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Some Hermite-Hadamard type integral inequalities for functions whose first derivatives are trigonometrically ρ-convex

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ABSTRACT. In this manuscript, we obtain refinements of the Hermite-Hadamard type inequalites for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is trigonometrically ρ-convex. We also show that our results coincide with previous results.

1. PRELIMINARIES

Throughout the paper I is a non-empty interval in $\mathbb{R}$.

Definition 1.1. A function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f\left(tx + (1-t)y\right) \leq tf\left(x\right) + \left(1-t\right)f\left(y\right)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. Let $f : [a,b] \to \mathbb{R}$ be a convex function, then the following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f\left(a\right)+f(b)}{2}$$

is known as the Hermite-Hadamard inequality (for more information, see [7]). Since then, some refinements of the Hermite-Hadamard inequality for convex functions have been obtained [4, 5, 6, 11].

Definition 1.2. ([10]) Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX\left(h, I\right)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0,1)$ we have

$$f\left(\alpha x + (1-\alpha)y\right) \leq h(\alpha)f\left(x\right) + h(1-\alpha)f\left(y\right).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV\left(h, I\right)$.

Definition 1.3 **([8]).** A non-negative function $f : I \to \mathbb{R}$ is called trigonometrically convex function on interval $[a,b]$, if for each $x, y \in [a,b]$ and $t \in [0,1]$,

$$f\left(tx + (1-t)y\right) \leq \left(\sin\frac{\pi t}{2}\right)f(x) + \left(\cos\frac{\pi t}{2}\right)f(y).$$

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We will denote by $TC(I)$ the class of all trigonometrically convex functions on interval I. For $h(t)=\sin\frac{\pi t}{2}$, every trigonometrically convex function is also h-convex function.

In [8], Kadakal obtained the following inequalities:

Theorem 1.1 ([8]). *Let $f: I \to \mathbb{R}$ be a continuously differentiable function, let $a<b$ in I and assume that $f' \in L[a,b]$. If $|f'|$ is trigonometrically convex function on interval $[a,b]$, then the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{2}{\pi}(b-a)\left[1-\frac{4}{\pi}\left(\sqrt{2}-1\right)\right] A\left(\left|f'(a)\right|,\left|f'(b)\right|\right) \tag{1.1}$$

holds for $t\in[0,1]$.

Theorem 1.2 ([8]). *Let $f: I \to \mathbb{R}$ be a continuously differentiable function, let $a<b$ in I and assume that $q>1$. If the mapping $|f'|^q$ is trigonometrically convex function on interval $[a,b]$, then the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} 2^{\frac{2}{q}}\pi^{-\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right). \tag{1.2}$$

holds for $t\in[0,1]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.3 ([8]). *Let $f: I\subseteq\mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, let $a<b$ in I and assume that $q>1$. If the mapping $|f'|^q$ is trigonometrically convex function on interval $[a,b]$, then the following inequality*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{3}{q}}\left[\frac{1}{\pi}-\frac{4\left(\sqrt{2}-1\right)}{\pi^2}\right]^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)
\end{aligned} \tag{1.3}$$

holds for $t\in[0,1]$.

In [1], Ali gave the following definition:

Definition 1.4 ([1]). A function $f: I\to\mathbb{R}$ is said to be trigonometrically ρ-convex, if for any arbitrary closed subinterval $[a,b]$ of I such that $0<\rho(b-a)<\pi$, we have

$$f(x) \leq \frac{\sin[\rho(b-x)]}{\sin[\rho(b-a)]}f(a)+\frac{\sin[\rho(x-a)]}{\sin[\rho(b-a)]}f(b) \tag{1.4}$$

for all $x\in[a,b]$. For the $x=ta+(1-t)b$, $t\in[0,1]$, then the condition (1.4) becomes

$$f(ta+(1-t)b) \leq \frac{\sin[\rho t(b-a)]}{\sin[\rho(b-a)]}f(a)+\frac{\sin[\rho(1-t)(b-a)]}{\sin[\rho(b-a)]}f(b). \tag{1.5}$$

If the inequality (1.4) holds with "$\geq$", then the function will be called trigonometrically ρ-concave on I.

