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Some convergence results for $G-$ mean nonexpansive mappings in uniformly convex Banach space endowed with graph

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ABSTRACT. In this paper, we deal with the strong convergence of the iteration scheme given by Akutsah et al., for mean nonexpansive mappings in uniformly convex Banach space endowed with directed graph. Also, we show fastness of Akutsah et al. iteration scheme with some well-known iteration schemes with the help of numerical examples. We, also present an application of fixed point theory in solution of integral equation.

1. INTRODUCTION

Fixed point theory has great implications not only in the field of analysis but also in to find solutions of different problems like an optimization problems, convex minimization problems, differential equations, integral equations, etc.

There are lots of fixed point results over different spaces. One of the significant and fruitful results in metric space was given by Banach [5] called "Banach Contraction Principle". This principle was generalized, and its several variants were studied by mathematicians over different spaces.

Note that a mapping $T : K \to K$ is called nonexpansive if

$$||Tx - Ty|| \leq ||x - y||,$$

for all $x, y \in K$, where K is non-empty subset of a Banach space $(X, ||.||)$. A point $x \in K$ is called a fixed point of T, if $Tx = x$. Here we denote the set of fixed points of T by $F(T)$.

Note that the Banach contraction principle is not true in the case of nonexpansive mappings in general. So the question is, when do nonexpansive type mappings have fixed points? In 1965, Browder [6] and Gohde [14] independently proved that " every nonexpansive self-mapping of a closed convex and bounded subset of a uniformly convex Banach space has a fixed point". Also in 1965, Kirk [21] proved that "if X is a reflexive Banach space with normal structure, then X has the fixed point property".

Several kinds of nonexpansive mappings have been studied so far. One of them is mean nonexpansive mapping. The concept of mean nonexpansive mapping was introduced by Zhang [38] in 1975, as a generalization of a nonexpansive mapping in Banach space. A mapping $T : K \to K$ is called mean nonexpansive if

$$||Tx - Ty|| \leq a||x - y|| + b||x - Ty||,$$

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where $a, b \geq 0$ and $a + b \leq 1$. Note that a nonexpansive mapping is mean nonexpansive for $a = 1$ and $b = 0$, but a mean nonexpansive mapping is not necessarily nonexpansive mapping.

Zhang [38] also proved the existence and uniqueness of fixed points of mean nonexpansive mappings in Banach spaces along with normal structure. After that, Wu and Zhang [37] investigated some properties of mean nonexpansive mappings, and they proved that if $a + b < 1$, then mean nonexpansive mapping has a unique fixed point.

So many fixed point results are established for mean nonexpansive mappings by several researchers. In 2012, Ouahab [25] proved the fixed point theorem for a strong semigroup of mean nonexpansive mappings in uniformly convex Banach space. In 2014, Zuo [39] proved that a mean nonexpansive mapping has approximately fixed point sequence, and under some suitable conditions, he proved some existence and uniqueness results of fixed point of mean nonexpansive mappings. In 2017, Gallagher [13] discussed fixed point properties of a new mapping associated with the class of mean nonexpansive mapping, which was introduced by Goebel and Pineda [15] in 2007. In 2022, Ezeora et al. [12] studied the approximation of common fixed points of mean nonexpansive mappings in hyperbolic spaces.

Fixed point theory plays an important role in different fields of mathematics. One of the fields is graph theory. The use of graph theory and fixed point theory has increased rapidly in the past few years. The study of fixed point theory by combining graph theory was started by Echenique [10] in 2005. Echenique proved to be an extension of Tarski's fixed point theorem, which is important in game theory. After that in 2006, Kirk and Espinola [11] established some fixed point results in graph theory. In 2008, Jachymski [17] used the concept of fixed point theory and graph theory to study fixed point theory for $G-$ contraction mapping in complete metric space endowed with a directed graph. He generalized the Banach contraction principle in a complete metric space endowed with a directed graph.

In 2012, Rezapour et al. [30] proved fixed point results for $G-$ nonexpansive mappings in metric space associated with the graph. In 2013, Shahzad et al. [31] proved fixed point results on subgraphs of directed graphs in metric space. In 2015, Alfuraidan [4] studied some fixed point results for nonexpansive type mappings in hyperbolic metric space endowed with graph and Tiammee [35] proved Browder's convergence theorem for nonexpansive type mappings in Hilbert spaces endowed with a directed graph. In 2016, Tripak [36] obtained common fixed points of $G-$ nonexpansive mappings on Banach spaces endowed with a directed graph. In 2020, Okeke and Abbas [24] proved the convergence of iterative schemes for $G-$ nonexpansive mappings in convex metric spaces endowed with graphs. In 2021, Panyanak et al. [26] studied fixed points results of Osilike-Berinde $G-$ nonexpansive mappings in metric spaces endowed with graphs.

