CREAT. MATH. INFORM.
Volume **33** (2024), No. 2,
Pages 203 - 216

Online version at https://creative-mathematics.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2024.02.06

Preserver Problems on Infinite Divisibility and Separability

INDU LETHA[1] and JILL K. MATHEW [2]

ABSTRACT. This paper is committed in characterizing the preserving maps for the class of separable matrices and a subclass of infinitely divisible matrices which are also separable. For this, an association between the classes of separable matrices and infinitely divisible matrices is established. Also, several properties and attributes of the above mentioned classes are stated and proved.

1. INTRODUCTION

Preserving problems can be regarded as century old problems. Many outstanding and interesting works have been carried out in this field and still it remains an area which can be explored much further.
Characterizing preserving maps for different sets, properties and classes is the main gist of preserving problems. A linear map from a matrix algebra into itself which preserve certain invariant properties can be called a linear preserver. Characterizing linear preserving maps mentioned in [11] drew our attention towards other works related to preserving problems like Commutativity Preserving Maps [15], Linear Maps on $M_n(R)$ Preserving Schur Stable Matrices [2] etc. We found in our survey that the characterization of preserving maps for the class of infinitely divisible matrices is an unexplored problem.
We focus our attention mainly on preserving infinitely divisibility and separability. For that, the concept of infinitely divisible matrix and separable matrix are discussed in detail. To define an infinitely divisible matrix, the notion of Hadamard product, Hadamard power of a matrix and fractional Hadamard power of a matrix is needed and is introduced in the sequel as follows.
Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $n \times n$ matrices. Then the Hadamard product (or the entry wise product) of A and B is the matrix $A \circ B = [a_{ij}b_{ij}]$, $0 \leq i, j \leq n$.
Example: $\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \circ \begin{bmatrix} 1 & 8 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -8 \\ 0 & 6 \end{bmatrix}$
For each non-negative integer m, the m^{th} Hadamard power of A is defined as $A^{\circ m} = [a_{ij}^m]$.
For example, $\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}^{\circ 3} = \begin{bmatrix} 64 & -1 \\ 0 & 8 \end{bmatrix}$.
For $a_{ij} \geq 0$ and any non-negative r, the fractional Hadamard power is defined as $A^{\circ r} = [a_{ij}^r]$.
A matrix whose fractional Hadamard power are all positive semidefinite is called infinitely divisible matrix ([6] & [10]). An example for an infinitely divisible matrix is Cauchy

Received: 22.04.2023. In revised form: 18.02.2024. Accepted: 26.02.2024
2020 *Mathematics Subject Classification.* 15A86.
Key words and phrases. *Infinitely divisible matrix, separable matrix, linear preserver, tensor product.*
Corresponding author: Indu Letha; indu051993@gmail.com

matrix. $\begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$ is a 3×3 Hilbert matrix which is a Cauchy matrix and it is infinitely divisible. A detailed study of infinitely divisible matrices can be found in [4], [5] and [8]. It is interesting to observe the techniques employed for proving infinite divisibility across various types of matrices. A detailed study for infinite divisibility of a tridiagonal matrix is carried out in [14]. It is proved that a tridiagonal matrix T is infinitely divisible if and only if T is a block diagonal matrix, where each non-zero diagonal block is a positive semi definite matrix of order 1 or 2. The underlying idea of infinitely divisible matrices is certainly the concept of positive semidefiniteness. At the starting point of the study, we gave more attention to uncover new properties of infinitely divisible matrix, the properties includes the closureness of the class of infinitely divisible matrices under the operations $+$ and $\otimes$. These properties are applied in Illustration 2.1 for obtaining the construction method of a 4×4 infinitely divisible matrix with real sequence as entries. Also, the necessary and sufficient condition for a matrix of the form $A \otimes B$ to be infinitely divisible is derived.

A whole new insight was obtained when we came across the concept of separability of matrices which is again a concept based on the idea of positive semidefiniteness. A matrix X is said to be separable [1] if $\exists\ A_i \in M_n$, $B_i \in M_m$ which are positive semidefinite matrices such that $X = \sum_{i=1}^{k}(A_i \otimes B_i)$.

The matrix $\begin{bmatrix} 20 & 7 & 10 & 2 \\ 7 & 26 & 2 & 12 \\ 10 & 2 & 23 & -6 \\ 2 & 12 & -6 & 28 \end{bmatrix}$ is separable, as it can be decomposed as follows:

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \otimes \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \otimes \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \otimes \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$, where each 2×2 matrix is positive semi definite.

The separability of a quantum system in quantum information theory is of great importance. The separability of a quantum system can be determined by the separability of its density matrix, which represents that quantum system. Many studies have been carried out in this area. [1], [12], [7] and [13] are few among them. Going through the concept of separability while gripping on to the idea of infinite divisibility, the notion of connecting these two concepts together became evident. It was intriguing to see that two completely unalike concepts are having some relation connecting them. All these connections were because of the underlying common characteristic of positive semidefiniteness.

In this article, the second section comprises of some properties of infinitely divisible matrix, including the closure property. Third section includes the result for separability condition of a positive semi definite matrix of the form $A \otimes B$ and its decomposition technique to the form $\sum_{i=1}^{k}(A_i \otimes B_i)$ and finally in the fourth section two preserver maps are characterized, one for the class of separable matrices and another for the class of matrices which is both separable and infinitely divisible.

2. SOME PROPERTIES OF INFINITELY DIVISIBLE MATRIX

In this section, we give the definition of infinitely divisible matrix with few detailed examples. Also, we have obtained some algebraic properties of infinitely divisible matrix. Also we have established the necessary and sufficient conditions for infinitely divisible matrix of the form $A \otimes B$.