Example 1.1. The function $f(x)=\cos x$ is a trigonometrically ρ-convex function on the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for all $\rho>1$ and trigonometrically ρ-convex function for $\rho\in(0,1)$.

Example 1.2. Consider the function $f:(0,\infty)\to(0,\infty)$, $f(x)=x^p$ with $p\in\mathbb{R}\setminus\{0\}$. If $p\in(-\infty,0)\cup[1,\infty)$ the function is convex and therefore trigonometrically ρ-convex function for any $\rho>0$.

The following Hermite-Hadamard type integral inequality that was obtained in 2013 in [2].

Theorem 1.4 ([2]). *Assume that the function $f : I \to \mathbb{R}$ is trigonometrically ρ-convex on I. Then for any $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$ we have*

$$\frac{2}{\rho} f\left(\frac{a+b}{2}\right) \sin\left[\frac{\rho(b-a)}{2}\right] \leq \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{\rho} \tan\left[\frac{\rho(b-a)}{2}\right].$$

Theorem 1.5 (Hölder inequality for integrals). *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|^q$, $|g|^q$ are integrable functions on interval $[a, b]$, $q > 1$ then*

$$\int_a^b |f(x)g(x)|\, dx \leq \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\, dx\right)^{\frac{1}{q}}$$

with equality holding if and only if $A\,|f(x)|^p = B\,|g(x)|^q$, almost everywhere, where A and B are constants [9].

Theorem 1.6 (Power-mean integral inequality). *Let $q \geq 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|$, $|f|\,|g|^q$ are integrable functions on interval $[a, b]$, then the following inequality holds:*

$$\int_a^b |f(x)g(x)|\, dx \leq \left(\int_a^b |f(x)|\, dx\right)^{1-\frac{1}{q}} \left(\int_a^b |f(x)|\,|g(x)|^q\, dx\right)^{\frac{1}{q}}.$$

The aim of this article is to obtain Hermite-Hadamard type integral inequalities for whose first derivatives are trigonometrically ρ-convex functions, using Hölder and Power-mean integral inequalities with the given identity, and show that these obtained inequalities coincide with those obtained previously in special cases.

2. Some new integral inequalities for trigonometrically ρ-convexity

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard type integral inequalities for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is trigonometrically ρ-convex. Dragomir and Agrawal [3] used the following lemma.

Lemma 2.1. *The following equality holds true:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t) f'\left(ta+(1-t)b\right) dt.$$

Note that we will use the followings in this section:

$$\begin{aligned}
& \int_0^1 \frac{\sin\left[\rho t\left(b-a\right)\right]}{\sin\left[\rho\left(b-a\right)\right]}dt \\
= & \int_0^1 \frac{\sin\left[\rho(1-t)\left(b-a\right)\right]}{\sin\left[\rho\left(b-a\right)\right]}dt = \frac{1-\cos\rho(b-a)}{\rho(b-a)\sin\rho(b-a)}, \\
& \int_0^1 \left|1-2t\right|\frac{\sin\left[\rho t\left(b-a\right)\right]}{\sin\left[\rho\left(b-a\right)\right]}dt \\
= & \frac{2\sin(\rho(b-a))-(b-a)\rho\left[\cos(\rho(b-a))-1\right]-4\sin\left(\frac{\rho(b-a)}{2}\right)}{(b-a)^2\rho^2\sin(\rho(b-a))}, \\
& \int_0^1 \left|1-2t\right|\frac{\sin\left[\rho(1-t)\left(b-a\right)\right]}{\sin\left[\rho\left(b-a\right)\right]}dt \\
= & \frac{2\sin(\rho(b-a))-(b-a)\rho\left[\cos(\rho(b-a))-1\right]-4\sin\left(\frac{\rho(b-a)}{2}\right)}{(b-a)^2\rho^2\sin(\rho(b-a))}, \\
\int_0^1 \left|1-2t\right|^p dt = & \frac{1}{p+1}, \int_0^1 \left|1-2t\right| dt = \frac{1}{2}, \\
A = & A\left(u,v\right) = \frac{u+v}{2} \text{ arithmetic mean}
\end{aligned}$$

