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# On Minimum Second neighborhood Degree Energy of Graphs

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ABSTRACT. In this paper we establish the second neighborhood degree polynomials of some graphs along with their energy and also, we obtain some bounds for the spectral radius and second neighborhood energy of graphs.

## 1. INTRODUCTION

Spectral graph theory has a deep and rich history due to its relevance in studying graph matrices using matrix theory and linear algebra. With its expeditious development, spectral graph theory has captured the interest of both pure and applied mathematicians whose concerns lie far from spectral graph theory, which may be surprising due to the fact that graph energy is a special kind of matrix norm. It was found that the concept of graph energy appears in a variety of apparently unassociated areas of their own insight. As astronomers examine stellar spectra to investigate the makeup of remote celestial bodies, one of the primary goals of graph theory is to figure out the principal characteristics and structure of a graph from its graph spectrum. The eigenvalues are strongly associated with almost all significant invariants of a graph, connecting one extremal property to another, and they play a crucial role in the fundamental understanding of graphs. Graph energy is an invariant that is associated with total $\pi-$electron energy in a molecular graph. It is defined by Gutman [8] in 1978 as the sum of absolute values of the eigenvalues of the associated adjacency matrix of a graph $G$. In literature, several matrices and associated polynomials for graphs, particularly the Laplacian matrix [14], signless Laplacian matrix [5], [17], degree sum matrix [20], Seidel matrix[4], Seidel Laplacian matrix [19], Seidel signless Laplacian matrix [18], etc were investigated.

In addition, numerous matrices are connected with chemical graphs, and these matrices play a vital role in chemical graph theory. An atomic graph is a graph where atoms in a molecule are represented as vertices and the chemical bonds between them are represented as edges. This graph provides a structural representation of a molecule. One of the important matrix in chemical graph theory is called the electronic connectivity matrix. It can serve as a basis for the construction of several topological indices, like the ZEP index [2], the RZ index [2], etc.

On the other hand, the neighborhood degree [1] of a vertex $v_i$ in $G$ is the sum of all degrees of its neighbors(first neighbors), defined as

$$\delta(v_i) = \sum_{u \in N(v_i)} d(u). \tag{1.1}$$

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Also, a number of energies and topological indices using the neighborhood degree were defined, such as neighborhood first and second Zagreb index [1], neighborhood Forgotten index [1], etc. Motivated by these works, we define the second neighborhood degree of a vertex $v_i$ in a graph $G$, represented by $\delta_2(v_i)$. It is defined as

$$\delta_2(v_i) = \sum_{u \in N(N(v_i))} d(u), \tag{1.2}$$

where $N(N(v_i)) = \{u \in V(G) : \ u \, is \, adjacent \, to \, N(v_i)\}$. The second neighborhood degree is a useful measure of local connectivity and can provide insights into the structural properties of the graph. For example, vertices with a high second neighborhood degree tend to be located in densely connected regions of the graph, while vertices with a low second neighborhood degree may be located in more isolated or sparsely connected regions. The second neighborhood degree has applications in fields such as network analysis, social network analysis, biological network analysis, etc. Furthermore, the relationship between ZEP index and the second neighborhood in graph theory lies in the way that, ZEP index counts and quantifies specific topological features in a molecular graph. ZEP index focus on 2-ECs that relates to the concept of pairs of atoms that are two edges apart, corresponding to the second neighborhood in graph theory. This relationship is essential for characterising and quantifying certain structural aspects of molecules for various chemoinformatics and computational chemistry applications.

Let $\delta_2(v_1) \leq \delta_2(v_2) \leq \cdots \leq \delta_2(v_n)$ be the second neighborhood degree sequence of the graph $G$. We define an $n \times n$ real symmetric matrix called the minimum second neighborhood degree matrix denoted by $M_S(G) = [m_{ij}]$, where

$$m_{ij} = \begin{cases} \min\{\delta_2(v_i), \ \delta_2(v_j)\}, \ if \ i \neq j \ , \ v_i \ and \ v_j \ are \ adjacent, \\ 0, \ otherwise. \end{cases} \tag{1.3}$$