The solution of a fixed point problem is difficult analytically, and hence there is a need for an approximate solution. Several iterative schemes have been developed by many researchers for solving fixed point problems for different operators over different spaces. Some of the well-known iterations are- Abbas [1], Agrawal [2], Ishikawa [16], Karakaya [19], Mann [22], Noor [23], Sintunavarat [27], Normal $S-$ iteration [28], Sintunavarat and Pitea [32], Thakur [34] etc.

Recently, Akutsah et al. [3] introduced the following iteration scheme in the framework of Banach space. Let $(X, ||.||)$ be a Banach space and K be a non-empty subset of X. For $x_1 \in K$, the sequence $\{x_k\}$ in K is defined by

$$\begin{cases} z_k = (1-\beta_k)x_k + \beta_k T x_k, \\ y_k = T z_k, \\ x_{k+1} = T((1-\alpha_k)y_k + \alpha_k T y_k), \ k \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are sequences in $(0,1)$. Akutsah et al. [3], proved that their iteration scheme is faster than other well-known iteration schemes, and also established convergence results of the iteration scheme defined by (1.1) for contractive-like mappings.

In this paper, our aim is to establish strong convergence of the iteration scheme defined by (1.1) for $G-$ mean nonexpansive type mappings in uniformly convex Banach space endowed with a directed graph. We have also shown the application of fixed point theory and graph theory in the solution of a nonlinear integral equation and the fastness of the iteration scheme (1.1).

2. Preliminaries

This section proceeds with some necessary concepts and useful results.

Definition 2.1. [33] Let X be a Banach space with the norm $||.||$ and K a convex subset of X. A mapping $T : K \to K$ with non-empty fixed point set $F(T)$ in K will be said to satisfy Condition (I), if there is a non-decreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r \in (0,\infty)$ such that $||x - Tx|| \geq f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = \inf\{||x - z|| : z \in F(T)\}$.

Definition 2.2. [18] A Banach space X is called uniformly convex Banach space if and only if for every choice of $r \in (0,2]$, one has a $s > 0$ such that for every $p, q \in X$, $\frac{1}{2}||p+q|| \leq (1-s)$, whenever $||p|| \leq 1$, $||q|| \leq 1$ and $||p-q|| > r$.

Definition 2.3. [20] Let K be a non-empty subset of a uniformly convex Banach space X. A sequence $\{x_k\}$ in X is said to be Fejer monotone with respect to subset K, if

$$||x_{k+1} - p|| \leq ||x_k - p||,$$

for all $p \in K$, $k \geq 1$.

Proposition 2.1. [20] *Let K be a non-empty subset of a uniformly convex Banach space X. Suppose that $\{x_k\}$ is Fejer monotone sequence with respect to K. Then the followings hold:*

(a) *Sequence $\{x_k\}$ is bounded.*
(b) *For every $x \in K$, $\{||x_k - x||\}$ converges.*

Definition 2.4. [11] A graph is an ordered pair (V, E), where V is a set and E is a binary relation on V, $E \subseteq V \times V$. Elements of E are called edges. Given a graph $G = (V, E)$, a path of G is a sequence $a_0 a_1 ... a_{n-1} ...$ with $(a_i, a_{i+1}) \in E$ for each $i = 0, 1, 2, ...$ A graph is connected if there is a finite path joining any two of its vertices.

Definition 2.5. [36] A directed graph, also called a digraph, is a graph in which the edges have a direction.

Definition 2.6. [4] Let X be a uniformly convex Banach space and K a nonempty subset of X. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = K$ and $E(G)$ contains all the loops, i.e., $(x,x) \in E(G)$ for any $x \in V(G)$. A mapping $T : K \to K$ is called an edge-preserving mapping (or $G-$ edge preserving mapping) if

$$for\ all\ x, y \in K, (x,y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G).$$

Definition 2.7. [35] Let $G = (V(G), E(G))$ be a directed graph. A graph G is called transitive if for any $x, y, z \in V(G)$ such that $(x, y) \in E(G)$ and $(y, z) \in E(G)$, then $(x, z) \in E(G)$.

Suppose that X is a uniformly convex Banach space. We can construct a graph in X by taking $V(G) = X$ or $V(G) =$ any subset of X, and $E(G) \supseteq \{(x, x) : x \in V(G)\}$, i.e., $E(G)$ contains all the loops. (For constructing a graph in an arbitrary space, refer [4, 31, 35]). Now we define $G-$ mean nonexpansive mapping.

Definition 2.8. Let K be non-empty convex subset of uniformly convex Banach space X, $G = (V(G), E(G))$ be a graph such that $V(G) = K$ and $T : K \to K$. Then T is said to be $G-$ mean nonexpansive if the following conditions hold:

(i) T is edge-preserving, i.e., for all $x, y \in K$ such that $(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)$;
(ii) T is mean nonexpansive, i.e., $||Tx - Ty|| \leq a||x - y|| + b||x - Ty||$, whenever $(x, y) \in E(G)$ for $x, y \in K$.

