QTAG$-module $M$ such that $S_1$ is $\Sigma$-uniserial in $M$. Then $M$ is a $\Sigma$-module, and $S_1 \cong S_2$.*

*Proof.* Since $\overline{S_1}$ and $\overline{S_2}$ are high submodules of $\overline{M}$, it is readily checked that $\overline{S_2}$ is a $\Sigma$-uniserial module and $\overline{S_1} \cong S_2$. But $S_1/\overline{S_1} \cong S_2/\overline{S_2}$, so that $S_2$ is a $\Sigma$-uniserial module and $S_1 \cong S_2$.

On the other hand, let us assume that $A$ is a $QTAG$-module for some submodules $B \subseteq M$ with $M = A \oplus B$. Let $S_1$ be high in $M$, and let $A_{S_1}$ be the $QTAG$-module of $A$ components of the elements of $S_1$. Then $S_1$ is a subdirect sum of $A_{S_1}$ and $B$, and

$$A_{S_1}/(S_1 \cap A) = A_{S_1}/\overline{S_1} \cong B/(\overline{S_1} \cap B).$$

Thus $B/(S_1 \cap B)$ is a $QTAG$-module, and if $\langle xR \rangle$ is a uniserial summand such that $xR \notin S_1 \cap B$, then

$$B/(S_1 \cap B) = \langle xR + S_1 \cap B \rangle \cong A_{S_1}/\overline{S_1}$$

is a finitely generated uniserial module. Since $\overline{S_1}$ is $h$-pure in $A_{S_1}$, this means that

$$A_{S_1} = \overline{S_1} \oplus C$$

for some submodule $C$ of $M$. Thus

$$S_1 \subseteq \overline{S_1} \oplus C \oplus B, \ \ S_1 = \overline{S_1} \oplus (S_1 \cap (C \oplus B)),$$

and so $\overline{S_1}$ is a summand of $S_1$. Since $S_1/\overline{S_1} \cong B$, we get $S_1 \cong \overline{S_1} \oplus B$. It is clear that any submodule high in $A$ is a basic submodule of $M$, whence any two high submodule of $A$ are isomorphic. It follows that every high submodule of $M$ is isomorphic to $\overline{S_1} \oplus B$. The proof is over. □

## 3. Open problems

We close the work by formulating the following problems.

**Problem 3.1.** *Determine under what additional circumstances in Theorem 2.2 the hypothesis that $M/\overline{M}$ be $h$-reduced is not required.*

**Problem 3.2.** *Describe those $QTAG$-modules such that all high submodules are endomorphic images?*

**Problem 3.3.** *Suppose $\{S_1, S_2\}$ is a pair of high submodules of a $QTAG$-module $M$ such that $S_1$ is $\Sigma$-uniserial in $M$. What are the conditions under which $f_{(S_1 \oplus S_2)}(t) = f_{(S_1)}(t) + f_{(S_2)}(t)$?*

**Acknowledgments.** The authors would like to thank the referees for the thoughtful comments on the article and to thank the editor for the valuable editorial work.

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