Definition 2.1. A matrix $A = [a_{ij}] \in M_n$, where $a_{ij} \geq 0$ is said to be infinitely divisible, if every fractional Hadamard power of A defined as $A^{\circ r} = [a_{ij}^r]\,\forall\, r \geq 0$ is positive semidefinite.

The following are few examples of infinitely divisible matrix:

$$\begin{bmatrix} 10 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 12 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 16 & 4 & 6 & 2 \\ 4 & 10 & 2 & 3 \\ 6 & 2 & 22 & 6 \\ 2 & 3 & 6 & 13 \end{bmatrix}$$

The method in which the aforementioned infinitely divisible matrices are obtained is similar to that demonstrated in Illustration 2.1. There are also many well known examples of infinitely divisible matrices like Cauchy matrix, Pascal matrix etc [4].

Cauchy matrix:

Consider the Cauchy matrix with entires defined as $C = [c_{ij}] = \left[\frac{1}{(\lambda_i + \lambda_j)}\right]$ associated with $\lambda_1, \lambda_2, \ldots, \lambda_n$ where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Let $L_2(0, \infty)$ be the Hilbert space, whose elements are the functions on $(0, \infty)$ that are square integrable with respect to the Lebesgue measure. Then the inner product between two elements $u_1, u_2 \in L_2(0, \infty)$ is defined by $< u_1, u_2 >= \int_0^\infty u_1(t)\overline{u_2(t)}dt$.

Let $u_i(t) = e^{-t\lambda_i}(1 \leq i \leq n)$. Then

$$< u_1, u_2 >= \int_0^\infty e^{-t\lambda_i} e^{-t\lambda_j} dt = \int_0^\infty e^{-t(\lambda_i + \lambda_j)} dt = \frac{1}{(\lambda_i + \lambda_j)} \tag{2.1}$$

Therefore the matrix $C = [c_{ij}] = [< u_i, u_j >] = \left[\frac{1}{(\lambda_i + \lambda_j)}\right]$ is a Gram matrix and hence it is positive semidefinite.

We have the gamma function $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ which can be applied for obtaining,

$$\frac{1}{(\lambda_i + \lambda_j)^r} = \frac{1}{\Gamma(r)} \int_0^\infty e^{-t(\lambda_i + \lambda_j)} t^{r-1} dt \quad ; r > 0$$

When $r = 1$, the above equation is reduced to Eq. (2.1).

Therefore, $C^{\circ r} = [c_{ij}^r] = \left[\frac{1}{(\lambda_i + \lambda_j)^r}\right]$ is also a Gram matrix associated with $u_i(t) = e^{-t\lambda_i}$ in $L_2(0, \infty)$ relative to the measure $d\mu(t) = \frac{t^{r-1}}{\Gamma(r)} dt$.

Thus proving $C^{\circ r}$ to be positive semidefinite for all $r > 0$ and hence C to be infinitely divisible.

Pascal Matrix:

The $n \times n$ Pascal matrix is defined as $P = [p_{ij}] = \left[\binom{i+j}{i}\right] = \frac{(i+j)!}{i!j!} \quad ; i, j = 0, 1, 2, \ldots, (n-1)$

For example a 3×3 Pascal matrix is $P' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$.

The Pascal matrix is positive semidefinite which can be proved by representing it as a Gram matrix. One such representation using gamma function is as follows:

For $x > 0, y > 0$, $\Gamma(x+y+1) = \int_0^\infty e^{-t} t^{(x+y)} dt$. Also, $\Gamma(n+1) = n!$
Thus, for a matrix Q with entries $[q_{ij}] = [(i+j)!] = [\Gamma(x+y+1)]$ is a gram matrix. A Pascal matrix $P = [p_{ij}] = \left[\frac{(i+j)!}{i!j!}\right]$ is congruent to $Q = [q_{ij}] = [\Gamma(i+j+1)]$. Therefore P is positive semidefinite. The infinite divisibility of P can be stated by proving that P is congruent to Cauchy matrix which can be referred in [4].

We also have examples of classes of infinitely divisible matrices like:

(1) Every 2×2 positive semidefinite matrix with non negative entries is infinitely divisible.
(2) Every positive semidefinite matrix with strictly positive entries is infinitely divisible.

The following theorem obtained shows that the set of infinitely divisible matrices is closed under addition.

Theorem 2.1. *Let $A_1, A_2, \ldots, A_n$ be infinitely divisible, then $A_1 + A_2 + \cdots + A_n$ is infinitely divisible.*

Proof. We prove the case for the sum of two matrices, similar proof can be said in the case of n matrices.
Suppose $A, B \in M_n$ are infinitely divisible. $(A+B)^{or}$, for any $r \geq 0$, can be decomposed as $A^{or} + B^{or} + S_1 + S_2 + \cdots$, where $S_i = \alpha_i A^{or_1} \circ B^{or_2}$ and $\alpha_i, r_1, r_2 \geq 0$. Now, since A and B are infinitely divisible matrices, A^{or} and B^{or} are positive semidefinite for any $r \geq 0$. And also by Schur's theorem, Hadamard product of positive semidefinite matrices is again positive semidefinite. Therefore, $S_i = \alpha_i A^{or_1} \circ B^{or_2}$, $\alpha_i, r_1, r_2 \geq 0$ is positive semidefinite. i.e., Every term in the decomposition of $(A+B)^{or}$ is positive semidefinite for any $r \geq 0$. Since the sum of positive semidefinite matrix is again positive semidefinite, $(A+B)^{or}$ is positive semidefinite $\forall\, r \geq 0$. Thus $A+B$ is infinitely divisible. □

The following definition is of tensor product.

Definition 2.2. The Tensor product (Kronecker product) of $A = [a_{ij}] \in M_n(F)$ and $B = [b_{ij}] \in M_m(F)$ is denoted by $X = A \otimes B \in M_n(M_m(F))$ and defined to be the block matrix

$$X = A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}$$

More results and details on tensor product can be obtained by referring [9].