Lemma 2.1. [29] *Let X be a uniformly convex Banach space, and $\{\alpha_k\}$ be a sequence in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$. Suppose that $\{x_k\}$ and $\{y_k\}$ are in X such that $\limsup_{k\to\infty} ||x_k|| \leq c$, $\limsup_{k\to\infty} ||y_k|| \leq c$, and $\limsup_{k\to\infty} ||\alpha_k x_k + (1 - \alpha_k)y_k|| = c$ for some $c \geq 0$. Then $\lim_{k\to\infty} ||x_k - y_k|| = 0$.*

3. Main results

The following results are the main outcomes of this section.

Lemma 3.2. *Let K be a non-empty closed convex subset of a uniformly convex Banach space X. Let $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = K$ and $E(G)$ is convex. Let $T : V(G) \to V(G)$ be $G-$ mean nonexpansive mapping. Fix $x_1 \in V(G)$ such that $(x_1, Tx_1) \in E(G)$. Let $\{x_k\}$ be a sequence in $V(G)$ defined by (1.1). Let $F(T) \neq \emptyset$ with $z \in F(T)$ such that $(x_1, z), (z, x_1) \in E(G)$. Then we have the following:*

(a) $(x_k, x_{k+1}) \in E(G)$ for any $k \geq 1$.
(b) $(x_k, Tx_k) \in E(G)$ for any $k \geq 1$.
(c) $(x_{k+1}, Tx_k) \in E(G)$ for any $k \geq 1$.

Proof. (a) By our assumption, $(x_1, z) \in E(G)$, hence by edge-preserving of T, we have $(Tx_1, z) \in E(G)$.
Note that $(x_1, z) \in E(G)$, $(Tx_1, z) \in E(G)$, so by convexity of $E(G)$,

$$((1 - \beta_1)x_1 + \beta_1 Tx_1, (1 - \beta_1)z + \beta_1 z) \in E(G),$$

i.e., $(z_1, z) \in E(G)$.
Also $(z_1, z) \in E(G) \Rightarrow (Tz_1, z) \in E(G)$(as T is an edge-preserving), i.e., $(y_1, z) \in E(G)$, hence $(Ty_1, z) \in E(G)$.
Again $(y_1, z) \in E(G)$, $(Ty_1, z) \in E(G)$, so by the convexity of $E(G)$, we have

$$((1 - \alpha_1)y_1 + \alpha_1 Ty_1, (1 - \alpha_1)z + \alpha_1 z) \in E(G),$$

i.e., $((1 - \alpha_1)y_1 + \alpha_1 Ty_1, z) \in E(G)$.
Therefore $(T((1 - \alpha_1)y_1 + \alpha_1 Ty_1), z) \in E(G)$, i.e. $(x_2, z) \in E(G)$.
As $(x_1, z), (x_2, z) \in E(G)$, by transitivity of $E(G)$, we have $(x_1, x_2) \in E(G)$. Continuing this process, we get $(x_k, x_{k+1}) \in E(G)$ for any $k \geq 1$.

(b) To show that $(x_k, Tx_k) \in E(G)$ for any $k \geq 1$, we proceed by induction on k. By our assumption $(x_1, Tx_1) \in E(G)$, hence induction is true for $k = 1$.
Now suppose that $(x_k, Tx_k) \in E(G)$ for $k \geq 2$. Since $(x_k, x_{k+1}) \in E(G)$, $(x_k, Tx_k) \in E(G)$, we have $(x_{k+1}, Tx_k) \in E(G)$, as G is transitive. Since $(x_k, x_{k+1}) \in E(G)$,

we have $(Tx_k, Tx_{k+1}) \in E(G)$ due to edge-preserving of T. Again, $(x_{k+1}, Tx_k) \in E(G)$, and $(Tx_k, Tx_{k+1}) \in E(G)$, which gives us that $(x_{k+1}, Tx_{k+1}) \in E(G)$.

(c) By part (b), we have $(x_{k+1}, Tx_k) \in E(G)$ for any $k \geq 1$.

□

Lemma 3.3. *Let K be a non-empty closed convex subset of a uniformly convex Banach space X. Let $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = K$ and $E(G)$ is convex. Let $T : V(G) \to V(G)$ be $G-$ mean nonexpansive mapping. Fix $x_1 \in V(G)$ such that $(x_1, Tx_1) \in E(G)$. Let $\{x_k\}$ be a sequence in $V(G)$ defined (1.1) and $\{\gamma_k\}$ be a sequence in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Let $F(T) \neq \emptyset$ with $z \in F(T)$ such that $(x_1, z), (z, x_1) \in E(G)$, then we have the following:*

(a) *(x_k, z) and (z, x_k) are in $E(G)$ for $k \geq 2$.*
(b) *$\lim_{k\to\infty} ||x_k - z||$ exists.*
(c) *$\lim_{k\to\infty} ||Tx_k - x_k|| = 0$.*

Proof. (a) We proceed by induction on k. Since T is an edge-preserving and $(x_1, z) \in E(G)$, we have $(Tx_1, z) \in E(G)$. By the convexity of $E(G)$ and $(x_1, z), (Tx_1, z) \in E(G)$, we have

$$((1-\beta_1)x_1 + \beta_1 Tx_1, z) \in E(G),$$

i.e, $(z_1, z) \in E(G)$, hence $(Tz_1, z) \in E(G)$, i.e., $(y_1, z) \in E(G)$. By edge preserving of T, we have $(Ty_1, z) \in E(G)$. Now $(y_1, z) \in E(G)$, $(Ty_1, z) \in E(G)$, so by convexity of $E(G)$, we have

$$((1-\alpha_1)y_1 + \alpha_1 Ty_1, z) \in E(G).$$

Therefore by edge preserving of T, we have $(T((1-\alpha_1)y_1 + \alpha_1 Ty_1, z) \in E(G)$, i.e., $(x_2, z) \in E(G)$.

