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A Note on Moritoh Transforms

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ABSTRACT. Some fundamental properties of the Moritoh wavelet are discussed in this paper. The Moritoh transform is approximated for ultra-distributions in generalised Sobolev space. The adjoint formula of the Fourier transform is extended to the Moritoh transform. The convolution for quaternion-valued functions is defined for a modified representation of quaternions. Furthermore, the quaternionic Moritoh transform is defined with the help of convolution. The inner product relation and the uncertainty principle are also established for the quaternionic Moritoh wavelet transform.

1. INTRODUCTION

Murenzi transforms and Moritoh transforms are n-dimensional wavelet transforms. The Moritoh wavelet transform was defined in [11], which differs from Murenzi's prior n-dimensional wavelet transform [12]. Moritoh transforms incorporate special type of rotational characteristics and Littlewood-Paley decompositions. The wavelet transform in Besov and Triebel-Lizarkin spaces has been discussed by S. Moritoh. Moritoh had discussed the Hormander's wave front sets using this transform.

Various setups have been used to examine quaternionic Fourier transforms and wavelet transforms throughout the last two decades. Jianxun He and Yu [6] conducted research on the wavelet transform in $L^2(\mathbb{R};\mathbb{H})$. They demonstrated that in general, quaternion-valued functions fails to satisfy the convolution theorem. Later, Akila and Roopkumar [1], developed convolution of quaternion-valued functions in a novel method and proved the convolution theorem. F. A. Shah *et al.* [15] employed this technique lately to examine the quaternionic Bendlet transform in 2-D. Quaternionic wavelet transforms have been investigated in several ways in 2-D [2, 3, 9]. Kundu and Prasad [7] investigated uncertainty principles and Young-Hausdroff inequalities for quaternion linear canonical transform. In [8], they also investigated the quaternion function space using quaternion pseudo-differential operators. Prasad *et. al* [14] developed the Lieb uncertainty principle, Donoho-Stark inequality, and local uncertainty principle for the quaternion windowed linear canonical transform. They also demonstrated its application in the linear time-varying TV system. The n-dimensional Moritoh wavelet transforms has been investigated in the quaternionic space in this article.

The paper is distributed in three sections. In the first section, we recall the definitions and some important results of the Moritoh wavelet. In the second section, the basic properties of Moritoh transform is derived. The approximation of the Moritoh transform for ultra distributions on the generalised Sobolev space $B^{\omega}_{p,k}$ and estimation of Moritoh wavelets have been investigated. The adjoint formula for Fourier transform [18] has been obtained for Mortioh transforms. Also, it has been shown that the Moritoh transform

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of the function in L^p is tempered distribution for $|\xi| < 1$. The fundamentals of quaternions have been examined in the third section. A new representation of quaternions have been proposed. The conjugate, product, inner-product of the two quaternions and the quaternion-valued functions are expressed in the proposed representation. The convolution of quaternion-valued functions in the proposed representation has been given, and with its help, the quaternionic Moritoh transform (QMT) is defined. Some basic properties of the quaternionic Moritoh transform are given. The relation between the quaternionic Fourier transform of QMT and the real-valued classical Moritoh transform has been established. In every representation, we have tried to represent the expressions and results in such a way that after expanding the expressions, the results coincide with those in the format of a real-valued coefficient having three pure quaternions as basis elements. Furthermore, some results related to the quaternionic Moritoh transform and inner product relations have been established. It has been shown that the quaternionic Moritoh transform satisfies the uncertainty principle. At the end, the conclusion of the article is discussed.

Definition 1.1 (Moritoh wavelet)**.** Let $\xi \in \mathbb{R}^n$ and r_ξ be any rotation on $\mathbb{R}^n$ sending unit $\xi/|\xi|$ to $(0,\cdots,0,1)$, then Moritoh wavelet can be defined by the family $\psi_{x,\xi}$ as:-

$$\psi_{x,\xi}(t) := |\xi|^{n/2}\psi(|\xi| r_\xi(t-x)), \text{ where } \xi,\, x \in \mathbb{R}^n.$$

Definition 1.2 (Moritoh wavelet transform [11])**.** Suppose that a function $\psi(x)$ (called a wavelet) such that: $\psi(x) \in \mathscr{S}(\mathbb{R}^n)$, $\hat{\psi}(\xi) \in C_0^\infty(\mathbb{R}^n)$ and $\hat{\psi}(\xi) \geqq 0$. Let $\Omega = \operatorname{supp}\hat{\psi}(\xi)$ be in a neighbourhood of $(0,\cdots,0,1)$. When $n = 1, \Omega \subset (0,\infty)$, while when $n \geqq 2, \Omega$ is connected, does not contain the origin **0** and $\psi(x) = \psi(rx)$ for $r \in SO(n)$ satisfying $r(0,\cdots,0,1) = (0,\cdots,0,1)$. Let r_ξ be any rotation which sends $\xi/|\xi|$ to $(0,\cdots,0,1)$. Then Moritoh wavelet transform is defined as follows: for $f \in \mathscr{S}'(\mathbb{R}^n)$, $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$W_\psi f(x,\xi) = \begin{cases} \int\limits_{\mathbb{R}} f(t)|\xi|^{\frac{1}{2}}\, \overline{\psi(\xi(t-x))}dt, & \text{if } n = 1 \\ \int\limits_{\mathbb{R}^n} f(t)|\xi|^{\frac{n}{2}}\, \overline{\psi(|\xi| r_\xi(t-x))}d^n t, & \text{if } n \geqq 2. \end{cases} \tag{1.1}$$

Here $\mathscr{S}(\mathbb{R}^n)$ stands for the Schwartz class $C_0^\infty(\mathbb{R}^n)$ consists of the functions which are smooth and compactly supported. The set $SO(n)$ represents the set of $n \times n$ orthogonal matrices.

If we take $\psi_\xi(t) = |\xi|^{\frac{n}{2}}\psi(|\xi| r_\xi(t))$, then the Moritoh wavelet can be written in the form of convolution as $W_\psi f(x,\xi) = (f * \overset{\vee}{\psi}_\xi)(x)$, where $\overset{\vee}{\psi}(x) = \psi(-x)$. Applying convolution theorem for Fourier transform, we get

$$\mathscr{F}[W_\psi f(x,\xi)](\tau) = \mathscr{F}[(f * \overset{\vee}{\psi}_\xi)(x)](\tau) = \hat{f}\overline{\hat{\psi}}_\xi.$$

In this paper, the terms Moritoh wavelet transform and Moritoh transform will be used interchangeably for the sake of simplicity.

Proposition 1.1 (Parseval's formula and the inversion formula [11])**.** *For $f, g \in L^2(\mathbb{R}^n)$, we have*

$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} W_\psi f(x,\xi)\overline{W_\psi g(x,\xi)} d^n x d^n \xi = C_\psi \int\limits_{\mathbb{R}^n} f(t)\overline{g(t)} d^n t,$$

and

$$f(t) = C_\psi^{-1}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} W_{\psi f}(x,\xi)|\xi|^{n/2}\psi(|\xi| r_\xi(t-x)) d^n x d^n \xi,\ n \geqq 2,$$

where

$$0 < C_\psi = \int_{\mathbb{R}^n} |\hat{\psi}(\xi)|^2 \frac{d^n\xi}{|\xi|^n} < \infty.$$

When $n = 1, |\xi| r_\xi(t-x) = \xi(t-x)$. *For* $f \in \mathscr{S}'(\mathbb{R}^n)$, *this inversion formula holds in the sense of distribution.*

Remark 1.1. The following result holds:-

$$\int_{\mathbb{R}^n} f(r_\theta x) e^{-2\pi i\langle \omega, x\rangle} d^n x = \hat{f}(r_\theta(\omega)).$$

Proposition 1.2. *The Fourier transform of* $\psi_{x,\xi}$ *is given by*

$$\mathscr{F}\{\psi_{x,\xi}\}(\omega) = \hat{\psi}_{x,\xi}(\omega) = |\xi|^{-n/2} \hat{\psi}(|\xi|^{-1} r_\xi \omega)\, e^{-2\pi i\langle \omega,\, x\rangle}. \tag{1.2}$$

Proof.