The following result obtained shows the preserving property of tensor product on the class of infinitely divisible matrices.

Theorem 2.2. *If $A \in M_n(F)$ and $B \in M_m(F)$ are infinitely divisible, then $A \otimes B$ is infinitely divisible.*

Proof. Since A and B are infinitely divisible, A^{or} and B^{or} are positive semidefinite $\forall\, r \geq 0$.
Therefore, $A^{or} \otimes B^{or}$ is positive semidefinite $\forall\, r \geq 0$.

$$(A \otimes B)^{or} = [a_{ij}B]^{or} = [a_{ij}^r B^{or}] = A^{or} \otimes B^{or} \;\forall\, r \geq 0$$

Therefore, by above relation $(A \otimes B)^{or}$ is positive semidefinite $\forall\, r \geq 0$. Thus $A \otimes B$ is infinitely divisible. □

Combining 2.1 and 2.2, we can construct infinite number of infinitely divisible matrices. Construction of a 4×4 infinitely divisible matrix with entries as real sequence is shown in the following illustration.

Illustration 2.1.

Consider $A_{2\times 2} = \left[\left(\frac{1}{n^{i+j}}\right)\right] = \begin{bmatrix} \frac{1}{n^2} & \frac{1}{n^3} \\ \frac{1}{n^3} & \frac{1}{n^4} \end{bmatrix}$; $n \in N$ and $B_{2\times 2} = \left[\left(\frac{1}{n}\right)\right] = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} \end{bmatrix}$; $n \in N$

Here A is positive semidefinite, since its eigenvalues are 0 and $\frac{n^2+1}{n^4}$. B is also positive semidefinite, since it is congruent to flat matrix. Clearly A and B are infinitely divisible matrices, since both are 2×2 positive semidefinite matrices.
Hence, in particular, $A_k = \left[\left(\frac{1}{n^{i+j}}\right)^k\right]$ and $B_k = \left[\left(\frac{1}{n}\right)^k\right]$ are all positive semidefinite for all $k \in Z^+$. Thus, $S = A_1 \otimes B_1 + A_2 \otimes B_2 + \cdots + A_m \otimes B_m$ is infinitely divisible.

i.e., $S = \begin{bmatrix} \frac{1}{n^2} & \frac{1}{n^3} \\ \frac{1}{n^3} & \frac{1}{n^4} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{n} & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} \end{bmatrix} + \begin{bmatrix} (\frac{1}{n^2})^2 & (\frac{1}{n^3})^2 \\ (\frac{1}{n^3})^2 & (\frac{1}{n^4})^2 \end{bmatrix} \otimes \begin{bmatrix} (\frac{1}{n})^2 & (\frac{1}{n})^2 \\ (\frac{1}{n})^2 & (\frac{1}{n})^2 \end{bmatrix} + \cdots + \begin{bmatrix} (\frac{1}{n^2})^m & (\frac{1}{n^3})^m \\ (\frac{1}{n^3})^m & (\frac{1}{n^4})^m \end{bmatrix} \otimes \begin{bmatrix} (\frac{1}{n})^m & (\frac{1}{n})^m \\ (\frac{1}{n})^m & (\frac{1}{n})^m \end{bmatrix}$

$$\implies S = \begin{bmatrix} \frac{1}{n^3} & \frac{1}{n^3} & \frac{1}{n^4} & \frac{1}{n^4} \\ \frac{1}{n^3} & \frac{1}{n^3} & \frac{1}{n^4} & \frac{1}{n^4} \\ \frac{1}{n^4} & \frac{1}{n^4} & \frac{1}{n^5} & \frac{1}{n^5} \\ \frac{1}{n^4} & \frac{1}{n^4} & \frac{1}{n^5} & \frac{1}{n^5} \end{bmatrix} + \begin{bmatrix} \frac{1}{n^6} & \frac{1}{n^6} & \frac{1}{n^8} & \frac{1}{n^8} \\ \frac{1}{n^6} & \frac{1}{n^6} & \frac{1}{n^8} & \frac{1}{n^8} \\ \frac{1}{n^8} & \frac{1}{n^8} & \frac{1}{n^{10}} & \frac{1}{n^{10}} \\ \frac{1}{n^8} & \frac{1}{n^8} & \frac{1}{n^{10}} & \frac{1}{n^{10}} \end{bmatrix} + \cdots + \begin{bmatrix} \frac{1}{n^{3k}} & \frac{1}{n^{3k}} & \frac{1}{n^{4k}} & \frac{1}{n^{4k}} \\ \frac{1}{n^{3k}} & \frac{1}{n^{3k}} & \frac{1}{n^{4k}} & \frac{1}{n^{4k}} \\ \frac{1}{n^{4k}} & \frac{1}{n^{4k}} & \frac{1}{n^{5k}} & \frac{1}{n^{5k}} \\ \frac{1}{n^{4k}} & \frac{1}{n^{4k}} & \frac{1}{n^{5k}} & \frac{1}{n^{5k}} \end{bmatrix}$$

$$\implies S = \begin{bmatrix} \sum_{i=1}^{m} \frac{1}{n^{3k}} & \sum_{i=1}^{m} \frac{1}{n^{3k}} & \sum_{i=1}^{m} \frac{1}{n^{4k}} & \sum_{i=1}^{m} \frac{1}{n^{4k}} \\ \sum_{i=1}^{m} \frac{1}{n^{3k}} & \sum_{i=1}^{m} \frac{1}{n^{3k}} & \sum_{i=1}^{m} \frac{1}{n^{4k}} & \sum_{i=1}^{m} \frac{1}{n^{4k}} \\ \sum_{i=1}^{m} \frac{1}{n^{4k}} & \sum_{i=1}^{m} \frac{1}{n^{4k}} & \sum_{i=1}^{m} \frac{1}{n^{5k}} & \sum_{i=1}^{m} \frac{1}{n^{5k}} \\ \sum_{i=1}^{m} \frac{1}{n^{4k}} & \sum_{i=1}^{m} \frac{1}{n^{4k}} & \sum_{i=1}^{m} \frac{1}{n^{5k}} & \sum_{i=1}^{m} \frac{1}{n^{5k}} \end{bmatrix} \text{ is infinitely divisible.}$$

□

Results on preserving properties of tensor product of rank-one Hermitian matrices can be referred from [17].
The following three results have been obtained, providing insight into the structure of tensor product of two matrices, their symmetric properties, condition for them to be infinitely divisible.