Next, we assume that $(x_k, z) \in E(G)$ for $k \geq 2$. Since T is an edge-preserving, we have $(Tx_k, z) \in E(G)$. By the convexity of $E(G)$ and $(x_k, z), (Tx_k, z) \in E(G)$, we have $((1-\beta_k)x_k + \beta_k Tx_k, (1-\beta_k)z + \beta_k Tz) \in E(G)$, i.e., $(z_k, z) \in E(G)$. By edge preserving of T, $(Tz_k, z) \in E(G)$, i.e., $(y_k, z) \in E(G) \Rightarrow (Ty_k, z) \in E(G)$. By convexity of $E(G)$, we have

$$((1-\alpha_k)y_k + \alpha_k Ty_k, z) \in E(G),$$

and by edge preserving of T, we have

$$(T((1-\alpha_k)y_k + \alpha_k Ty_k), z) \in E(G),$$

i.e., $(x_{k+1}, z) \in E(G)$.

Therefore, $(x_k, z) \in E(G)$ for $k \geq 2$. Using a similar argument, we can show that $(z, x_k) \in E(G)$.

(b) Let $z \in F(T)$. By (a), we have $(x_k, z) \in E(G)$ for $k \geq 2$. Note that

$$\begin{aligned}
||z_k - z|| &= ||(1-\beta_k)x_k + \beta_k Tx_k - z|| \\
&\leq (1-\beta_k)||x_k - z|| + \beta_k ||Tx_k - z|| \\
&\leq (1-\beta_k)||x_k - z|| + \beta_k(a||x_k - z|| + b||x_k - Tz|| \\
&(1-\beta_k)||x_k - z|| + \beta_k(a+b)||x_k - z|| \\
&\leq ||x_k - z||,
\end{aligned}$$

$$\begin{aligned}
||y_k - z|| &= ||Ty_k - z|| \\
&\leq a||y_k - z|| + b||y_k - Tz|| \\
&\leq ||y_k - z||,
\end{aligned}$$

$$\begin{aligned}||x_{k+1}-z|| &= ||T((1-\alpha_k)y_k+\alpha_k Ty_k)-z||\\ &\le a||(1-\alpha_k)y_k+\alpha_k Ty_k-z||+b||(1-\alpha_k)y_k+\alpha_k Ty_k-Tz||\\ &\le a(1-\alpha_k)||y_k-z||+a\alpha_k||Ty_k-z||+b(1-\alpha_k)||y_k-Tz||\\ &+b\alpha_k||Ty_k-Tz||\\ &=(a+b)(1-\alpha_k)||y_k-z||+(a+b)\alpha_k||Ty_k-Tz||\\ &\le(1-\alpha_k)||y_k-z||+\alpha_k(a||y_k-z||+b||y_k-Tz||)\\ &\le ||y_k-z||\\ &\le ||x_k-z||.\end{aligned}$$

It follows that the sequence $\{x_k\}$ is Fejer monotone sequence with respect to $F(T)$. Hence from Proposition 2.1, sequence $\{x_k\}$ is bounded and $\{||x_k-z||\}$ converges, i.e., $\lim_{k\to\infty}||x_k-z||$ exists for all $z\in F(T)$.

(c) Since $\lim_{k\to\infty}||x_k-z||$ exists for all $z\in F(T)$, so suppose that $\lim_{k\to\infty}||x_k-z||=p$, where $p\ge 0$.
If $p=0$, then

$$\begin{aligned}||x_k-Tx_k|| &\le ||x_k-z||+||z-Tx_k||\\ &\le ||x_k-z||+a||x_k-z||+b||x_k-Tz||\\ &\le (1+a+b)||x_k-z|| \ \to\ 0 \ as\ k\to\infty.\end{aligned}$$

Let $p>0$. Since $\lim_{k\to\infty}||x_k-z||=p\Rightarrow\limsup_{k\to\infty}||x_k-z||\le p$. Also,

$$\begin{aligned}||Tx_k-z|| &\le (a+b)||x_k-z||\\ &\Rightarrow \limsup_{k\to\infty}||Tx_k-z||\le p.\end{aligned}$$

Let $\{\gamma_k\}$ be a sequence in $[\delta,1-\delta]$ for some $\delta\in(0,1)$, then we have

$$\begin{aligned}\limsup_{k\to\infty}||(1-\gamma_k)(x_k-z)+\gamma_k(Tx_k-z)|| &\le (1-\gamma_k)\limsup_{k\to\infty}||x_k-z||\\ &+\gamma_k\limsup_{k\to\infty}||Tx_k-z||\\ &\le (1-\gamma_k)\limsup_{k\to\infty}||x_k-z||\\ &+\gamma_k\limsup_{k\to\infty}(a||x_k-z||+b||x_k-Tz||)\\ &\le p.\end{aligned}$$

So from Lemma 2.1, we have $\lim_{k\to\infty}||x_k-Tx_k||=0$.