Theorem 2.7. *Let $f : I \to \mathbb{R}$ be a continuously differentiable function, let $a < b$ in I such that $0 < \rho\left(b-a\right) < \pi$ and assume that $f' \in L\left[a,b\right]$. If $\left|f'\right|$ is trigonometrically ρ-convex function on interval $\left[a,b\right]$, then the following inequality*

$$\begin{aligned}
& \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq A\left(\left|f'(a)\right|,\left|f'(b)\right|\right) \\
& \times \left[\frac{2\sin(\rho(b-a))-(b-a)\rho\left[\cos(\rho(b-a))-1\right]-4\sin\left(\frac{\rho(b-a)}{2}\right)}{(b-a)\rho^2\sin(\rho(b-a))}\right]
\end{aligned}$$

holds for $t \in [0,1]$.

Proof. Using Lemma 2.1 and the inequality

$$\left|f'(ta+(1-t)b)\right| \leq \frac{\sin\left[\rho t\left(b-a\right)\right]}{\sin\left[\rho\left(b-a\right)\right]}\left|f'(a)\right| + \frac{\sin\left[\rho(1-t)\left(b-a\right)\right]}{\sin\left[\rho\left(b-a\right)\right]}\left|f'(b)\right|,$$

we get

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq &\left|\frac{b-a}{2}\int_0^1 (1-2t)f'\left(ta+(1-t)b\right)dt\right| \\
\leq &\frac{b-a}{2}\int_0^1 |1-2t|\left|f'\left(ta+(1-t)b\right)\right|dt \\
\leq &\frac{b-a}{2}\int_0^1 |1-2t|\left[\left(\frac{\sin\left[\rho t\left(b-a\right)\right]}{\sin\left[\rho\left(b-a\right)\right]}\right)|f'(a)|+\left(\frac{\sin\left[\rho(1-t)\left(b-a\right)\right]}{\sin\left[\rho\left(b-a\right)\right]}\right)|f'(b)|\right]dt \\
= &\frac{b-a}{2}\frac{2\sin(\rho(b-a))-(b-a)\rho\left[\cos(\rho(b-a))-1\right]-4\sin\left(\frac{\rho(b-a)}{2}\right)}{(b-a)^2\rho^2\sin(\rho(b-a))}|f'(a)| \\
&+\frac{b-a}{2}\frac{2\sin(\rho(b-a))-(b-a)\rho\left[\cos(\rho(b-a))-1\right]-4\sin\left(\frac{\rho(b-a)}{2}\right)}{(b-a)^2\rho^2\sin(\rho(b-a))}|f'(b)| \\
= &\left[\frac{2\sin(\rho(b-a))-(b-a)\rho\left[\cos(\rho(b-a))-1\right]-4\sin\left(\frac{\rho(b-a)}{2}\right)}{(b-a)\rho^2\sin(\rho(b-a))}\right]\left(\frac{|f'(a)|+|f'(b)|}{2}\right) \\
= &\left[\frac{2\sin(\rho(b-a))-(b-a)\rho\left[\cos(\rho(b-a))-1\right]-4\sin\left(\frac{\rho(b-a)}{2}\right)}{(b-a)\rho^2\sin(\rho(b-a))}\right]A\left(|f'(a)|,|f'(b)|\right).
\end{aligned}
$$

This completes the proof of theorem. □

Corollary 2.1. *Under the assumption of Theorem 2.7 with $\rho=\frac{\pi}{2(b-a)}$, we get the following inequality:*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
(2.6)\qquad \leq &\frac{2}{\pi}(b-a)\left[1+\frac{4}{\pi}\left(1-\sqrt{2}\right)\right]A\left(|f'(a)|,|f'(b)|\right)
\end{aligned}
$$

This inequality coincides with the inequality (1.1) in Theorem 1.1.