$$\mathscr{F}\{\psi_{x,\xi}\}(\omega) = \int_{\mathbb{R}^n} \psi_{x,\xi}(t)\ e^{-2\pi i\langle \omega, t\rangle} d^n t = \int_{\mathbb{R}^n} |\xi|^{n/2} \psi(|\xi| r_\xi(t-x)) e^{-2\pi i\langle \omega, t\rangle} d^n t.$$

Put $|\xi|(t-x) = u$, then $t = |\xi|^{-1} u + x$ and $d^n t = |\xi|^{-n} d^n u$.

$$\begin{aligned}\mathscr{F}\{\psi_{x,\xi}\}(\omega) &= \int_{\mathbb{R}^n} |\xi|^{-n/2} \psi(r_\xi(u)) e^{-2\pi i\langle \omega, |\xi|^{-1} u + x\rangle} d^n u \\ &= |\xi|^{-n/2} \hat{\psi}(|\xi|^{-1} r_\xi \omega)\, e^{-2\pi i\langle \omega,\, x\rangle}.\end{aligned}$$

□

Remark 1.2. In view of Parseval's formula and the equation (1.2), the Moritoh wavelet transform can be rewritten as:-

$$W_\psi f(x,\xi) = \langle f(t), \psi_{x,\xi}(t)\rangle = \langle \hat{f}(t), \hat{\psi}_{x,\xi}(t)\rangle = \int_{\mathbb{R}^n} \hat{f}(\tau) |\xi|^{-n/2} \overline{\hat{\psi}(|\xi|^{-1} r_\xi \tau)}\, e^{2\pi i\langle \tau,\, x\rangle} d^n \tau.$$

1.1. **Moritoh wavelet in 1D.** We take $\psi(t) = t(1-t^2)e^{-1.5t^2}$ then Moritoh wavelet translated by 3 and dilated by 2 is given in Figure 1.

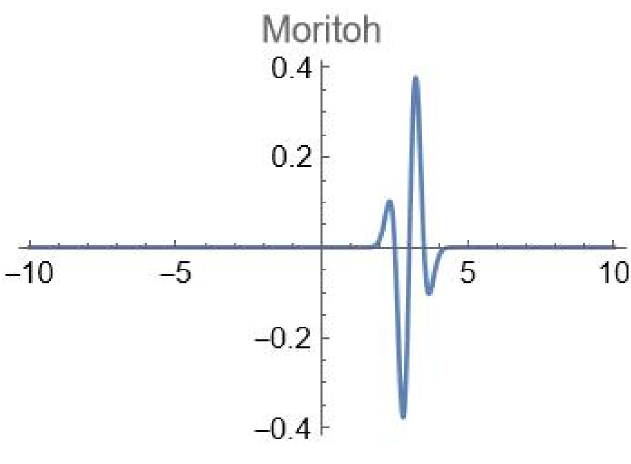

FIGURE 1. 1D-Moritoh

2. Basic Properties of Moritoh Wavelet transform

Theorem 2.1. *Let $\alpha,\ \beta \in \mathbb{R}$ and functions $f,\ g$ be defined on $\mathbb{R}^n$. Following results hold for the Moritoh wavelet transform*

(1) W_ψ is linear, i.e.

$$W_\psi(\alpha f + \beta g)(x,\xi) = \alpha W_\psi f(x,\xi) + \beta W_\psi g(x,\xi).$$

(2) Parity

$$W_{P\psi} P f(x,\xi) = W_\psi f(-x,\xi),$$

where $Pf(t) = f(-t)$ is parity operator.

(3) Anti-linear

$$W_{\alpha\psi+\beta\phi} f(x,\xi) = \overline{\alpha} W_\psi f(x,\xi) + \overline{\beta} W_\phi f(x,\xi).$$

(4)

$$W_{D_c\psi} f(x,\xi) = \sqrt{|c|} W_\psi f(x,\xi/c),$$

where $D_c\psi(x) = \frac{1}{\sqrt{|c|}}\psi(\frac{x}{c})$ is dilation operator and $c > 0$.

(5) Translation property

$$W_\psi(T_c f)(x,\xi) = W_\psi f(x-c,\xi),$$

where $T_c f(x) = f(x-c)$ is translation operator and $c > 0$.

(6) Dilation property

$$W_\psi(D_c f)(x,\xi) = c^{\frac{2n-3}{2}} W_\psi f\left(\frac{x}{c}, c\xi\right).$$

(7) If f is a homogeneous function of degree n and $\lambda \in \mathbb{R}^+$, then we have,

$$W_\psi f(\lambda x, \lambda\xi) = \frac{1}{|\lambda|^{\frac{3n}{2}}} W_\psi f(\lambda^2 x, \xi).$$

Proof. The proofs of (1)-(4) are straight forward.

(5)

$$\begin{aligned} W_\psi(T_c f)(x,\xi) &= \int\limits_{\mathbb{R}^n} f(t-c)\overline{|\xi|\psi(|\xi| r_\xi(t-x))} d^n t \\ &= \int\limits_{\mathbb{R}^n} f(u)\overline{|\xi|^{\frac{n}{2}}\psi(|\xi| r_\xi(u+c-x))} d^n u \\ &= \int\limits_{\mathbb{R}^n} f(u)\overline{|\xi|^{\frac{n}{2}}\psi(|\xi| r_\xi(u-(x-c)))} d^n u \\ &= W_\psi f(x-c,\xi). \end{aligned}$$

(6)

$$\begin{aligned}
W_\psi(D_c f)(x,\xi) &= \int\limits_{\mathbb{R}^n} \frac{1}{\sqrt{|c|}} f(t/c)\overline{|\xi|^{\frac{n}{2}}\psi(|\xi|r_\xi(t-x))}d^n t\\
&= \int\limits_{\mathbb{R}^n} \frac{1}{\sqrt{|c|}} f(u)\overline{|\xi|^{\frac{n}{2}}\psi(|\xi|r_\xi(cu-x))}c^n d^n u\\
&= |c|^{\frac{2n-1}{2}} \int\limits_{\mathbb{R}^n} f(u)\overline{|\xi|^{\frac{n}{2}}\psi(|\xi|r_\xi(c(u-x/c)))}d^n u\\
&= |c|^{\frac{2n-1}{2}} \int\limits_{\mathbb{R}^n} f(u)\overline{|\xi|^{\frac{n}{2}}\psi(|c\xi|r_\xi(u-x/c))}d^n u\\
&= |c|^{\frac{n-1}{2}} \int\limits_{\mathbb{R}^n} f(u)\overline{|c\xi|^{\frac{n}{2}}\psi(|c\xi|r_\xi(u-x/c))}d^n u\\
&= |c|^{\frac{n-1}{2}} W_\psi f(x/c, c\xi).
\end{aligned}$$

(7)

$$\begin{aligned}
W_\psi f(\lambda x, \lambda\xi) &= \int\limits_{\mathbb{R}^n} f(t)\,|\lambda\xi|^{n/2}\,\overline{\psi(|\lambda\xi|r_\xi(t-\lambda x))}d^n t\\
&= \int\limits_{\mathbb{R}^n} f\left(\frac{u}{|\lambda|}\right)|\lambda\xi|^{n/2}\,\overline{\psi(|\xi|r_\xi(u-\lambda^2 x))}\,\frac{d^n u}{|\lambda|^n}\\
&= \int\limits_{\mathbb{R}^n} \frac{1}{|\lambda|^n}\cdot|\lambda|^{n/2}\cdot\frac{1}{|\lambda|^n} f(u)\,|\xi|^{n/2}\overline{\psi(|\xi|r_\xi(u-\lambda^2 x))}\,d^n u\\
&\quad (\because f \text{ is homogeneous function of degree } n)\\
&= \frac{1}{|\lambda|^{\frac{3n}{2}}}\,W_\psi f(\lambda^2 x, \xi).
\end{aligned}$$

□

2.1. **Generalized Sobolev Space.**

Definition 2.3 (The space $\mathscr{M}_c$)**.** The space $\mathscr{M}_c$ is the collection of real-valued continuous functions ω on $\mathbb{R}^n$ having following properties

(i) $0 = \omega(0) \le \omega(\xi+\eta) \le \omega(\xi)+\omega(\eta)$,
(ii) $\int\limits_{\mathbb{R}^n} \frac{\omega(\xi)}{(1+|\xi|)^{n+1}} d^n\xi < \infty$,
(iii) $\omega(\xi) \ge a + b\log(1+|\xi|)$, for all $\xi \in \mathbb{R}^n, a \in \mathbb{R}, b \in \mathbb{R}^+$, and
(iv) $\omega(\xi) = \Omega(|\xi|)$, where Ω is concave function on $[0,\infty)$.