□

Now, we prove strong convergence of the iteration scheme defined by (1.1).

Theorem 3.1. *Let K be a non-empty closed convex subset of a uniformly convex Banach space X. Let $G=(V(G),E(G))$ be a directed transitive graph such that $V(G)=K$ and $E(G)$ is convex. Let $T:V(G)\to V(G)$ be $G-$ mean nonexpansive mapping. Fix $x_1\in V(G)$ such that $(x_1,Tx_1)\in E(G)$. Let $\{x_k\}$ be a sequence in $V(G)$ defined by (1.1). Let $F(T)\neq\emptyset$ with $z\in F(T)$ such that $(x_1,z),(z,x_1)\in E(G)$. Then the sequence $\{x_k\}$ converges strongly to a fixed point of T if and only if $\lim_{k\to\infty}d(x_k,F(T))=0$, where $d(x_k,F(T))=\inf\{||x_k-z||: z\in F(T)\}$.*

Proof. If the sequence $\{x_k\}$ converges strongly to a fixed point of T, then it is obvious that $\lim_{k\to\infty}d(x_k,F(T))=0$.

For the converse part, suppose that $\lim_{k\to\infty} d(x_k, F(T)) = 0$. Then for $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$,

$$d(x_k, F(T)) < \frac{\varepsilon}{4}.$$

In particular, there must $p \in F(T)$ such that

$$||x_{k_0} - p|| < \frac{\varepsilon}{2}.$$

For $k, m \geq k_0$, we have

$$\begin{aligned}
||x_{k+m} - x_k|| &\leq ||x_{k+m} - p|| + ||p - x_k|| \\
&= 2||x_{k_0} - p|| \\
&< \varepsilon.
\end{aligned}$$

It follows that $\{x_k\}$ is a Cauchy sequence in K. Since K is closed subset of uniformly convex Banach space X, so there exists a point say $x \in K$ such that $||x_k - x|| \to 0$ as $k \to \infty$.
By our assumption, $\lim_{k\to\infty} d(x_k, F(T)) = 0$, it gives that

$$d(x, F(T)) = 0 \Rightarrow x \in F(T).$$

□

Theorem 3.2. *Let K be a non-empty closed convex subset of a uniformly convex Banach space X. Let $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = K$ and $E(G)$ is convex. Let $T : V(G) \to V(G)$ be $G-$ mean nonexpansive mapping such that it satisfies Condition (I). Fix $x_1 \in V(G)$ such that $(x_1, Tx_1) \in E(G)$. Let $\{x_k\}$ be a sequence in $V(G)$ defined by (1.1). Let $F(T) \neq \emptyset$ with $z \in F(T)$ such that $(x_1, z), (z, x_1) \in E(G)$. Let $\{\alpha_k\}$, $\{\beta_k\}$ are sequences in $[\delta, 1-\delta]$, for some $\delta \in (0, 1)$. Then the sequence $\{x_k\}$ converges strongly to a fixed point of T.*

Proof. From Lemma 3.3, we have $\lim_{k\to\infty} ||x_k - Tx_k|| = 0$ and T satisfy Condition (I), so we have $\lim_{k\to\infty} d(x_k, F(T)) = 0$. From the Theorem 3.1, we conclude that $\{x_k\}$ converges strongly to a fixed point of T. □

Theorem 3.3. *Let K be a non-empty compact convex subset of a uniformly convex Banach space X. Let $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = K$ and $E(G)$ is convex. Let $T : V(G) \to V(G)$ be $G-$ mean nonexpansive mapping. Fix $x_1 \in V(G)$ such that $(x_1, Tx_1) \in E(G)$. Let $\{x_k\}$ be a sequence in $V(G)$ defined by (1.1). Let $F(T) \neq \emptyset$ with $z \in F(T)$ such that $(x_1, z), (z, x_1) \in E(G)$. Then the sequence $\{x_k\}$ converges strongly to a fixed point of T.*

Proof. Let $z \in F(T)$. From Lemma 3.3, we have $\lim_{k\to\infty} ||x_k - Tx_k|| = 0$, and K is compact, there must exists a subsequence $\{x_{k_p}\}$ of $\{x_k\}$ such that $x_{k_p} \to z$ for some $z \in K$. Note that

$$\begin{aligned}
||x_{k_p} - Tz|| &\leq ||x_{k_p} - Tx_{k_p}|| + ||Tx_{k_p} - Tz|| \\
&\leq ||x_{k_p} - Tx_{k_p}|| + a||x_{k_p} - z|| + b||x_{k_p} - Tz|| \\
&\leq ||x_{k_p} - Tx_{k_p}|| + ||x_{k_p} - z|| \to \ 0 \ as \ k \to \infty.
\end{aligned}$$