Also, define the second neighborhood degree polynomial $P_{SND}(G; \ \mu)$ as

$$P_{SND}(G; \ \mu) = |\mu I - M_S(G)|. \tag{1.4}$$

Since $M_s(G)$ is a real symmetric matrix, all the eigenvalues of $M_S(G)$ are real. Thus we can arrange the eigenvalues as, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The largest non-negative eigenvalue $\mu_1$ of $M_S(G)$ is known as the spectral radius of $M_S(G)$. Define $E_{SND}(G)$ to be the minimum second neighborhood degree energy as the sum of the absolute values of the eigenvalues of the matrix $M_S(G)$, that is

$$E_{SND}(G) = \sum_{i=1}^{n} |\mu_i|. \tag{1.5}$$

The motivation of studying $E_{SND}(G)$ of a graph $G$ comes from the energy of the graph $G$, which is denoted by $\mathcal{E}(G)$, a notion introduced by Gutman [8]. We must bear in mind that the energy of a graph $G$ is the sum of the absolute values of the eigenvalues of the adjacency matrix $A(G)$. Moreover, it is noteworthy that the graph energy $\mathcal{E}(G)$ can be derived, if $E_{SND}(G)$ is known for some classes of graphs. For example, $\mathcal{E}(K_n) = \dfrac{E_{SND}(K_n)}{(n-1)^2}$, where $K_n$ is the complete graph on $n > 2$ vertices, and $\mathcal{E}(K_{m,\,n}) = \dfrac{E_{SND}(K_{m,\,n})}{n(m-1)}$, where $K_{m,\,n}$ is a complete bipartite graph with $1 < m \leq n$. Furthermore, for the cycle $C_n, n \geq 3$, we have $\mathcal{E}(C_n) = \dfrac{E_{SND}(C_n)}{2}$.

In this paper, we evaluated the characteristic polynomial of the minimum second neighborhood degree matrix and obtained some bounds for the largest minimum degree eigenvalue and minimum second neighborhood degree energy. This paper is structured as follows: First, we study the second neighborhood degree polynomial of graphs obtained by some graph operations. Then we obtain some bounds for the largest eigenvalue of the minimum second neighborhood degree matrix and the minimum second neighborhood energy of a graph $G$.

## 2. Preliminaries

A graph $G$ consists of a non-empty finite set of $n$ vertices known as the vertex set $V(G)$ and another prescribed set of $m$ pairs of distinct members of $V(G)$ known as the edge set $E(G)$. Two vertices are said to be adjacent if they share a common edge, and that edge is said to be incident to those vertices. Two edges are said to be adjacent if they have a common end vertex. Throughout this paper, we only consider simple connected graph with $n$ vertices and $m$ edges. Denote $d_i$ for the degree of a vertex $v_i$, which is the number of edges incident to it in $G$. The graph $G$ is said to be $r$-regular if and only if $d_i = r$ for every vertex $v_i$ in $G$. We refer to [3] for any graph theory keywords that are undefined. The following results are relevant to the next sections:

**Lemma 2.1.** [15] *Let $a_i$ and $b_i$ be non-negative real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{n^2}{4}\left(M_1 M_2 - m_1 m_2\right),$$

*where $M_1 = \max\{a_i\},\ M_2 = \max\{b_i\},\ m_1 = \min\{a_i\}$ and $m_2 = \min\{b_i\}$.*

**Lemma 2.2.** [7] *Let $a_i$ and $b_i$ be non-negative real numbers, then*

$$\sum_{i=1}^{n} b_i^2 + pP \sum_{i=1}^{n} a_i^2 \leq (p+P) \sum_{i=1}^{n} a_i b_i,$$

*where $p$ and $P$ are real constants such that $pa_i \leq b_i \leq Pa_i$ for all $1 \leq i \leq n$.*

**Lemma 2.3.** [13] *Let $B$ be a symmetric matrix of order $n$ and $B_k$ be its $k \times k$ submatrix. Then, for any integer $1 \leq i \leq n$,*

$$\rho_{n-k+i}(B) \leq \rho_i(B_k) \leq \rho_i(B),$$

*where $\rho_i(B),\ \rho_i(B_k)$ are the $i^{th}$ largest eigenvalues of $B$ and $B_k$, respectively.*

**Lemma 2.4.** [16] *For positive numbers $x_1,\ x_2,\ \cdots,\ x_n$,*

$$\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} \leq \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}.$$

**Lemma 2.5.** [6] *For real numbers $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$, then*

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \leq n \sum_{i=1}^{n} a_i b_i.$$

**Lemma 2.6.** [16] *For non-negative numbers $x_1,\ x_2,\ \cdots,\ x_n$ and $k \geq 2$*

$$\sum_{i=1}^{n} (x_i)^k \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{k}{2}}.$$

## 3. MAIN RESULTS

In this section, we determine the minimum second neighborhood degree polynomial of graphs obtained from various graph operations. We then obtain some bounds for the spectral radius $\mu_1$ of $M_s(G)$ and minimum second neighborhood energy $E_{SND}(G)$ of a graph $G$.

### 3.1. Minimum Second neighborhood Degree Polynomial of Graphs Obtained by Graph Operations.

In this section, we compute the minimum second neighborhood degree polynomial of graphs obtained from various graph operations. We begin this section with the following proposition, which is quite useful for evaluating the second neighborhood degree polynomials of regular graphs.