Definition 2.4 (Ultra distribution)**.** For $\omega \in \mathscr{M}_c$, the collection of all functions $\phi \in L^1(\mathbb{R}^n)$ with following properties

(i) $\phi, \hat{\phi} \in C^\infty$,
(ii) for multi-index α and non-negative number λ, the function ϕ satisfies,

$$p_{\alpha,\lambda}(\phi) = \sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)}|D^\alpha\phi(x)| < \infty,$$

(iii) for multi-index α and non-negative number λ, the function ϕ satisfies,

$$\pi_{\alpha,\lambda}(\phi) = \sup_{\xi\in\mathbb{R}^n} e^{\lambda\omega(\xi)}\left|D^\alpha\hat{\phi}(\xi)\right| < \infty$$

is denoted by $\mathscr{S}_\omega$. The space $\mathscr{S}_\omega$ forms topological space generated by semi norms $p_{\alpha,\lambda}$ and $\pi_{\alpha,\lambda}$. The dual space $\mathscr{S}'_\omega$ of the space of test functions $\mathscr{S}_\omega$ is called the space of **ultra distributions** and its members are known as ultra distributions.

Definition 2.5 (Generalized Sobolev space [13])**.** Let k be positive function on $\mathbb{R}^n$ such that $k(\xi+\eta) \leq e^{\lambda\omega(-\xi)}k(\eta)$ for all $\xi,\eta\in\mathbb{R}^n$ and $\lambda>0$. For $1\leq p<\infty$ we have

$$\|f\|_{p,k}^p = \int_{\mathbb{R}^n}\left|k(\xi)\hat{f}(\xi)\right|^p d^n\xi \quad \text{and} \quad \|f\|_{\infty,k} = ess\sup k(\xi)\left|\hat{f}(\xi)\right|.$$

The collection of all ultra distributions $f\in\mathscr{S}'_\omega$ for which $\|f\|_{p,k}<\infty$ is denoted by $B^\omega_{p,k}(\mathbb{R}^n)$ and known as generalized Sobolev space. For $\omega=\log(1+|\xi|)$, $k(\xi)=(1+|\xi|^2)^{s/2}$ and $p=2$ it becomes the Sobolev space H^s.

In the following theorem, we will see the approximation of Moritoh transform of the ultra-distribution on the generalized Sobolev space $B^\omega_{p,k}$.

Theorem 2.2. *For admissible Moritoh wavelets ψ,ϕ and ultra-distributions $F,G\in B^\omega_{p,k}(\mathbb{R}^n)$, we have*

$$\|(W_\psi F)(x,\xi)-(W_\phi G)(x,\xi)\|_{p,k} \leq |\xi|^{\frac{-n}{2}}\left(\|\psi-\phi\|_{L^1}\|F\|_{p,k}+\|\phi\|_{L^1}\|F-G\|_{p,k}\right).$$

Consequently, $\|(W_\psi F)(x,\xi)\| = o(|\xi|^{-n/2})$.

Proof.

$$\begin{aligned}&\|(W_\psi F)(x,\xi)-(W_\phi G)(x,\xi)\|_{p,k}\\ &= \|(W_\psi F)(x,\xi)-(W_\phi F)(x,\xi)+(W_\phi F)(x,\xi)-(W_\phi G)(x,\xi)\|_{p,k}\\ &\leq \|(W_\psi F)(x,\xi)-(W_\phi F)(x,\xi)\|_{p,k}+\|(W_\phi F)(x,\xi)-(W_\phi G)(x,\xi)\|_{p,k}.\end{aligned} \tag{2.3}$$

Using definition of $\|\cdot\|_{p,k}$ we have,

$$\begin{aligned}&\|(W_\psi F)(x,\xi)-(W_\phi F)(x,\xi)\|_{p,k}\\ &= \left(\int_{\mathbb{R}^n}|((W_\psi F)(x,\xi)-(W_\phi F)(x,\xi))\hat{}(\omega)|^p|k(\omega)|^p d^n\omega\right)^{\frac{1}{p}}\\ &= \left(\int_{\mathbb{R}^n}\left|\hat{F}(\omega)|\xi|^{-n/2}\overline{\hat{\psi}}(|\xi|^{-1}r_\xi\omega)-\hat{F}(\omega)|\xi|^{-n/2}\overline{\hat{\phi}}(|\xi|^{-1}r_\xi\omega)\right|^p|k(\omega)|^p d^n\omega\right)^{\frac{1}{p}}\\ &= \left(\int_{\mathbb{R}^n}\left|\hat{F}(\omega)\right|^p|\xi|^{-np/2}\left|\overline{\hat{\psi}}(|\xi|^{-1}r_\xi\omega)-\overline{\hat{\phi}}(|\xi|^{-1}r_\xi\omega)\right|^p|k(\omega)|^p d^n\omega\right)^{\frac{1}{p}}.\end{aligned} \tag{2.4}$$

By assumption in Moritoh wavelet we have $\psi(x)=\psi(rx)$ which implies that $\hat{\psi}(\omega)=\hat{\psi}(r\omega)$. As $|\xi|^{-1}r_\xi$ is special rotation therefore, $\hat{\psi}(\omega)=\hat{\psi}(|\xi|^{-1}r_\xi\omega)$. Since, $\left|\overline{\hat{\psi}}(|\xi|^{-1}r_\xi\omega)-\overline{\hat{\phi}}(|\xi|^{-1}r_\xi\omega)\right|\leq\|\psi-\phi\|_{L^1}$. Using this in the equation (2.4), we get

$$\|(W_\psi F)(x,\xi)-(W_\phi F)(x,\xi)\|_{p,k}\leq|\xi|^{-n/2}\|\psi-\phi\|_{L^1}\|F\|_{p,k}. \tag{2.5}$$

Similarly, we get,

$$\|(W_\phi F)(x,\xi)-(W_\phi G)(x,\xi)\|_{p,k}\leq|\xi|^{-n/2}\|\phi\|_{L^1}\|F-G\|_{p,k}. \tag{2.6}$$

Using (2.5) and (2.6) in the equation (2.3) we get,

$$\|(W_\psi F)(x,\xi)-(W_\phi G)(x,\xi)\|_{p,k}\le|\xi|^{\frac{-n}{2}}\left[\|\psi-\phi\|_{L^1}\|F\|_{p,k}+\|\phi\|_{L^1}\|F-G\|_{p,k}\right].$$

□

Theorem 2.3. *Let $f\in L^1(\mathbb{R}^n)$. Let ψ be an admissible Moritoh wavelet with estimates*

$$|\psi(r_\xi y)|\le\frac{A|\xi|^{n/2}}{1+B|y|^n}\ \forall\, y\in\mathbb{R}^n. \tag{2.7}$$

For Moritoh transform, we have the following relation,

$$\|(W_{P\psi}f)(x,\xi)\|_p\le\int\limits_{\mathbb{R}^n}\frac{A}{1+B|v|^n}\left\|\Delta_{-|\xi|^{-1}v}f\right\|_p d^nv.$$

Proof. By the admissibility condition of the Moritoh wavelet we have $\int\limits_{\mathbb{R}^n}|\xi|^{n/2}\psi(|\xi|r_\xi y)d^ny=0$.

$$\begin{aligned}(W_{P\psi}f)(x,\xi)&=\int\limits_{\mathbb{R}^n}f(t)|\xi|^{n/2}\overline{P\psi(|\xi|r_\xi(t-x))}d^nt\\&=\int\limits_{\mathbb{R}^n}f(t)|\xi|^{n/2}\overline{\psi(|\xi|r_\xi(x-t))}d^nt\\&=\int\limits_{\mathbb{R}^n}f(y+x)|\xi|^{n/2}\overline{\psi(|\xi|r_\xi(-y))}d^ny\\&=\int\limits_{\mathbb{R}^n}(f(y+x)-f(x))|\xi|^{n/2}\overline{\psi(|\xi|r_\xi(-y))}d^ny\\&=\int\limits_{\mathbb{R}^n}\Delta_yf(x)|\xi|^{n/2}\overline{\psi(|\xi|r_\xi(-y))}d^ny.\end{aligned}$$