Lemma 2.1. *If $X = A \otimes B \in M_n(M_m(F))$ is symmetric with atleast one non-zero diagonal entry then $A \in M_n(F)$ and $B \in M_m(F)$ are symmetric.*

Proof. Let $X = A \otimes B \in M_n(M_m(F))$ be symmetric.
Therefore, $X^T = X$

$$\Rightarrow A^T \otimes B^T = (A \otimes B) \Rightarrow \begin{bmatrix} a_{11}B^T & a_{21}B^T & \cdots & a_{n1}B^T \\ a_{12}B^T & a_{22}B^T & \cdots & a_{n2}B^T \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n}B^T & a_{2n}B^T & \cdots & a_{nn}B^T \end{bmatrix} = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}$$

Clearly for X to be symmetric, if B is symmetric then A should be symmetric and if B is skew-symmetric then A should be skew-symmetric. Since there is alteast one diagonal entry of X which is of the form $a_{ii}b_{jj} \neq 0$,where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, mn$; then there exist atleast one $a_{ii} \neq 0$. Then by equating corresponding matrix block, $a_{ii}B^T = a_{ii}B \Rightarrow B^T = B \Rightarrow B$ is symmetric$\Rightarrow A$ is symmetric. □

Lemma 2.2. *If $X = A \otimes B \neq 0 \in M_n(M_m(F))$ is positive semidefinite, then A and B are symmetric.*

Proof. Let $X = A \otimes B \neq 0 \in M_n(M_m(F))$ be positive semidefinite. Then,
Sum of diagonal entries of $X =$ Sum of the eigenvalues of $X \geq 0$.
If sum of diagonal entries of $X = 0$
$\Rightarrow$Sum of eigenvalues of $X = 0$
$\Rightarrow$Every eigenvalues of X are zero
$\Rightarrow X$ is nil potent matrix, which is a contradiction, since X is symmetric.
Therefore, sum of diagonal entries is greater than 0. Thus there exist atleast one non-zero diagonal entry. Then by Lemma 2.1, A and B are symmetric. □

Corollary 2.1. *If $X = A \otimes B \neq 0 \in M_n(M_m(F))$ is infinitely divisible, then A and B are symmetric.*

Proof. Clearly, $X = A \otimes B \neq 0$ being infinitely divisible implies X is positive semidefinite. Then by Lemma 2.2, A and B are symmetric. □

The subsequent theorem (Theorem 2.3) obtained offers a necessary and sufficient condition for a matrix of order 4×4 of the form $X = A \otimes B \in M_n(M_m(F))$ to be infinitely divisible.

Theorem 2.3. *$X = A \otimes B \in M_n(M_m(F))$ such that $mn = 4$ is an infinitely divisible matrix if and only if either A and B are infinitely divisible or $-A$ and $-B$ are infinitely divisible.*

Proof. Let $X = [x_{ij}] = A \otimes B \in M_n(M_m(F))$ be infinitely divisible. Clearly, by Corollary 2.1, A and B are symmetric and $x_{ij} \geq 0$.
Therefore, entries of A and B are either all non-negative or all non-positive. We split the proof into three cases based on orders of A and B.
Case 1: If $A \in M_1(F)$ and $B \in M_4(F)$, then $X = B$, nothing to prove.
Case 2: If $A \in M_4(F)$ and $B \in M_1(F)$, then $X = A$, nothing to prove.
Case 3: Let $A, B \in M_2(F)$
Since X is infinitely divisible,the eigenvalues of $X^{or} = (A^{or} \otimes B^{or})$ is non-negative. Note that the eigenvalue of $X^{or} = (A^{or} \otimes B^{or})$ are product of eigenvalues of A^{or} and B^{or} for each r.
Now, for any r_0, some eigenvalues of A^{or} or B^{or} are positive and the rest are negative is a non-occuring case, since this will make the eigenvalues of $(A^{or} \otimes B^{or})$ negative.
For $(A^{or} \otimes B^{or})$ to be positive semidefinite $\forall\, r \geq 0$, the occuring possibilities can be classified into three cases.

Case (i): A^{or} and B^{or} are positive semidefinite $\forall\, r \geq 0$.
i.e. A and B are infinitely divisible.
Case (ii): A^{or} and B^{or} are negative semidefinite $\forall\, r \geq 0$.
$-A^{or}$ and $-B^{or}$ are positive semidefinite $\forall\, r \geq 0$.
$-A$ and $-B$ are infinitely divisible.
Case (iii): A^{or} and B^{or} are positive semidefinite for some r_0 and negative semidefinite for remaining $r \geq 0$.
Consider the case for $r = 1$.
Suppose A and B are 2×2 positive semidefinite matrices.
$\Rightarrow A$ and B are infinitely divisible.
Similarly, if A and B are 2×2 negative semidefinite matrices.
$\Rightarrow -A$ and $-B$ are 2×2 positive semidefinite matrices.
$\Rightarrow -A$ and $-B$ are infinitely divisible.