Corollary 2.2. *Under the assumption of Theorem 2.7 with $\rho=1$, we get the following inequality:*

$$
\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq &\left[\frac{2\sin(b-a)-(b-a)\left[\cos(b-a)-1\right]-4\sin\frac{(b-a)}{2}}{(b-a)\sin(b-a)}\right] \\
&\times A\left(|f'(a)|,|f'(b)|\right).
\end{aligned}
$$

Theorem 2.8. *Let $f:I\to\mathbb{R}$ be a continuously differentiable function, let $a<b$ in I such that $0<\rho(b-a)<\pi$ and assume that $q>1$. If the mapping $|f'|^q$ is trigonometrically ρ-convex function on interval $[a,b]$, then the following inequality*

$$
\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq &\frac{b-a}{2}2^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1-\cos\rho(b-a)}{\rho(b-a)\sin\rho(b-a)}\right)^{\frac{1}{q}} \\
&\times A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right)
\end{aligned}
$$

holds for $t\in[0,1]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.1, Hölder's integral inequality and inequality

$$|f'(ta+(1-t)b)|^q \leq \frac{\sin[\rho t(b-a)]}{\sin[\rho(b-a)]}|f'(a)|^q + \frac{\sin[\rho(1-t)(b-a)]}{\sin[\rho(b-a)]}|f'(b)|^q$$

which is the trigonometrically ρ-concexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt \\
\leq\ & \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[\frac{\sin[\rho t(b-a)]}{\sin[\rho(b-a)]}|f'(a)|^q + \frac{\sin[\rho(1-t)(b-a)]}{\sin[\rho(b-a)]}|f'(b)|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[|f'(a)|^q\int_0^1\frac{\sin[\rho t(b-a)]}{\sin[\rho(b-a)]}dt + |f'(b)|^q\int_0^1\frac{\sin[\rho(1-t)(b-a)]}{\sin[\rho(b-a)]}dt\right]^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{1-\cos\rho(b-a)}{\rho(b-a)\sin\rho(b-a)}|f'(a)|^q + \frac{1-\cos\rho(b-a)}{\rho(b-a)\sin\rho(b-a)}|f'(b)|^q\right]^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}2^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1-\cos\rho(b-a)}{\rho(b-a)\sin\rho(b-a)}\right)^{\frac{1}{q}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}2^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1-\cos\rho(b-a)}{\rho(b-a)\sin\rho(b-a)}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right).
\end{aligned}$$

This completes the proof of theorem. □

Corollary 2.3. *Under the assumption of Theorem 2.8 with* $\rho = \frac{\pi}{2(b-a)}$, *we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}2^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}(\pi)^{-\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right).$$

This inequality coincides with the inequality (1.2) in Theorem 1.2.

Corollary 2.4. *Under the assumption of Theorem 2.8 with* $\rho = 1$, *we get the following inequality:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq\ & \left(\frac{b-a}{2}\right)^{\frac{1}{p}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1-\cos(b-a)}{\sin(b-a)}\right)^{\frac{1}{q}} \\ & \times A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right).\end{aligned}$$

Theorem 2.9. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a continuously differentiable function, let* $a < b$ *in* I *such that* $0 < \rho(b-a) < \pi$ *and assume that* $q \geq 1$. *If the mapping* $|f'|^q$ *is trigonometrically* ρ*-convex function on interval* $[a, b]$, *then the following inequality holds for* $t \in [0, 1]$:

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}} A^{\frac{1}{q}}\left(|f'(a)|^q, |f'(b)|^q\right) \\ \times\left[\frac{2\sin(\rho(b-a)) - (b-a)\rho\left[\cos(\rho(b-a))-1\right] - 4\sin\left(\frac{\rho(b-a)}{2}\right)}{(b-a)^2\rho^2\sin(\rho(b-a))}\right]^{\frac{1}{q}}\end{aligned}$$