**Proposition 3.1.** *Let $G$ be an $r-$regular graph of order $n$, then*

$$\begin{aligned} P_{SND}(G;\ \mu) &= \left[\mu - r^2(r-1)(n-1)\right]\left[\mu + r^2(r-1)\right]^{(n-1)},\ \textit{and} \\ E_{SND}(G) &= 2r^2(r-1)(n-1). \end{aligned}$$

*Proof.* It is easy to see that $M_S(G) = r^2(r-1)J - r^2(r-1)I$, where $J$ is an $n \times n$ matrix of all ones and $I$ is the identity matrix of order $n$. Therefore, we have

$$P_{SND}(G;\ \mu) = \left[\mu - r^2(r-1)(n-1)\right]\left[\mu + r^2(r-1)\right]^{(n-1)}.$$

Thus we obtain $E_{SND}(G) = 2r^2(r-1)(n-1)$. □

**Example 3.1.** The cycle $C_n$ is a two-regular graph of $n$ vertices. Then, by Proposition 3.1, we have $P_{SND}(G;\ \mu) = [\mu - 4(n-1)]\,[\mu + 4]^{(n-1)}$. Therefore, $E_{SND}(C_n) = 8(n-1)$.

**Theorem 3.1.** *For every integer $n \geq 3$, there exists an $r-$regular graph $G$ such that $|G| > n$ and which satisfies $\dfrac{E_{SND}(G)}{\mathcal{E}(G)^3} = n$.*

*Proof.* Let $n \geq 3$ be the given integer. Now set $p = 2(2n+1)$ and let $G$ be $K_p$, the complete graph on $p$ vertices. Then, by Proposition 3.1, we get

$$P_{SND}(G;\ \mu) = \left[\mu - (p-1)^3(p-2)\right]\left[\mu + (p-1)^2(p-2)\right]^{(p-1)},$$

which implies, $E_{SND}(K_p) = 2(p-2)(p-1)^3$. Moreover, observe that $\mathcal{E}(K_p) = 2(p-1)$, thus we obtain $\dfrac{E_{SND}(K_p)}{\mathcal{E}(K_p)^3} = n$, as required. □

For the simple graph $G$, denote $L(G)$ to be the line graph of $G$, and $L^k(G)$ be the $k^{th}$ iterated line graph [21], which is obtained by the relation $L^{k+1}(G) = L(L^k(G)),\ k \geq 1$.

**Theorem 3.2.** *Let $G$ be an $r-$regular graph of order $n$ and $n_k$ be the order of the $k^{th}$ iterated line graph $L^k(G)$ $(k \geq 1)$. Then,*

$$\begin{aligned} P_{SND}(L^k(G);\ \mu) &= \left[\mu - (n_k-1)(t^2+3t+2)\right]\left[\mu + (t^2+3t+2)\right]^{(n_k-1)}\ \textit{and} \\ E_{SND}(L^k(G)) &= 2(n_k-1)(t^2+3t+2),\ \textit{where}\ t = 2^k(r-2). \end{aligned}$$

*Proof.* Since $G$ is $r-$regular, the $k^{th}$ iterated line graph $L^k(G)$ of order $n_k$ is $(t+2)-$regular, where $t = 2^k(r-2)$. Hence the theorem follows from Proposition 3.1.

**Lemma 3.7.** [17] *If $a,\ b,\ c$ and $d$ are real numbers, then the determinant of the form*

$$\begin{vmatrix} (x+a)I_{n_1} - aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (x+b)I_{n_2} - aJ_{n_2} \end{vmatrix}$$

*can be expressed in a simplified form as:*

$$(x+a)^{(n_1-1)}(x+b)^{(n_2-1)}\left[(x-a(n_1-1))(x-b(n_2-1)) - n_1n_2cd\right].$$

The subdivision graph $S(G)$ [9] of a graph $G$ is the graph obtained by inserting a new vertex on each edge of $G$.

**Theorem 3.3.** *Let $G$ be an $r-$regular graph of order $n$ and size $m$. Let $SD(G)$ be the subdivision graph of $G$, then*

$$P_{SND}(SD(G);\ \mu) = \begin{cases} (\mu+r^2)^{(n-1)}(\mu+4r)^{(m-1)}\left[(\mu-r^2(n-1))(\mu-4r(m-1)) - 16mnr^2\right], r \geq 4, \\ \\ (\mu+r^2)^{(n-1)}(\mu+4r)^{(m-1)}\left[(\mu-r^2(n-1))(\mu-4r(m-1)) - mnr^4\right], r = 2,\ 3, \end{cases}$$