Therefore, from Case (1), (2) and (3), we can conclude that X is infinitely divisible only when A and B are infinitely divisible or $-A$ and $-B$ are infinitely divisible.
Conversely, if A and B are infinitely divisible $\Rightarrow A \otimes B$ is infinitely divisible, by Theorem 2.2.
Or if $-A$ and $-B$ are infinitely divisible $\Rightarrow -A \otimes -B = A \otimes B$ is infinitely divisible, by Theorem 2.2. □

The following theorem (Theorem 2.4) obtained is a generalization of the preceding theorem.

Theorem 2.4. *Let $X = A \otimes B \in M_n(M_m(F))$ with strictly positive entries is infinitely divisible matrix if and only if either A and B are infinitely divisible with strictly positive entries or $-A$ and $-B$ are infinitely divisible with strictly positive entries.*

Proof. The method of proving this theorem is very much similar to the previous theorem but with more cases and possibilities.
Let $X = [x_{ij}] = A \otimes B \in M_n(M_m(F))$ be infinitely divisible with $x_{ij} > 0$. Clearly, by Corollary 2.1, A and B are symmetric and entries of A and B are either all positive or negative. We split the proof into three cases based on orders of A and B.
Case 1: If $A \in M_1(F)$ and $B \in M_m(F)$ then $X = B$, nothing to prove.
Case 2: If $A \in M_n(F)$ and $B \in M_1(F)$ then $X = A$, nothing to prove.
Case 3: Let $A \in M_n(F)$ and $B \in M_m(F)$ where $m, n \neq 1$
Since X is infinitely divisible, the eigenvalues of $X^{or} = (A^{or} \otimes B^{or})$ is non-negative. As in Theorem 2.3, for $(A^{or} \otimes B^{or})$ to be positive semidefinite $\forall\, r \geq 0$, the occuring possibilities can be classified into three cases.

Case (i): A^{or} and B^{or} are all positive semidefinite $\forall\, r \geq 0$.
i.e. A and B are infinitely divisible.
Case (ii): A^{or} and B^{or} are negative semidefinite $\forall\, r \geq 0$.
$-(A^{or})$ and $-(B^{or})$ are positive semidefinite $\forall\, r \geq 0$.
In particular, for $r = 1$, i.e. $-A$ and $-B$ are positive semidefinite.
Here again we consider two cases:
Case (a): All the entries of $-A$ and $-B$ are strictly positive.
$-A$ and $-B$ are infinitely divisible.
Case (b): All the entries of $-A$ and $-B$ are strictly negative.
$\Rightarrow Trace(-A) < 0$.
But $Trace\,(-A) = Sum\,of\,eigenvalues\,of\,A \geq 0$,
which is a contradiction.

Similar conclusion can be obtained for B.
Thus, *Case (b)* is not possible.

Case (iii): A^{or} and B^{or} are positive semidefinite for some r_0 and negative semidefinite for remaining $r \geq 0$. Here again, we consider two cases.

Case (a): Suppose entries of A and B are all positive. Clearly, A^{or_0} and B^{or_0} are infinitely divisible. Therefore, $A^{om} = (A^{r_0})^{o\frac{m}{r_0}}$ is positive semidefinite $\forall\, m \geq 0$. Thus A is infinitely divisible.
Similarly, B is also infinitely divisible.
$\Rightarrow A$ and B are infinitely divisible.

Case (b): Suppose entries of A and B are all negative.
Considering the case for $r = 1$.
Again, it is required to consider two cases.

Case (b1): A and B are negative semidefinite.
$-A$ and $-B$ are positive semidefinite with all entries of $-A$ and $-B$ all positive.
$\Rightarrow -A$ and $-B$ are infinitely divisible.

Case (b2): A and B are positive semidefinite.
But $Case(b2)$ is not possible.
Proof similar to $Case(3(ii(b)))$ of this theorem.

Therefore, from Case (1), (2) and (3), we can conclude that X is infinitely divisible with positive entries only when A and B are infinitely divisible with positive entries or $-A$ and $-B$ are infinitely divisible with positive entries.
Converse part of the theorem is similar to the proof of Theorem 2.3. □

3. Decomposition of a separable matrix

In this section, we introduce the concept of separability of a matrix. As mentioned earlier the separability of a matrix is of much importance. Here, the separability of a positive semi definite matrix of the form $A \otimes B$ is proved using a decomposition techniques which we have obtained.

Definition 3.3. A matrix $X \in M_n(M_m(F))$ is said to be separable, if there exists positive semidefinite matrices $A_i \in M_n(F)$ and $B_i \in M_m(F)$ such that $X = \sum_{i=1}^{k} A_i \otimes B_i$.

Notice from the above form of separable matrix[1] that it is always positive semidefinite. Combining this with the result that positive semidefinite matrices with positive entries are infinitely divisible [4], we can conclude that *separable matrix with positive entries is always infinitely divisible.*

One of the important existing result on separability is the Peres-Horodecki separability criterion for density matrices which is a necessary condition for separability[16], stating that the matrix (A^{T_p}) obtained by partial transposition of A is positive semi-definite. The following is an illustration of judging the separability of 2-qubit family of Werner states, $\rho = q|\psi^-\rangle\langle\psi^-| + \frac{(1-q)}{4} I_{(4)}$. In [3], Azuma and Ban has given the density matrix for the 2-qubit family of Werner states. It has a single real parameter and varies from inseparable state to separable state according to the value of its parameter. Using Peres-Horodecki criterion, we can fix the critical point of the parameter between the separable and inseparable states. Its density matrix is $W(q) = \frac{1}{4}\begin{bmatrix} 1-q & 0 & 0 & 0 \\ 0 & 1+q & -2q & 0 \\ 0 & -2q & 1+q & 0 \\ 0 & 0 & 0 & 1-q \end{bmatrix}$.