Proof. From Lemma 2.1, well known power-mean integral inequality and trigonometrically ρ-convexity of the function $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-a}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 |1-2t|\left[\frac{\sin[\rho t(b-a)]}{\sin[\rho(b-a)]}|f'(a)|^q+\frac{\sin[\rho(1-t)(b-a)]}{\sin[\rho(b-a)]}|f'(b)|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \\
&\times\left(|f'(a)|^q\int_0^1 |1-2t|\frac{\sin[\rho t(b-a)]}{\sin[\rho(b-a)]}dt+|f'(b)|^q\int_0^1 |1-2t|\frac{\sin[\rho(1-t)(b-a)]}{\sin[\rho(b-a)]}dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{2\sin(\rho(b-a))-(b-a)\rho[\cos(\rho(b-a))-1]-4\sin\left(\frac{\rho(b-a)}{2}\right)}{(b-a)^2\rho^2\sin(\rho(b-a))}|f'(a)|^q\right. \\
&\left.+\frac{2\sin(\rho(b-a))-(b-a)\rho[\cos(\rho(b-a))-1]-4\sin\left(\frac{\rho(b-a)}{2}\right)}{(b-a)^2\rho^2\sin(\rho(b-a))}|f'(a)|^q\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left[\frac{2\sin(\rho(b-a))-(b-a)\rho[\cos(\rho(b-a))-1]-4\sin\left(\frac{\rho(b-a)}{2}\right)}{(b-a)^2\rho^2\sin(\rho(b-a))}\right]^{\frac{1}{q}} \\
&\times A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right).
\end{aligned}$$

For $q=1$ we use the estimates from the proof of Theorem 2.7, which also follow step by step the above estimates. This completes the proof of theorem. □

Corollary 2.5. *Under the assumption of Theorem 2.9 with $q=1$, we get the following inequality:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq A(|f'(a)|,|f'(b)|) \\
&\times\left[\frac{2\sin(\rho(b-a))-(b-a)\rho[\cos(\rho(b-a))-1]-4\sin\left(\frac{\rho(b-a)}{2}\right)}{(b-a)\rho^2\sin(\rho(b-a))}\right]
\end{aligned}$$

Corollary 2.6. *Under the assumption of Theorem 2.9 with $\rho=1$, we get the following inequality:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}A^{\frac{1}{q}}\left(|f'(a)|^q,|f'(b)|^q\right) \\
&\times\left[\frac{2\sin(b-a)-(b-a)[\cos(b-a)-1]-4\sin\frac{b-a}{2}}{(b-a)^2\sin(b-a)}\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 2.7. *Under the assumption of Theorem 2.9 with $q=1$ and $\rho=1$ we get the following inequality:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq A\left(\left|f'(a)\right|,\left|f'(b)\right|\right) \\ &\times\left[\frac{2\sin(b-a)-(b-a)\left[\cos(b-a)-1\right]-4\sin\frac{(b-a)}{2}}{(b-a)\sin(b-a)}\right]\end{aligned}$$

Corollary 2.8. *Under the assumption of Theorem 2.9 with $\rho=\frac{\pi}{2(b-a)}$, we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{3}{q}}\left[\frac{1}{\pi}-\frac{4\left(\sqrt{2}-1\right)}{\pi^2}\right]^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right).$$

This inequality coincides with the inequality (1.3) in Theorem 1.3.

Corollary 2.9. *Under the assumption of Theorem 2.9 with $q=1$ and $\rho=\frac{\pi}{2(b-a)}$, we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)\,A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)\left[\frac{2}{\pi}-\frac{8\left(\sqrt{2}-1\right)}{\pi^2}\right]$$

This inequality coincides with the inequality (2.6).

Conclusion

In this paper, we obtain Hermite-Hadamard type inequalites for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is trigonometrically ρ-convex. We also show that our results in this study are consistent with previous results. We would also like to point out that the inequalities we obtained coincide with the inequalities obtained previously in the special cases of $q=1$ and $\rho=\frac{\pi}{2(b-a)}$. This shows that our paper is a more general study. A similar method can be applied to other classes of convexity.

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