This shows that $x_{k_p} \to Tz$ as $k \to \infty$. By uniqueness of limits, we have $z = Tz$.
Also by Lemma 3.3, $\lim_{k\to\infty} ||x_k - z||$ exists, thus z is the strong limit of the sequence $\{x_k\}$ itself . □

4. Numerical examples

Now with the help of the Matlab software program, we show the fastness of the iteration scheme defined by (1.1) with some well-known iteration schemes by considering following examples:

Example 4.1. Let K be a closed unit ball of the space l_1 with the norm $||\{x_k\}|| = \sum_k |x_k|$. Let $G = (V(G), E(G))$ be a graph such that $V(G) = K$ and

$$E(G) = \{(\{x_k\}, \{y_k\}) : |x_k| + |y_k| \leq 1 \text{ } and \text{ } ||\{x_k\} - \{y_k\}|| \leq \frac{2}{7}\}.$$

Let $T : V(G) \to V(G)$ be a mapping defined by

$$T(\{x_k\}) = \{x_k^2\}.$$

Then T is an edge-preserving mapping as

$$\begin{aligned} ||T(\{x_k\}) - T(\{y_k\})|| &= ||\{x_k^2\} - \{y_k^2\}|| \\ &\leq \frac{2}{7}||\{x_k\} + \{y_k\}|| \\ &\leq \frac{2}{7}\sum_k(|x_k| + |y_k|) \\ &\leq \frac{2}{7}. \end{aligned}$$

But, T is not a nonexpansive mapping for $\{x\} = \{\frac{1}{3}, 0, 0, ...\}$ and $\{y\} = \{1, 0, 0, ...\}$ because $||Tx - Ty|| > ||x - y||$. However, T is a mean nonexpansive mapping for $a = 1$ and $b = 0$. Since

$$\begin{aligned} ||T(\{x_k\}) - T(\{y_k\})|| &\leq ||\{x_k\} - \{y_k\}|| \sum_k(|x_k| + |y_k|) \\ &\leq (a||\{x_k\} - \{y_k\}|| + b||\{x_k\} - T(\{y_k\})||) \sum_k(|x_k| + |y_k|) \\ &\leq a||\{x_k\} - \{y_k\}|| + b||\{x_k\} - T(\{y_k\})||. \end{aligned}$$

Iteration	Akutsah (1)	Thakur (2)	Abbas (3)	Karakaya (4)
0	0.90000000	0.90000000	0.90000000	0.90000000
1	0.00036394	0.08503056	-0.41552352	0.25220484
2	0.00000000	0.00006567	-0.02220020	0.00066228
3	0.00000000	0.00000000	-0.00002163	0.00000000
4	0.00000000	0.00000000	0.00000000	0.00000000
5	0.00000000	0.00000000	0.00000000	0.00000000
6	0.00000000	0.00000000	0.00000000	0.00000000
7	0.00000000	0.00000000	0.00000000	0.00000000
8	0.00000000	0.00000000	0.00000000	0.00000000

Table 1. Strong convergence of Akutsah iteration scheme (1.1), Abbas [1], Karakaya [19], and Thakur [34] iterations to the fixed point $x = 0$ of T in Example 4.1

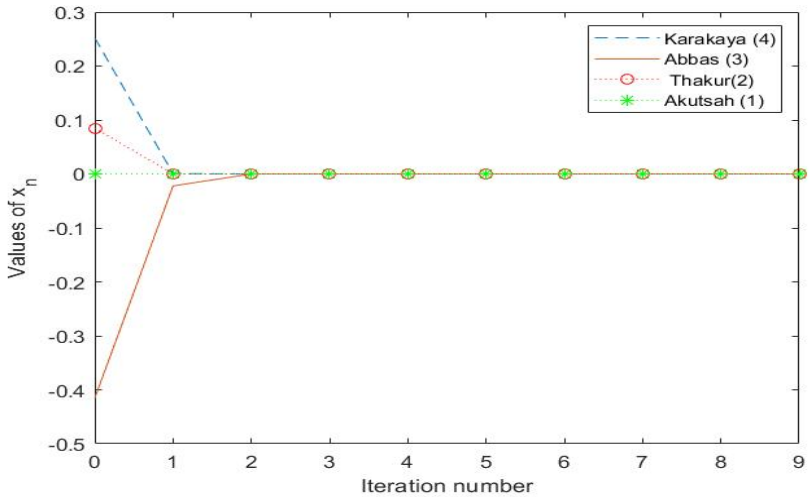

FIGURE 1. Behaviors of Akutsah iteration (green), Thakur iteration (red), Abbas iteration (carrot orange), Karakaya iterations (cyan) to the fixed point $x = 0$ of the mapping T in Example 4.1