Its partial transposition is $\tilde{W}(q) = \frac{1}{4}\begin{bmatrix} 1-q & 0 & 0 & 0 \\ 0 & 1+q & -2q & 0 \\ 0 & -2q & 1+q & 0 \\ 0 & 0 & 0 & 1-q \end{bmatrix}$. The least eigen value of $\tilde{W}(q)$ is $\frac{(1-3q)}{4}$. Hence, $W(q)$ is inseparable for $\frac{1}{3} < q \leq 1$. If inseparable, then it is called entangled.

It is interesting to see that every infinitely divisible matrix, X of the the form $A \otimes B$ satisfy Peres-Horodecki criterion.

The next two results obtained (Theorem 3.5 and Corollary 3.2) prove the separability of such a matrix along with its decomposition to the form $\sum_{i=1}^{k} A_i \otimes B_i$.

Theorem 3.5. *A positive semidefinite matrix $X = A \otimes B \in M_n(M_m(R))$ is always separable and also X can be decomposed as $X = \sum_{i=1}^{k} A_i \otimes B_i$, where A_i and B_i are positive semidefinite rank one matrices.*

Proof. Suppose $X = A \otimes B \in M_n(M_m(R))$ is positive semidefinite implying that A and B are symmetric.Also, A and B are both either positive semidefinite or negative semidefinite.

Case 1: A and B are both positive semidefinite. A being diagonalizable A can be written as $A = PDP^{-1} = PDP^T$, where D is a diagonal matrix with eigenvalues $\lambda_i's$ of A as diagonal entries and P is a orthogonal matrix.

Now, $D = D_1 + D_2 + \cdots + D_n$, where D_i is a diagonal matrix with non-zero element only at $(i,i)^{th}$ position and all other entries are zero.

Therefore, $A = PDP^T = PD_1P^T + PD_2P^T + \cdots + PD_nP^T$.

Let $PD_iP^T = A_i$, where $i = 1, 2, \cdots, n$. We will prove each A_i is positive semidefinite.

Consider $A_1 = PD_1P^T$.

Taking $P = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}$ and $D_1 = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$

We will have $A_1 = \lambda_1 \begin{bmatrix} x_{11}^2 & x_{11}x_{21} & \cdots & x_{11}x_{n1} \\ x_{21}x_{11} & x_{21}^2 & \cdots & x_{21}x_{n1} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1}x_{11} & x_{n1}x_{21} & \cdots & x_{n1}^2 \end{bmatrix}$ = $\lambda_1 P_1$(say), where $\lambda_1 \geq 0$.

Clearly, every principal minors of P_1 is non-negative making P_1 positive semidefinite. Therefore A_1 is also positive semidefinite. Similarly, every A_i is positive semidefinite, for $i = 1, 2, \cdots, n$.

Therefore, $A = A_1 + A_2 + \cdots + A_n$, where each A_i is positive semidefinite.

Similarly, $B = B_1 + B_2 + \cdots + B_m$, where each B_i is positive semidefinite.

$$\begin{aligned}X &= A \otimes B\\ &= (A_1 + \cdots + A_n) \otimes (B_1 + \cdots + B_n)\\ &= (A_1 \otimes (B_1 + \cdots + B_m)) + \cdots + (A_n \otimes (B_1 + \cdots + B_m))\\ &= (A_1 \otimes B_1 + \cdots + A_1 \otimes B_m) + \cdots + (A_n \otimes B_1 + \cdots + A_n \otimes B_m)\\ &= \sum_{i=1}^{mn} A_i^0 \otimes B_i^0, \text{where } A_i^0 \ \& \ B_i^0 \text{ are positive semidefinite matrices for } i = 1, 2, \ldots, mn\end{aligned}$$

Case 2: A and B are negative semidefinite.
Clearly, $-A$ and $-B$ are positive semidefinite. Proving as in *Case 1*,
$-A = A_1^{'} + A_2^{'} + \cdots + A_{\bar{n}}^{'}$ and $-B = B_1^{'} + B_2^{'} + \cdots + B_{\bar{m}}^{'}$, where $A_i^{'}$'s and $B_j^{'}$'s are positive semidefinite matrices for $i = 1, 2, \ldots, \bar{n}$ and $j = 1, 2, \ldots, \bar{m}$.
Now, $X = A \otimes B = (-A \otimes -B) = \sum_{i=1}^{mn} A_i^{00} \otimes B_i^{00}$,where A_i^{00} & B_i^{00} are positive semidefinite matrices for $i = 1, 2, \ldots, \bar{m}\bar{n}$

Therefore, in both cases X is separable. □

Corollary 3.2. *An infinitely divisible matrix $X = A \otimes B \in M_n(M_m(R))$ is always separable and also X can be decomposed as $X = \sum_{i=1}^{k} A_i \otimes B_i$, where A_i and B_i are positive semidefinite rank one matrices.*

Proof. An infinitely divisible matrix is always positive semidefinite. Then by Theorem 3.5, $X = A \otimes B$ is always separable and hence X can be decomposed as $X = \sum_{i=1}^{k} A_i \otimes B_i$, where A_i and B_i are positive semidefinite rank one matrices. □

4. Maps preserving separable matrices and infinitely divisible matrices

The aim of this section is to preserve the classes of separable matrices and a sub-class of infinitely divisible matrices which is also separable.
The following theorem obtained describes the characterization of a preserving map which preserves the class of separable matrices.