Using definition of L^p-norm and Minkowski's inequality, we get,

$$\begin{aligned}\|(W_{P\psi}f)(x,\xi)\|_p&=\left(\int\limits_{\mathbb{R}^n}\left|\int\limits_{\mathbb{R}^n}(\Delta_yf(x))|\xi|^{n/2}\overline{\psi(|\xi|r_\xi(-y))}d^ny\right|^pd^nx\right)^{\frac{1}{p}}\\&\le\int\limits_{\mathbb{R}^n}|\xi|^{n/2}\left|\overline{\psi(|\xi|r_\xi(-y))}\right|\|\Delta_yf\|_pd^ny.\end{aligned}$$

Let $-y|\xi|=v$, then $d^ny=|\xi|^{-n}d^nv$.

$$\|(W_{P\psi}f)(x,\xi)\|_p\le\int\limits_{\mathbb{R}^n}\left|\overline{\psi(r_\xi v)}\right|\left\|\Delta_{-|\xi|^{-1}v}f\right\|_p|\xi|^{-n/2}d^nv.$$

Suppose the wavelet satisfies that following estimate

$$|\psi(r_\xi y)|\le\frac{A|\xi|^{n/2}}{1+B|y|^n},$$

where A and B are constants. Then

$$\|(W_{P\psi}F)(x,\xi)\|_p\le\int\limits_{\mathbb{R}^n}\frac{A}{1+B|v|^n}\left\|\Delta_{-|\xi|^{-1}v}f\right\|_p d^nv.$$

□

The estimate of Moritoh wavelet given in (2.7) is smooth and the plotted figure 2 shows that the Moritoh wavelet satisfying estimate (2.7) have narrow width. For example, $A = 1 = B$ and $\xi, y \in [-10, 10]$, the graph of estimate function can be seen in the Fig. 2.

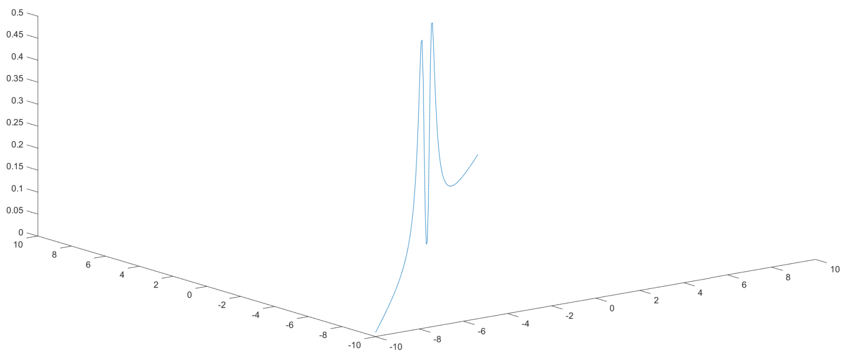

FIGURE 2. Estimation Function

The adjoint formula for the Fourier transform [18] have been generalised for the Moritoh transforms.

Theorem 2.4. *Let ψ be admissible Moritoh wavelet and $f, g \in \mathscr{S}'(\mathbb{R}^n)$, then we have,*

$$\iint \mathscr{F}[W_\psi f(x,\xi)] W_\phi g(x,\xi) d^n x d^n \xi = \iint W_\phi f(x,\xi) \mathscr{F}[W_\psi\, g(x,\xi)] d^n x d^n \xi.$$

Proof. We know that $\mathscr{F}[W_\psi f(x,\xi)] = \hat{f}(\omega)|\xi|^{\frac{-n}{2}}\overline{\hat{\psi}}(|\xi|^{-1} r_\xi \omega)$ and hence $W_\psi f(x,\xi)$ $= \int\limits_{\mathbb{R}^n} \hat{f}(\omega)|\xi|^{\frac{-n}{2}}\overline{\hat{\psi}}(|\xi|^{-1} r_\xi \omega) e^{2\pi i \langle x,\omega\rangle} d^n\omega$. Using these results, commutativity of complex numbers and Fubini theorem, the result is straight forward. □

Now, in the following result we claim that for $|\xi| < 1$ and the function in L^p space, the Moritoh transform $W_\psi f$ is tempered distribution.

Theorem 2.5. *Let $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^n)$ then for $|\xi| < 1$, the Moritoh transform $W_\psi f(x,\xi)$ is tempered distribution.*

Proof. As $f \in L^p(\mathbb{R}^n)$, therefore $\int\limits_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^N} d^n x < \infty$ for some positive integer N. Hence, f is tempered function.

$$\begin{aligned} |W_\psi f(x,\xi)| &\leq \int\limits_{\mathbb{R}^n} |f(t)||\xi|^{\frac{n}{2}} |\psi(|\xi| r_\xi (t-x))| d^n t \\ &\leq \left(\int\limits_{\mathbb{R}^n} \frac{|f(t)|}{(1+|t|)^N} d^n t \right) |\xi|^{\frac{n}{2}} \sup_{t\in\mathbb{R}^n} \left\{ (1+|t|)^N |\psi(|\xi| r_\xi (t-x))| \right\} < \infty. \end{aligned}$$

Let $\{\psi_j\}$ be a sequence of the Moritoh wavelets in $\mathscr{S}$ converging to 0 in $\mathscr{S}$ as $j \to \infty$, then

$$\sup_{t\in\mathbb{R}^n} \{(1+|t|)^N |\psi_j(|\xi| r_\xi (t-x))|\} \to 0 \text{ as } j \to \infty. \tag{2.8}$$

Since,

$$\begin{aligned}\left|(W_{\psi_j}f)(x,\xi)\right| &\le \int\limits_{\mathbb{R}^n} |f(t)||\xi|^{\frac{n}{2}}\psi_j(||\xi|r_\xi(t-x)|)d^nt \\ &\le \left(\int\limits_{\mathbb{R}^n} \frac{|f(t)|}{(1+|t|)^N}d^nt\right)|\xi|^{\frac{n}{2}} \sup_{t\in\mathbb{R}^n}\{(1+|t|)^N|\psi_j(|\xi|r_\xi(t-x))|\}.\end{aligned}$$

Using, the equation (2.8), we conclude that $\left|(W_{\psi_j}f)(x,\xi)\right| \to 0$ as $j\to\infty$. Hence, $W_\psi f$ is tempered distribution. □

3. Quaternion-valued Moritoh wavelets

Definition 3.6 (Quaternions)**.** The generalization of complex numbers namely, the quaternions was introduced by Sir William Roman Hamilton in 1843. The set of quaternions is denoted by $\mathbb{H}$ and each element $q\in\mathbb{H}$ is written as

$$\mathbb{H} = \{q = a_0 + i\,a_1 + j\,a_2 + k\,a_3 \mid a_0,a_1,a_2,a_3\in\mathbb{R}\},$$

where $i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$

Jianxun He *et al.* [6] and also L. Akila *et al.* [1] have represented the quaternions with the help of complex-valued coefficients and only one pure quaternion, namely, j. Their representation for quaternions can be mathematically described as: for $q = a_0 + i\,a_1 + j\,a_2 + k\,a_3$ where a_ℓ are real numbers for $\ell = 0,1,2,3$, they used the representation as $q = z_0 + jz_1$ where z_0 and z_1 are complex numbers. Here, we always have to specify that the value of z_1 is of conjugate type. In order to get back the same quaternion, we should put the value of conjugate in place of z_1. To get rid of the problem, we propose the new representation, i.e., $q = z_0 + j\overline{z}_1$. For proposed representation of quaternions, we have the following lemma:

Lemma 3.1. *The basic properties of quaternions can be represented in our proposed representation as*

(1) The conjugate $\overline{q}$ of $q = a_0 + i\,a_1 + j\,a_2 + k\,a_3$ is defined as $\overline{q} := a_0 - i\,a_1 - j\,a_2 - k\,a_3$.
(2) A quaternion $q = a_0 + i\,a_1 + j\,a_2 + k\,a_3$ can be represented by $q = z_0 + j\overline{z}_1$ where $z_0, z_1\in\mathbb{C}$.