Iteration	Akutsah (1)	MPIH (2)	Sintunavarat (3)	Agrawal (4)
0	0.70000000	0.70000000	0.70000000	0.70000000
1	0.00437012	0.01517824	0.13221376	-0.33320000
2	0.00000000	0.00001369	0.00035621	0.11095563
3	0.00000000	0.00000000	0.00000001	-0.00353953
4	0.00000000	0.00000000	0.00000000	-0.00000212
5	0.00000000	0.00000000	0.00000000	0.00000000
6	0.00000000	0.00000000	0.00000000	0.00000003
7	0.00000000	0.00000000	0.00000000	0.00000000
8	0.00000000	0.00000000	0.00000000	0.00000000

TABLE 2. Strong convergence of Akutsah iteration scheme (1.1), MPIH [27], Agrawal [2], and Sintunavarat [32] iterations to the fixed point $x = 0$ of T in Example 4.1.

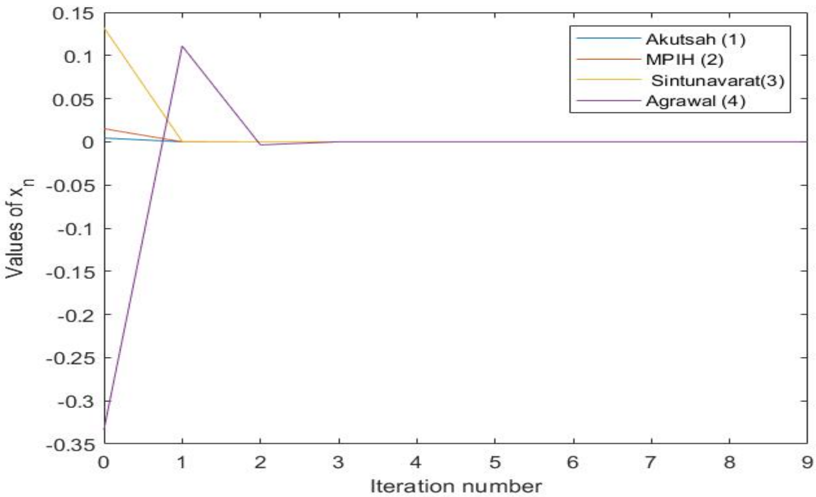

FIGURE 2. Behaviors of Akutsah iteration (cyan), Modified Picard Ishikawa hybrid iteration (carrot orange), Sintunavarat iteration (yellow), Agrawal iteration (magenta) to the fixed point $x = 0$ of the mapping T in Example 4.1.

Iteration	Akutsah (1)	MPH (2)	GM (3)	Farajzadeh (4)
0	0.30000000	0.30000000	0.30000000	0.30000000
1	0.00019559	0.01440000	-0.57351936	-0.27120000
2	0.00000000	0.00004746	0.92396200	-0.14763874
3	0.00000000	0.00000000	0.91314953	-0.09329058
4	0.00000000	0.00000000	0.88341178	-0.06986061
5	0.00000000	0.00000000	0.82875562	-0.05648219
6	0.00000000	0.00000000	0.73538372	-0.04100303
7	0.00000000	0.00000000	0.36547811	-0.03607353

TABLE 3. Strong convergence of Akutsah iteration scheme (1.1), Farajzadeh [7], Generalised M (GM) [8], and Modified Picard Hybrid (MPH) [9], iterations to the fixed point $x = 0$ of T in Example 4.1.

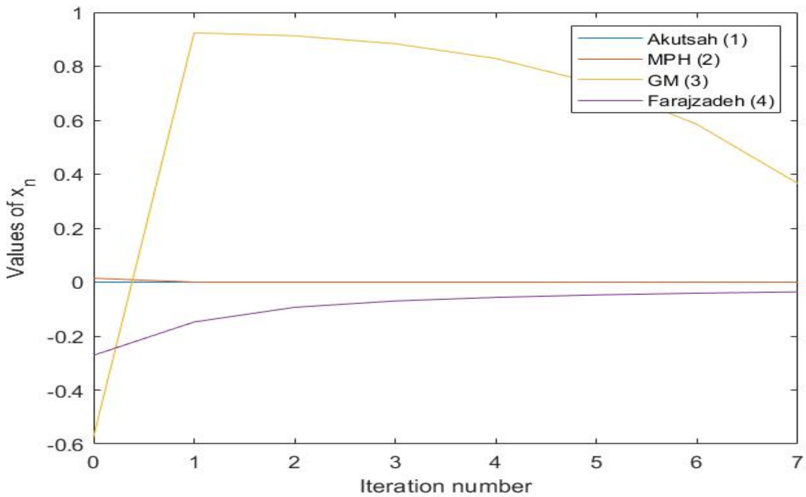

FIGURE 3. Behaviors of Akutsah iteration (cyan), Modified Picard hybrid (MPH) iteration (carrot orange), Generalized M (yellow), Farajzadeh iteration (magenta) to the fixed point $x = 0$ of the mapping T in Example 4.1.