Theorem 4.6. *Let $X \in M_n(M_m(C))$ be separable. Then $\phi : M_n(M_m(C)) \to M_n(M_m(C))$ preserves separability if and only if $\phi(X) = M^* X M$ where $M = P \otimes Q$, where $P \in M_n(C)$ and $Q \in M_m(C)$.*

Proof. Let $X \in M_n(M_m(C))$ be separable. Therefore, $X = \sum_{i=1}^{k} A_i \otimes B_i$, where A_i's and B_i's are positive semidefinite matrices. Then

$$\begin{aligned}
\phi(X) &= M^* X M \\
&= M^*(\sum_{i=1}^{k} A_i \otimes B_i)M \\
&= (P \otimes Q)^*[(A_1 \otimes B_1) + \cdots + (A_k \otimes B_k)](P \otimes Q) \\
&= (P^* \otimes Q^*)(A_1 \otimes B_1)(P \otimes Q) + \cdots + (P^* \otimes Q^*)(A_k \otimes B_k)(P \otimes Q) \\
&= P^* A_1 P \otimes Q^* B_1 Q + \cdots + P^* A_k P \otimes Q^* B_k Q \\
&= \sum_{i=1}^{k} P^* A_i P \otimes Q^* B_i Q
\end{aligned}$$

Here, $P^* A_i P$'s and $Q^* B_i Q$'s are positive semidefinite, since A_i's and B_i's are always positive semidefinite, for $i = 1, 2, \ldots k$. This makes $\phi(X)$ separable. Similar proof can be obtained for $\phi(X) = M^* X^T M$. Thus ϕ preserves separability. Conversely, suppose $\phi : M_n(M_m(C)) \to M_n(M_m(C))$ preserves separability.
i.e., $\phi(X) = X'$, where X and X' are separable. Therefore $\exists$ positive semidefinite matrices A_i and B_i such that $X = \sum_{i=1}^{k} A_i \otimes B_i$. Therefore,

$$\begin{aligned}
\phi(X) &= \phi(\sum_{i=1}^{k} A_i \otimes B_i) \\
&= \phi(A_1 \otimes B_1 + \cdots + A_n \otimes B_n) \\
&= \phi(A_1 \otimes B_1) + \cdots + \phi(A_n \otimes B_n) \\
&= Q_1 + \cdots + Q_n
\end{aligned}$$

$Q_1 \ldots Q_n$ are all are all separable: since $A_i \otimes B_i$ are separable and ϕ preserves separability. So, in general $Q_i = \sum_{j=1}^{\alpha_i} C_{ij} \otimes D_{ij}$, where C_{ij} and D_{ij} are rank 1 matrices and α_i is the number of elements in the summation of separability which are decomposed as rank 1 matrices.
Therefore, $A_i \otimes B_i$ is a positive semidefinite matrix taken to another positive semidefinite matrix $C_{i1} \otimes D_{i1} + \cdots + C_{i\alpha_i} \otimes D_{i\alpha_i}$. Since C_{ij} and D_{ij} are rank one matrices the summation can be written as $C_i \otimes D_i$.
i.e., $\phi(A_i \otimes B_i) = C_i \otimes D_i$, where A_i, B_i, C_i, D_i are all positive semidefinite. In short we can say positive semidefinite A_i is taken to another positive semidefinite C_i and a positive semidefinite B_i is taken to another positive semidefinite D_i.
Thus, there exist P and Q such that $C_i = P^* A_i P$ and $D_i = Q^* B_i Q$

Therefore,

$$\begin{aligned}
\phi(X) &= Q_1 + \cdots + Q_n \\
&= C_1 \otimes D_1 + \cdots + C_n \otimes D_n \\
&= P^* A_1 P \otimes Q^* B_1 Q + \cdots + P^* A_n P \otimes Q^* B_n Q \\
&= (P^* \otimes Q^*)(A_1 \otimes B_1)(P \otimes Q) + \cdots + (P^* \otimes Q^*)(A_n \otimes B_n)(P \otimes Q) \\
&= (P \otimes Q)^*[(A_1 \otimes B_1) + \cdots + (A_n \otimes B_n)](P \otimes Q) \\
&= (P \otimes Q)^*(\sum_{i=1}^{k} A_i \otimes B_i)(P \otimes Q) \\
&= M^* X M
\end{aligned}$$

Thus, $\phi(X) = M^*XM$, where $P \otimes Q$, is the preserving map. □

Combining Theorem 4.6 with Schur triangularization theorem, there always exist a unitary matrix U which is likely to be of the form $U = P \otimes Q$ such that $\phi(X) = U^*XU$ or $\phi(X) = U^*X^TU$ preserves separability and is a diagonal separable matrix.

The next theorem proved here gives a class of matrices which is both separable and infinitely divisible.

Theorem 4.7. *Suppose* $X = \sum_{i=1}^{k} A_i \otimes B_i \in M_n(M_m(R))$ *is separable with* $x_{ij} \geq 0$ *and* A_i*'s and* $B_i's$ *being infinitely divisible. Then* X *is infinitely divisible.*

Proof. Since A_i and B_i are infinitely divisible, $A_i \otimes B_i$ is infinitely divisible.
By Theorem 2.1, $X = \sum_{i=1}^{k} A_i \otimes B_i$ is infinitely divisible. □

A linear map ϕ defined $\phi : M_n(M_m(R)) \to M_n(M_m(R))$ as $\phi(X) = Q^TXQ$ or $\phi(X) = Q^TX^TQ$ where Q is an orthogonal matrix of the form $Q = A \otimes B$ and the column of Q contains an eigen vector of X, maps separable and infinitely divisible matrix X to a diagonal matrix which is again separable and infinitely divisible.

From now on, the class of matrices in Theorem 4.7 will be denoted as ρ_{inf}. The matrix S constructed in Illustration 2.1 is an element in ρ_{inf}. S is infinitely divisible as well as separable and all the matrix involved in the decomposed form is also infinitely divisible. The next theorem acquired yields the characterization of a map preserving ρ_{inf}.