$$\begin{aligned}q &= a_0 + i\,a_1 + j\,a_2 + k\,a_3 = a_0 + i\,a_1 + j\,a_2 - ji\,a_3 \\ &= (a_0 + ia_1) + j(a_2 - ia_3) = z_0 + j\overline{z}_1.\end{aligned}$$

Also, conjugate of quaternion can be written as $\overline{q} = \overline{z}_0 - j\overline{z}_1$ for some $z_0, z_1\in\mathbb{C}$.

$$\begin{aligned}\overline{q} &= a_0 - ia_1 - ja_2 - ka_3 = a_0 - ia_1 - ja_2 + jia_3 \\ &= (a_0 - ia_1) - j(a_2 - ia_3) = \overline{z}_0 - j\overline{z}_1.\end{aligned}$$

(3) The product of two quaternions say $q_1 = a_0 + i\,a_1 + j\,a_2 + k\,a_3 = z_0 + j\overline{z}_1$ and $q_2 = b_0 + i\,b_1 + j\,b_2 + k\,b_3 = w_0 + j\overline{w}_1$ is represented as

$$q_1q_2 = z_0w_0 - z_1\overline{w}_1 + j[\overline{z}_1w_0 + \overline{z}_0\overline{w}_1]. \tag{3.9}$$

(4) The inner product of two quaternions $q_1 = a_0 + i\,a_1 + j\,a_2 + k\,a_3 = z_0 + j\overline{z}_1$ and $q_2 = b_0 + i\,b_1 + j\,b_2 + k\,b_3 = w_0 + j\overline{w}_1$ is represented as

$$\langle q_1, q_2\rangle = q_1\overline{q}_2 = z_0\overline{w}_0 + z_1\overline{w}_1 + j[\overline{z}_1\overline{w}_0 - \overline{z}_0\overline{w}_1]. \tag{3.10}$$

(5) *The conjugate of product of quaternions and the product of the conjugate of two quaternions have the following relations:-*

$$\overline{q_1 q_2} = \overline{q}_2 \overline{q}_1 \neq \overline{q}_1 \overline{q}_2. \tag{3.11}$$

The product of the conjugate of two quaternions is given by

$$\overline{q}_2 \overline{q}_1 = \overline{w}_0 \overline{z}_0 - w_1 \overline{z}_1 - j(\overline{w}_1 \overline{z}_0 + w_0 \overline{z}_1). \tag{3.12}$$

(6) *A quaternion-valued function $f : \mathbb{R}^n \to \mathbb{H}$ can be expressed as*

$$F(x) = u_0(x) + i\, u_0(x) + j\, u_0(x) + k\, u_0(x) = f_0(x) + j\overline{f}_1(x),$$

where $u_0(x), u_1(x), u_2(x), u_3(x)$ are real-valued functions while f_0, f_1 are complex-valued functions.

3.1. **Convolution for the quaternions in terms of the proposed representation.** With the above observations, we have to define new convolution for our proposed representation for the quaternions. This new convolution is different from that given in [6, 1].

Definition 3.7 (The convolution of quaternion-valued functions)**.** Let $F, G \in L^1(\mathbb{R}^n; \mathbb{H}) \cap L^2(\mathbb{R}^n; \mathbb{H})$ with $F = F_0 + j\overline{F}_1$ and $G = G_0 + j\overline{G}_1$, then their convolution can be denoted as $F \star G$ and defined as

$$\begin{aligned}(F \star G)(t) :=& \int_{\mathbb{R}^n} F_0(x) G_0(t-x) d^n x - \int_{\mathbb{R}^n} \overset{\vee}{F}_1(x) \overline{G}_1(t-x) d^n x \\ &+ j\left(\int_{\mathbb{R}^n} \overline{F}_1(x) G_0(t-x) d^n x + \int_{\mathbb{R}^n} \overset{\vee}{\overline{F}}_0(x) \overline{G}_1(t-x) d^n x \right).\end{aligned} \tag{3.13}$$

Remark 3.3. Let $F, \Psi \in L^1(\mathbb{R}^n; \mathbb{H}) \cap L^2(\mathbb{R}^n; \mathbb{H})$ with $F = F_0 + j\overline{F}_1$ and $\Psi = \Psi_0 + j\overline{\Psi}_1$, then their convolution can be represented as

$$(F \star \overset{\vee}{\overline{\Psi}})(t) := \left((F_0 * \overset{\vee}{\overline{\Psi}}_0) + (\overset{\vee}{F}_1 * \overset{\vee}{\overline{\Psi}}_1) \right)(t) + j\left((\overline{F}_1 * \overset{\vee}{\overline{\Psi}}_0) - (\overset{\vee}{\overline{F}}_0 * \overset{\vee}{\overline{\Psi}}_1) \right)(t).$$

Definition 3.8 (The space $L^p(\mathbb{R}^n; \mathbb{H})$)**.** The space $L^p(\mathbb{R}^n; \mathbb{H})$, $1 \le p < \infty$ denotes the space of all measurable quaternion-valued functions f on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |f(x)|^p \, d^n x < \infty.$$

This space is a normed linear space.

Definition 3.9. The Fourier transform of a quaternion-valued function $F = f_0 + j\overline{f}_1$ is defined as

$$\mathscr{F}_{\mathbb{H}}[F](\xi) = \mathscr{F}[f_0](\xi) + j\mathscr{F}[\overline{f}_1](\xi),$$

where $\mathscr{F}[f](\xi) = \int_{\mathbb{R}^n} f(t)\, e^{-2\pi i \langle \xi, t\rangle} d^n t$, provided the integral exists. The inverse Fourier transform is defined as

$$\mathscr{F}_{\mathbb{H}}^{-1}[\mathscr{F}_{\mathbb{H}}[F]](t) = F(t) = \mathscr{F}^{-1}[\hat{f}_0](t) + j\mathscr{F}^{-1}[\overset{\vee}{\overline{\hat{f}}}_1](t),$$

where $\mathscr{F}^{-1}[\hat{f}](t) = \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{2\pi i \langle \xi, t\rangle} d^n \xi$, provided the integral exists.

Remark 3.4. Let $F, G \in L^1(\mathbb{R}^n;\mathbb{H}) \cap L^2(\mathbb{R}^n;\mathbb{H})$ with $F = F_0 + j\overline{F}_1$ and $G = G_0 + j\overline{G}_1$, then the convolution theorem for quaternion-valued function expressed in proposed representation holds:-

$$\mathscr{F}_{\mathbb{H}}[(F \star G)(t)](\omega) = \mathscr{F}_{\mathbb{H}}[F](\omega)\mathscr{F}_{\mathbb{H}}[G](\omega). \tag{3.14}$$

Proposition 3.3. *Let $\Psi \in L^2(\mathbb{R}^n;\mathbb{H})$ and $\xi, x \in \mathbb{R}^n$. Let $r_\xi \in SO(n)$ be a rotation which sends $\xi/|\xi|$ to $(0,...,0,1) \in \mathbb{R}^n$. Then we have*

$$\mathscr{F}_{\mathbb{H}}[\Psi_{x,\xi}](\omega) = |\xi|^{-\frac{n}{2}}\left[\hat{\Psi}_0(|\xi|^{-1}r_\xi\omega)e^{-2\pi i\langle\omega,x\rangle} + j\check{\hat{\Psi}}_1(|\xi|^{-1}r_\xi\omega)e^{2\pi i\langle\omega,x\rangle}\right]. \tag{3.15}$$

Proof.

$$\begin{aligned}\mathscr{F}_{\mathbb{H}}[\Psi_{x,\xi}](\omega) &= \mathscr{F}[\Psi_{0\,x,\xi}](\omega) + j\mathscr{F}[\overline{\Psi}_{1\,x,\xi}](\omega)\\ &= \mathscr{F}[\Psi_{0\,x,\xi}](\omega) + j\overline{\mathscr{F}[\Psi_{1\,x,\xi}]}^{\vee}(\omega).\end{aligned}$$

We know that $\Psi_{\ell\,x,\xi}(t) = |\xi|^{\frac{n}{2}}\Psi_\ell(|\xi|r_\xi(t-x)) \quad \forall\ \ell = 0,1$.