5. APPLICATION INTO SOLUTION OF AN INTEGRAL EQUATION

Consider the following nonlinear integral equation-

$$x(t) = \lambda \int_a^1 f(t, s, x(s))ds + y(t), \ a \in [0, 1], \tag{5.2}$$

where $\lambda \in (0, 1]$, $y : [0, 1] \to \mathbb{R}$ and $f(s, t, x(s)) : [0, 1] \times [0, 1] \times [0, 1] \to \mathbb{R}$ all are continuous. Let $X = \mathbb{R}$. We define $||.|| : \mathbb{R} \to \mathbb{R}$ by

$$||x - y|| = \sup\{|x(s) - y(s)| : s \in [0, 1]\},$$

for all $x \in \mathbb{R}$.

Theorem 5.4. *Let $X = \mathbb{R}$, $K = [0, 1]$ and $T : K \to K$ defined by*

$$Tx(t) = \begin{cases} 0, \ x \in [0, a); \\ \lambda \int_a^1 f(t, s, x(s))ds + y(t), \ x \in [a, 1]. \end{cases} \tag{5.3}$$

where $y : [0, 1] \to \mathbb{R}$ and $f(s, t, x(s)) : [0, 1] \times [0, 1] \times [0, 1] \to \mathbb{R}$ are continuous. Suppose that $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = K$. Let $x, y \in V(G)$ such that $(x, y) \in E(G)$ with $||x - y|| \leq \frac{1}{2}$. Suppose that the following conditions are satisfied-

(i) *There exists a continuous mapping $F : X \times X \to [0, \infty)$ such that*

$$|f(t, s, x(s)) - f(t, s, y(s))| \leq F(x, y)|x(s) - y(s)|,$$

for all $s \in [0, 1]$, $x, y \in X$;

(ii) *There exists a point $\zeta \in \mathbb{R}$ such that $\int_a^1 F(x, y) \leq \zeta$;*

(iii) $\lambda\zeta \leq (a' + b' \frac{||x-Ty||}{||x-y||})$.

Let $\{x_k\}$ be a sequence in K defined by (1.1). Then the sequence $\{x_k\}$converges to a point of T and this point will be solution of the integral equation (5.3).

Proof. Let $x, y \in [0,1]$. It is clear that T is an edge-preserving mapping. Also, T is discontinuous at $x = a$, hence T is not nonexpansive mapping. Now we show that T is mean nonexpansive mapping for $a^{'} + b^{'} \leq 1$. Consider the following cases:

Case I: when $x, y \in [0, a)$. Obviously T is mean nonexpansive for $a^{'} = \frac{1}{2}, b^{'} = \frac{1}{2}$.

Cae II: when $x, y \in [a, 1]$. Then we have,

$$\begin{aligned}
|Tx(s) - Ty(s)| &= |\lambda \int_a^1 f(t,s,x(s))dt - \lambda \int_a^1 f(t,s,y(s))dt| \\
&= |\lambda \int_a^1 (f(t,s,x(s)) - f(t,s,y(s)))dt| \\
&\leq \lambda \int_a^1 F(x,y)|x(s) - y(s)|dt \\
\sup_{s\in[0,1]} |Tx(s) - Ty(s)| &\leq \sup_{s\in[0,1]} |x(s) - y(s)|\lambda \int_a^1 F(x,y)dt \\
&\leq ||x - y||\lambda\zeta \\
&\leq a^{'}||x - y|| + b^{'}||x - Ty|| \\
&\Rightarrow ||Tx - Ty|| \leq a^{'}||x - y|| + b^{'}||x - Ty||.
\end{aligned}$$

Hence, we conclude that T is mean nonexpansive mapping for $a^{'} = \frac{1}{2}, b^{'} = \frac{1}{2}$. Also

$$\begin{aligned}
||x - Tx|| &= \sup\{|x(s) - Tx(s)| : s \in [0,1]\} \\
&= \sup\{|x(s) - \lambda \int_a^1 f(t,s,x(s))dt - y(s)||\} \\
&= \sup\{|x(s) - x(s)| : s \in [0,1]\} \\
&\Rightarrow ||x - Tx|| = 0.
\end{aligned}$$

From Theorem 3.3, we have $F(T) \neq \emptyset$, and T is $G-$ mean nonexpansive mapping, therefore $\{x_k\}$converges to a point of T and this point will be solution of the integral equation (5.3). □

6. Conclusion

Starting with Akutsah iteration scheme,we have some fixed point results for $G-$ mean nonexpansive mappings in Uniformly convex Banach space. With the help of an example, we proved that the Akutsah et al. iteration scheme is faster than the iteration scheme given by Thakur, Abbas, Karakaya, Modified Picard Ishikawa hybrid, Modified Picard hybrid, Sintunavarat, Generalized M, Farajzadeh and Agrawal, etc.

From the comparison table and corresponding graph, it is clear that Akutsah et al. iteration scheme is faster than other three-step iteration schemes and at last, there is an application of fixed point theory.

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