Theorem 4.8. *Let* $X \in \rho_{inf}$ *and* $\phi : M_n(M_m(R)) \to M_n(M_m(R))$ *preserves* ρ_{inf} *if and only if* $\phi(X) = M^TXM$ *where* $M = P \otimes Q$*, where* $P \in M_n(R)$ *and* $Q \in M_m(R)$ *and entries of* $P \geq 0$ *or entries of* $P \leq 0$ *and entries of* $Q \geq 0$ *or entries of* $Q \leq 0$*.*

Proof. Assuming $M = P \otimes Q$ and by the properties of P and Q mentioned in the theorem, entries of M is always non-negative which provides entries of $\phi(X)$ to be always non-negative. Let $X \in \rho_{inf}$. Since X is separable,

$$\begin{aligned} X &= \sum_{i=1}^{k} A_i \otimes B_i \\ \phi(X) &= M^T X M \\ &= (P^T \otimes Q^T)(\sum_{i=1}^{k} A_i \otimes B_i)(P \otimes Q) \\ &= P^T A_1 P \otimes Q^T B_1 Q + \cdots + P^T A_k P \otimes Q^T B_k Q \end{aligned}$$

By Theorem 4.6, $\phi(X)$ is always separable. Also, $P^T A_i P$ and $Q^T B_i Q$ are Gram matrices which are always positive semidefinite. Clearly, both are infinitely divisible as well. This implies $\phi(X)$ to be infinitely divisible, by Theorem 2.1 and Theorem 2.2. Thus $\phi(X) \in \rho_{inf}$. Therefore, ϕ preserves the matrix class ρ_{inf}. Conversely, suppose ϕ preserves ρ_{inf}. By Theorem 4.6,

$$\begin{aligned} \phi(X) &= M^* X M, \quad \textit{since separability preserving} \\ &= (P \otimes Q)^* X (P \otimes Q) \\ &= \sum_{i=1}^{k} A_i^{'} \otimes B_i^{'} \end{aligned}$$

where, $A_i^{'}$ and $B_i^{'}$ are infinitely divisible, since $\phi(X) \in \rho_{inf}$. Clearly, entries of $A_i^{'}$ and $B_i^{'}$ are ≥ 0, which implies $entries\ of\ P \geq 0$ or $entries\ of\ Q \geq 0$ and $entries\ of\ P \geq 0$ or $entries\ of\ Q \geq 0$. □

5. Concluding remarks

In this paper, we have seen some properties of classes of infinitely divisible matrices and separable matrices which further led to the results on relations between these classes of matrices. Also the main results were focused on the preserving maps on classes of infinitely divisible and separable matrices. The study continues with more preserving maps and determination of different forms of separable matrix which helps in the separability of a quantum system in quantum information theory.

References

[1] Alfsen, E.; Shultz, F. Finding decompositions of a class of separable states. *Linear Algebra Appl.* **437** (2012), no. 10, 2613-2629.

[2] Arumugasamy, C.; Jayaraman, S. Linear Maps on $M_n(R)$ Preserving Schur Stable Matrices. *Bull. Kerala Math. Assoc.* **16**, (2018), no. 1, 127-137.

[3] Azuma, H.; Ban, M. A convex combination of product states for the separable Werner state. *arXiv preprint quant-ph/0601044* (2008).

[4] Bhatia, R. Infinitely divisible Matrices. *Amer. Math. Monthly* **113** (2006), no. 3, 221-235.

[5] Bhatia, R.; Kosaki, H. Mean matrices and infinite divisibility. *Linear Algebra Appl.* **424** (2007), no. 1, 36–54.

[6] FitzGerald, C. H.; Horn, R. A. On fractional Hadamard powers of positive definite matrices. *J. Math. Anal. Appl.* **61** (1977), no. 3, 633-642.

[7] Hiai, F. Monotonicity for entrywise functions of matrices. *Linear Algebra Appl.* **431** (2009), no. 8, 1125-1146.

[8] Horn, R. A. The Theory of Infinitely Divisible Matrices and Kernels. *Trans. Amer. Math. Soc.* **136** (1969), 269-286.

[9] Horn, R. A.; Johnson, C. R. *Topics in Matrix Analysis*. Cambridge University Press, New York, 1991.

[10] Jain, T. Hadamard powers of some positive matrices. *Linear Algebra Appl.* **528** (2017), 147-158.

[11] Li, C. K.; Tsing, N. K. Linear Preserver Problem: A Brief Introduction and Some Special Techniques. *Linear Algebra Appl.*, **162** (1992), 217-235.

[12] McWeeny, R. Separability of quantum systems: A density matrix approach. *Adv. Quantum Chem.*, **31**, (1998), 15-35.

[13] Nie, J.; Zhang, X. Positive maps and separable matrices. *SIAM J. Optim.*, **26** (2016), no. 2, 1236-1256.

[14] Panwar, V. S.; Reddy, A. S. Positivity of Hadamard powers of a few band matrices. *Electron. J. Linear Algebra*, **38** (2022), 85-90.

[15] Šemrl, P. Commutativity Preserving Maps. *Linear Algebra Appl.*, **429** (2008), no. 5-6, 1051-1070.

[16] Simon, R. Peres-Horodecki separability criterion for continuous variable systems. *Phys. Rev. Lett.*, **84** (2000), no. 12, 2726.

[17] Xu, J.; Zheng,B.; Fosner,A. Linear maps preserving tensor product of rank-one Hermitian matrices. *J. Aust. Math. Soc.*, **98** (2015), no. 3, 407–428.

[1] Department of Mathematics
College of Engineering Trivandrum
Affiliated to APJ Abdul Kalam Technological University
Thiruvananthapuram, Kerala, India - 695016.
Email address: indu051993@gmail.com

[2] Department of Mathematics
Mar Ivanios College (Autonomous)
Affiliated to University of Kerala
Thiruvananthapuram, Kerala, India - 695015.
Email address: jill.mathew@mic.ac.in