$$\begin{aligned}\mathscr{F}[\Psi_{\ell\,x,\xi}(t)](\omega) &= \int_{\mathbb{R}^n} |\xi|^{\frac{n}{2}}\Psi_\ell(|\xi|r_\xi(t-x))e^{-2\pi i\langle\omega,t\rangle}d^nt\\ &= \int_{\mathbb{R}^n} |\xi|^{\frac{n}{2}}\Psi_\ell(r_\xi u)e^{-2\pi i\langle\omega,|\xi|^{-1}u+x\rangle}|\xi|^{-n}d^nu\\ &= \left(\int_{\mathbb{R}^n} |\xi|^{\frac{-n}{2}}\Psi_\ell(r_\xi u)e^{-2\pi i\langle|\xi|^{-1}\omega,u\rangle}d^nu\right)e^{-2\pi i\langle\omega,x\rangle}\\ &= |\xi|^{-\frac{n}{2}}\hat{\Psi}_\ell(|\xi|^{-1}r_\xi\omega)\,e^{-2\pi i\langle\omega,x\rangle}.\end{aligned}$$

Therefore, we have

$$\mathscr{F}_{\mathbb{H}}[\Psi_{x,\xi}](\omega) = |\xi|^{-\frac{n}{2}}\left[\hat{\Psi}_0(|\xi|^{-1}r_\xi\omega)e^{-2\pi i\langle\omega,x\rangle} + j\check{\hat{\Psi}}_1(|\xi|^{-1}r_\xi\omega)e^{2\pi i\langle\omega,x\rangle}\right].$$

□

Definition 3.10 (Quaternionic Moritoh transfrom)**.** Let F and $\psi \in L^1(\mathbb{R}^n;\mathbb{H}) \cap L^2(\mathbb{R}^n;\mathbb{H})$. Then the quaternion Moritoh wavelet transform or simply quaternionic Moritoh transform can be denoted by $(\mathbb{H}\mathbb{W}_\psi F)(x,\xi)$ and defined as

$$(\mathbb{H}\mathbb{W}_\Psi F)(x,\xi) = (F \star \check{\overline{\Psi}}_\xi)(x). \tag{3.16}$$

Where $\check{\overline{\Psi}}_\xi(t) = \check{\overline{\Psi}}_0(|\xi|r_\xi t) - j\check{\overline{\Psi}}_1(|\xi|r_\xi t)$.

Definition 3.11 (Admissibility condition for quaternionic Moritoh transform)**.** An quaternion Moritoh wavelet $\psi \in L^1(\mathbb{R}^n;\mathbb{H}) \cap L^2(\mathbb{R}^n;\mathbb{H})$ is said to be admissible quaternion Moritoh wavelet if it satisfy the following admissibility condition

$$C_\Psi = \int_{\mathbb{R}^n} |\xi|^{-n}\left|\mathscr{F}_{\mathbb{H}}[\Psi](|\xi|^{-1}r_\xi\omega)\right|^2 d^n\xi < \infty. \tag{3.17}$$

Applying the change of variable from ξ to $\tau = |\xi|^{-1}r_\xi\omega$ we get,

$$C_\Psi = \int_{\mathbb{R}^n} |\tau|^{-n}|\mathscr{F}_{\mathbb{H}}[\Psi](\tau)|^2 d^n\tau < \infty. \tag{3.18}$$

Theorem 3.6. *For any $F \in L^1(\mathbb{R}^n;\mathbb{H}) \cap L^2(\mathbb{R}^n;\mathbb{H})$, the quaternion Moritoh transform defined by* (3.16) *satisfies the following identity:-*

$$\begin{aligned}(\mathbb{HW}_\Psi F)(x,\xi) =&(W_{\Psi_0}F_0)(x,\xi) + (W_{\Psi_1}\overset{\vee}{F}_1)(x,\xi)\\ &+ j\left((W_{\Psi_0}\overline{F}_1)(x,\xi) - (W_{\Psi_1}\overset{\vee}{\overline{F}}_0)(x,\xi)\right).\end{aligned}$$

Proof. Using definition of convolution, we have

(3.19)

$$\begin{aligned}(\mathbb{HW}_\Psi F)(x,\xi) &= \big(F \star \overset{\vee}{\overline{\Psi}}_\xi\big)(x)\\ &= \big[\big(F_0 + j\overline{F}_1\big) \star \big(\overset{\vee}{\overline{\Psi}}_{0\,\xi} - j\overset{\vee}{\overline{\Psi}}_{1\,\xi}\big)\big](x)\\ &= (F_0 * \overset{\vee}{\overline{\Psi}}_{0\,\xi})(x) + (\overset{\vee}{F}_1 * \overset{\vee}{\overline{\Psi}}_{1\,\xi})(x) + j\left((\overline{F}_1 * \overset{\vee}{\overline{\Psi}}_{0\,\xi}) - (\overset{\vee}{\overline{F}}_0 * \overset{\vee}{\overline{\Psi}}_{1\,\xi})\right)(x).\end{aligned}$$

Now,

$$\left(\overset{\vee}{\overline{F}}_0 * \overset{\vee}{\overline{\Psi}}_{1\,\xi}\right)(x) = \int_{\mathbb{R}^n} \overset{\vee}{\overline{F}}_0(t)\overset{\vee}{\overline{\Psi}}_{1\,\xi}(x-t)d^n x = (W_{\Psi_1}\overset{\vee}{\overline{F}}_0)(x,\xi).$$

Using this observation, in the equation (3.19), we get the result. □

By the straight forward calculation, one can get the following basic properties of QMT.

Proposition 3.4. *For $F, G \in L^1(\mathbb{R}^n;\mathbb{H}) \cap L^2(\mathbb{R}^n;\mathbb{H})$, $\alpha \in \mathbb{H}$, and admissible quaternionic Moritoh wavelets $\Psi, \Phi \in L^1(\mathbb{R}^n;\mathbb{H}) \cap L^2(\mathbb{R}^n;\mathbb{H})$ following are true*

(i) $(\mathbb{HW}_\Psi(F+G))(x,\xi) = (\mathbb{HW}_\Psi F)(x,\xi) + (\mathbb{HW}_\Psi G)(x,\xi)$
(ii) $(\mathbb{HW}_\Psi \alpha F)(x,\xi) = \alpha(\mathbb{HW}_\Psi F)(x,\xi)$
(iii) $(\mathbb{HW}_{\alpha\Psi} F)(x,\xi) = (\mathbb{HW}_\Psi F)(x,\xi)\overline{\alpha}$
(iv) $(\mathbb{HW}_{\Psi+\Phi} F)(x,\xi) = (\mathbb{HW}_\Psi F)(x,\xi) + (\mathbb{HW}_\Phi F)(x,\xi)$.

Lemma 3.2. *Suppose $F, \Psi \in L^1(\mathbb{R}^n;\mathbb{H}) \cap L^2(\mathbb{R}^n;\mathbb{H})$. The quaternionic Moritoh transform satisfies the following property:-*

(3.20)
$$\begin{aligned}\mathscr{F}_{\mathbb{H}}[(\mathbb{HW}_\psi f)(x,\xi)](\omega) = |\xi|^{\frac{-n}{2}}\Bigg[&\hat{f}_0(\omega)\overline{\hat{\Psi}}_0(|\xi|^{-1}r_\xi\omega) + \overset{\vee}{\hat{f}}_1(\omega)\overline{\hat{\Psi}}_1(|\xi|^{-1}r_\xi\omega)\\ &+ j\left\{\overset{\vee}{\hat{f}}_1(\omega)\overline{\hat{\Psi}}_0(|\xi|^{-1}r_\xi\omega) - \overline{\hat{f}}_0(\omega)\overline{\hat{\Psi}}_1(|\xi|^{-1}r_\xi\omega)\right\}\Bigg].\end{aligned}$$

However, for the sake of brevity in the calculation, we will use the following equation

$$\mathscr{F}_{\mathbb{H}}[(\mathbb{HW}_\psi f)(x,\xi)](\omega) = |\xi|^{\frac{-n}{2}}\mathscr{F}_{\mathbb{H}}[F](\omega)\overline{\mathscr{F}_{\mathbb{H}}[\Psi]}(|\xi|^{-1}r_\xi\omega). \tag{3.21}$$

Proof. Using the convolution theorem for the quaternionic Fourier transform, we have

$$\begin{aligned}\mathscr{F}_{\mathbb{H}}[(\mathbb{HW}_\psi F)(x,\xi)](\omega) &= \mathscr{F}_{\mathbb{H}}[(F \star \overset{\vee}{\overline{\Psi}}_\xi)(x)](\omega) = \mathscr{F}_{\mathbb{H}}[F](\omega)\mathscr{F}_{\mathbb{H}}[\overset{\vee}{\overline{\Psi}}_\xi](\omega)\\ \mathscr{F}_{\mathbb{H}}[(\mathbb{HW}_\Psi F)(x,\xi)](\omega) &= |\xi|^{\frac{-n}{2}}\mathscr{F}_{\mathbb{H}}[F](\omega)\overline{\mathscr{F}_{\mathbb{H}}[\Psi]}(|\xi|^{-1}r_\xi\omega).\end{aligned}$$

In view of the equation (3.10) we get,

$$\begin{aligned}\mathscr{F}_{\mathbb{H}}[(\mathbb{HW}_\Psi F)(x,\xi)](\omega) = |\xi|^{\frac{-n}{2}}\Bigg[&\hat{f}_0(\omega)\overline{\hat{\Psi}}_0(|\xi|^{-1}r_\xi\omega) + \overset{\vee}{\hat{f}}_1(\omega)\overline{\hat{\Psi}}_1(|\xi|^{-1}r_\xi\omega)\\ &+ j\left\{\overset{\vee}{\hat{f}}_1(\omega)\overline{\hat{\Psi}}_0(|\xi|^{-1}r_\xi\omega) - \overline{\hat{f}}_0(\omega)\overline{\hat{\Psi}}_1(|\xi|^{-1}r_\xi\omega)\right\}\Bigg].\end{aligned}$$

□

Remark 3.5. The equation (3.21) can be re-written as

$$(\mathbb{HW}_\psi F)(x,\xi) = \int_{\mathbb{R}^n} |\xi|^{\frac{-n}{2}} \mathscr{F}_{\mathbb{H}}[F](\omega)\overline{\mathscr{F}_{\mathbb{H}}[\Psi]}(|\xi|^{-1} r_\xi \omega) e^{2\pi i\langle\omega,x\rangle} d^n\omega. \tag{3.22}$$

Theorem 3.7 (Quaternionic orthogonality relation). *Let $\psi,\phi \in L^1(\mathbb{R}^n;\mathbb{H})\cap L^2(\mathbb{R}^n;\mathbb{H})$. Suppose $(\mathbb{HW}_\psi F)(x,\xi)$, $(\mathbb{HW}_\phi G)(x,\xi)$ are quaternionic Moritoh transforms of F and G as defined in* (3.16) *then for any pair of quaternion-valued functions $F,G \in L^1(\mathbb{R}^n;\mathbb{H})\cap L^2(\mathbb{R}^n;\mathbb{H})$ we have*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\mathbb{HW}_\psi F)(x,\xi)\overline{(\mathbb{HW}_\phi G)(x,\xi)} d^n x d^n\xi = C_{\psi,\phi}\langle F,G\rangle, \tag{3.23}$$

where $C_{\psi,\phi}$ is the admissibility condition associated with the quaternion-valued wavelets ψ and ϕ is given by:-

$$C_{\psi,\phi} = \left(\int_{\mathbb{R}^n} |\omega|^{-n}\overline{\mathscr{F}_{\mathbb{H}}[\psi]}(\omega)\mathscr{F}_{\mathbb{H}}[\phi](\omega) d^n\omega\right).$$

Proof. In view of the equation (3.22), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\mathbb{HW}_\psi F)(x,\xi)\overline{(\mathbb{HW}_\phi G)(x,\xi)} d^n x d^n\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |\xi|^{\frac{-n}{2}} \mathscr{F}_{\mathbb{H}}[F](\tau)\overline{\mathscr{F}_{\mathbb{H}}[\psi]}(|\xi|^{-1} r_\xi\tau) e^{2\pi i\langle\tau,x\rangle} d^n\tau\right] \\
&\qquad \overline{\left[\int_{\mathbb{R}^n} |\xi|^{\frac{-n}{2}} \mathscr{F}_{\mathbb{H}}[G](\sigma)\overline{\mathscr{F}_{\mathbb{H}}[\phi]}(|\xi|^{-1} r_\xi\sigma) e^{2\pi i\langle\sigma,x\rangle} d^n\sigma\right]} d^n x d^n\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\xi|^{-n} \mathscr{F}_{\mathbb{H}}[F](\tau)\overline{\mathscr{F}_{\mathbb{H}}[\psi]}(|\xi|^{-1} r_\xi\tau)\big(\delta(\tau-\sigma)\big)\mathscr{F}_{\mathbb{H}}[\phi](|\xi|^{-1} r_\xi\sigma) \\
&\qquad \overline{\mathscr{F}_{\mathbb{H}}[G](\sigma)} d^n\tau d^n\sigma d^n\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\xi|^{-n} \mathscr{F}_{\mathbb{H}}[F](\tau)\overline{\mathscr{F}_{\mathbb{H}}[\psi]}(|\xi|^{-1} r_\xi\tau)\mathscr{F}_{\mathbb{H}}[\phi](|\xi|^{-1} r_\xi\tau)\overline{\mathscr{F}_{\mathbb{H}}[G](\tau)} d^n\tau d^n\xi \\
&= \int_{\mathbb{R}^n} \mathscr{F}_{\mathbb{H}}[F](\tau)\left(\int_{\mathbb{R}^n} |\xi|^{-n}\overline{\mathscr{F}_{\mathbb{H}}[\psi]}(|\xi|^{-1} r_\xi\tau)\mathscr{F}_{\mathbb{H}}[\phi](|\xi|^{-1} r_\xi\tau) d^n\xi\right)\overline{\mathscr{F}_{\mathbb{H}}[G](\tau)} d^n\tau \\
&= \int_{\mathbb{R}^n} \mathscr{F}_{\mathbb{H}}[F](\tau)\left(\int_{\mathbb{R}^n} |\omega|^{-n}\overline{\mathscr{F}_{\mathbb{H}}[\psi]}(\omega)\mathscr{F}_{\mathbb{H}}[\phi](\omega) d^n\omega\right)\overline{\mathscr{F}_{\mathbb{H}}[G](\tau)} d^n\tau \\
&= C_{\psi,\phi}\int_{\mathbb{R}^n} \mathscr{F}_{\mathbb{H}}[F](\tau)\overline{\mathscr{F}_{\mathbb{H}}[G](\tau)} d^n\tau \\
&= C_{\psi,\phi}\int_{\mathbb{R}^n} F(x)\overline{G(x)} d^n x.
\end{aligned}$$

□

Corollary 3.1.

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\mathbb{HW}_\psi F)(x,\xi)\overline{(\mathbb{HW}_\psi G)(x,\xi)}d^nxd^n\xi = C_\psi\langle F,G\rangle_{L^2(R^n;\mathbb{H})}. \tag{3.24}$$

Corollary 3.2. *For $F = G$ and $\psi = \phi$ we have the following result:-*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|(\mathbb{HW}_\psi F)(x,\xi)|^2_{\mathbb{H}}d^nxd^n\xi = C_\psi\|F\|^2_{L^2(R^n;\mathbb{H})}. \tag{3.25}$$

Consequently, $\|(\mathbb{HW}_\psi F)(x,\xi)\|_{L^2(\mathbb{R}^n;\mathbb{H})} = \sqrt{C_\psi}\|F\|_{L^2(\mathbb{R}^n;\mathbb{H})}$.

Theorem 3.8 (Reconstruction formula)**.** *The quaternion-valued functions $F \in L^2(\mathbb{R}^n;\mathbb{H})$ can be reconstructed from their quaternionic Moritoh transforms by*

$$F(x) = \frac{1}{C_\psi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\mathbb{HW}_\psi F)(x,\xi)\,|\xi|^{\frac{n}{2}}\psi_\xi(|\xi|r_\xi(t-x))d^nxd^n\xi.$$

Proof. By orthogonality relation (3.24), we have

$$\begin{aligned}
C_\psi\langle F,G\rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\mathbb{HW}_\psi F)(x,\xi)\,\overline{(\mathbb{HW}_\psi G)(x,\xi)}d^nxd^n\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\mathbb{HW}_\psi F)(x,\xi)\left(\overline{\int_{\mathbb{R}^n} G(t)\check{\psi}_\xi(x-t)d^nt}\right)d^nxd^n\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\mathbb{HW}_\psi F)(x,\xi)\left(\overline{\int_{\mathbb{R}^n} G(t)\overline{\psi}_\xi(t-x)d^nt}\right)d^nxd^n\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\mathbb{HW}_\psi F)(x,\xi)\left(\int_{\mathbb{R}^n} \overline{\overline{\psi}_\xi(t-x)}\,\overline{G(t)}d^nt\right)d^nxd^n\xi \\
&= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\mathbb{HW}_\psi F)(x,\xi)\,\psi_\xi(t-x)d^nxd^n\xi\right)\overline{G(t)}d^nt.
\end{aligned}$$

Consequently, we get the result. □

3.2. **Uncertainty Principle.** The uncertainty principle says that if we are trying to get sharp localization of a signal in time domain then we have compromise with sharp localization property in frequency domain and vice-versa. A non-zero function and its Fourier transform cannot both be sharply localized [5]. In this section we will discuss the uncertainty principle of quaternion-valued functions and quaternion Moritoh wavelet transform.

Lemma 3.3. *For $\omega, b, x, \xi \in \mathbb{R}^n$ and quaternionic Moritoh transform $(\mathbb{HW}_\psi F)(x,\xi)$ of the quaternion-valued function F, following result holds good*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|(\omega-b)\mathscr{F}_{\mathbb{H}}(\mathbb{HW}_\psi F)(x,\xi)|^2d^n\omega d^n\xi = C_\psi\int_{\mathbb{R}^n}|(\omega-b)\mathscr{F}_{\mathbb{H}}[F](\omega)|^2d^n\omega. \tag{3.26}$$

Proof. By the equations (3.21) and (3.17), we have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} |(\omega-b)\mathscr{F}_{\mathbb{H}}(\mathbb{HW}_{\psi}F)(x,\xi)|^2 d^n\omega d^n\xi \\
&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} |\omega-b|^2 \left| |\xi|^{-n/2}\mathscr{F}_{\mathbb{H}}[F](\omega)\overline{\mathscr{F}_{\mathbb{H}}[\Psi]}(|\xi|^{-1}r_{\xi}\omega)\right|^2 d^n\omega d^n\xi \\
&= \int\limits_{\mathbb{R}^n} |(\omega-b)\mathscr{F}_{\mathbb{H}}[F](\omega)|^2 C_{\psi}\ d^n\omega.
\end{aligned}$$

□

Theorem 3.9. *Let $\Psi \in L^2(\mathbb{R}^n;\mathbb{H})$ be an admissible quaternion Moritoh wavelet, then for any function $F \in L^2(\mathbb{R}^n;\mathbb{H})$ and $a, b \in \mathbb{R}^n$, the following uncertainty inequality holds:*

$$\begin{aligned}
&\left\{\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} |(x-a)(\mathbb{HW}_{\Psi}F)(x,\xi)|^2 d^n x\, d^n\xi\right\}^{\frac{1}{2}} \\
&\times\left\{\int\limits_{\mathbb{R}^n} |(\omega-b)\mathscr{F}_{\mathbb{H}}[F](\omega)|^2 d^n\omega\right\}^{\frac{1}{2}} \geq \frac{n}{4\pi}\sqrt{C_{\psi}}\|F\|^2_{L^2(\mathbb{R}^n;\mathbb{H})}
\end{aligned} \tag{3.27}$$

Proof. By Corollary 2.8 of [5], for $F \in L^2(\mathbb{R}^n)$ and $a, b \in \mathbb{R}^n$, we have,

$$\left(\int\limits_{\mathbb{R}^n} |x-a|^2|F(x)|^2 d^n x\right)\left(\int\limits_{\mathbb{R}^n} |\omega-b|^2\left|\hat{F}(\omega)\right|^2 d^n\omega\right) \geq \frac{n^2}{16\pi^2}\|F\|^4.$$

As $\mathbb{HW}_{\psi}F(x,\xi) \in L^2(\mathbb{R}^{2n},\mathbb{H})$ we can extend above result as follows:

$$\begin{aligned}
&\left(\int\limits_{\mathbb{R}^n} |x-a|^2|\mathbb{HW}_{\psi}F(x,\xi)|^2 d^n x\right)^{\frac{1}{2}}\left(\int\limits_{\mathbb{R}^n} |\omega-b|^2|\mathscr{F}_{\mathbb{H}}[(\mathbb{HW}_{\psi}F)(x,\xi)](\omega)|^2 d^n\omega\right)^{\frac{1}{2}} \\
&\geq \frac{n}{4\pi}\int\limits_{\mathbb{R}^n} |(\mathbb{HW}_{\psi}F)(x,\xi)|^2 d^n x.
\end{aligned}$$

Integrating w.r.t. ξ we get,

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}^n} |x-a|^2|\mathbb{HW}_{\psi}F(x,\xi)|^2 d^n x\right)^{\frac{1}{2}}\left(\int\limits_{\mathbb{R}^n} |\omega-b|^2|\mathscr{F}_{\mathbb{H}}[(\mathbb{HW}_{\psi}F)(x,\xi)](\omega)|^2 d^n\omega\right)^{\frac{1}{2}} d^n\xi \\
&\geq \frac{n}{4\pi}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} |(\mathbb{HW}_{\psi}F)(x,\xi)|^2 d^n x d^n\xi.
\end{aligned}$$

Applying Cauchy- Schwartz inequality and using the equation (3.25) we get,

$$\begin{aligned}
&\left(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} |x-a|^2|\mathbb{HW}_{\psi}F(x,\xi)|^2 d^n x d^n\xi\right)^{\frac{1}{2}} \\
&\times\left(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} |\omega-b|^2|\mathscr{F}_{\mathbb{H}}[(\mathbb{HW}_{\psi}F)(x,\xi)](\omega)|^2 d^n\omega d^n\xi\right)^{\frac{1}{2}} \geq \frac{n}{4\pi}C_{\psi}\|F\|^2_{L^2(\mathbb{R}^n;\mathbb{H})}.
\end{aligned}$$

Using (3.26) in second term of L.H.S. we get,

$$\left(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|x-a|^2|\mathbb{HW}_\psi F(x,\xi)|^2 d^n x d^n \xi\right)^{\frac{1}{2}}\left(C_\psi\int\limits_{\mathbb{R}^n}|(\omega-b)\mathscr{F}_{\mathbb{H}}[F](\omega)|^2 d^n\omega\right)^{\frac{1}{2}} \geq \frac{n}{4\pi}C_\psi\|F\|^2_{L^2(\mathbb{R}^n;\mathbb{H})}.$$

We achieve the required result (3.27) after simplification. □

4. Conclusion

The approximation of the Moritoh transform of the ultra distribution on generalised Sobolev space and the estimation of admissible Moritoh wavelets with narrow width have been established. The adjoint formula for the Fourier transform has been extended to the Moritoh transform. It has been shown that for $|\xi| < 1$, the Moritoh transform of the function in L^p space is a tempered distribution. The one-dimensional quaternionic transform and the two-dimensional quaternionic transform have been defined, and relevant theory has been developed in [6, 2, 3, 1, 9, 15]. We are presenting the study of the quaternionic transform for n-dimension. Generally, we write the quaternions as an expression having three pure quaternions with a real coefficient. We are giving a new representation of quaternions that consists of only one pure quaternion with the coefficient in the complex number field. The advantage of our representation is that we need only one pure quaternion, namely, j, which enables us to get rid of the complexity of performing algebraic operations on two quaternions, especially the product of quaternions.

With the help of our representation, one can get the convolution and other important theorems for quaternionic integral transforms in n-dimension. Although we have derived the theorems or results for quaternion-valued functions and quaternionic transforms, we can also get these results for complex-valued Moritoh transforms.

- The results boil down to a complex-valued Moritoh transform if we take $f_1 = 0$ and $\psi_1 = 0$ in the results of QMT;
- For real-valued functions f_0, ψ_0 and by making other components of the function and wavelet zero, we get the classical Moritoh transform.

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