*and*

$$E_{SND}(SD(G)) = \begin{cases} r\left[r(n-1) + 4(m-1) + \sqrt{8m(2m-p)+q^2}\right] + 2s,\ r \geq 4, \\ \\ r\left[r(n-1) + 4(m-1) + \sqrt{4m(4m+t)+q^2}\right] + 2s,\ r = 2,\ 3, \end{cases}$$

*where $p = n(r-8)-r+4,\ q = r(n-1)+4,\ s = 4(m-1)+r(n-1)$ and $t = nr^2-2r(n-1)-8$.*

□

*Proof.* Let the vertices of $G$ be $V(G) = \{v_1,\ v_2,\ \cdots,\ v_n\}$ and the vertices of $SD(G)$ be $V(SD(G)) = V(G) \cup \{u_1,\ u_2,\ \cdots,\ u_m\}$. Then observe that $\delta_2(v_i) = r^2$ for $1 \leq i \leq n$ and $\delta_2(u_j) = 4r$ for $1 \leq j \leq m$. Now consider the following cases:

**Case 1.** If $r \geq 4$.

In this case we have $\min\{\delta_2(v_i),\ \delta_2(u_i)\} = 4r$. Hence, the matrix $M_S(G)$ is given by,

$$M_S(G) = \begin{bmatrix} r^2J_{n\times n} - r^2I_{n\times n} & 4rJ_{n\times m} \\ 4rJ_{m\times n} & 4rJ_{m\times m} - 4rI_{m\times m} \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{SND}(SD(G);\ \mu) &= |\mu I - M_S(G)| \\ &= \begin{vmatrix} (\mu+r^2)I_{n\times n} - r^2J_{n\times n} & -4rJ_{n\times m} \\ -4rJ_{m\times n} & (\mu+4r)I_{m\times m} - 4rJ_{m\times m} \end{vmatrix}. \end{aligned}$$

Again, using Lemma 3.7, we have

$$P_{SND}(SD(G);\ \mu) = (\mu+r^2)^{(n-1)}(\mu+4r)^{(m-1)}\left[(\mu-r^2(n-1))(\mu-4r(m-1)) - 16mnr^2\right].$$

Thus,

$$E_{SND}(G) = r\left[r(n-1) + 4(m-1) + \sqrt{8m(2m-p)+q^2}\right] + 2s,$$

where $p = n(r-8) - r + 4,\ q = r(n-1) + 4$ and $s = 4(m-1) + r(n-1)$.

**Case 2:** If $r = 2$ or 3.
In this case we have $\min\{\delta_2(v_i),\ \delta_2(u_i)\} = r^2$. Hence, the matrix $M_S(G)$ is given by,

$$M_S(G) = \begin{bmatrix} r^2 J_{n\times n} - r^2 I_{n\times n} & r^2 J_{n\times m} \\ r^2 J_{m\times n} & 4rJ_{m\times m} - 4rI_{m\times m} \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{SND}(SD(G);\ \mu) &= |\mu I - M_S(G)| \\ &= \begin{vmatrix} (\mu + r^2)I_{n\times n} - r^2 J_{n\times n} & -r^2 J_{n\times m} \\ -r^2 J_{m\times n} & (\mu + 4r)I_{m\times m} - 4rJ_{m\times m} \end{vmatrix} \end{aligned}$$

Again, using Lemma 3.7, we have

$$P_{SND}(SD(G);\ \mu) = (\mu + r^2)^{(n-1)}(\mu + 4r)^{(m-1)} \left[(\mu - r^2(n-1))(\mu - 4r(m-1)) - mnr^4\right].$$

Thus,

$$E_{SND}(G) = r\left[r(n-1) + 4(m-1) + \sqrt{4m(4m+t) + q^2}\right] + 2s,$$

where $q = r(n-1) + 4,\ s = 4(m-1) + r(n-1)$ and $t = nr^2 - 2r(n-1) - 8$. Hence the proof. □

The semitotal point graph [20] $T_2(G)$ is a graph that is obtained from the graph $G$ by inserting a vertex corresponding to each edge of $G$ and by joining each new vertex to the end vertices of the edge corresponding to it.

**Theorem 3.4.** *Let $G$ be an $r-$regular graph order $n$ and size $m$ and let $T_2(G)$ be the semitotal point graph of $G$. Then $M_S(T_2(G)) = 2r(m+n)A(T_2(G))$, where $A(T_2(G))$ is the adjacency matrix of $T_2(G)$. Hence $E_{SND}(G) = 2r(m+n)\mathcal{E}(T_2(G))$.*

*Proof.* Let the vertices of $G$ be $V(G) = \{v_1,\ v_2,\ \cdots,\ v_n\}$ and let the vertices of $T_2(G)$ be $V(T_2(G)) = V(G) \cup \{u_1,\ u_2,\ \cdots,\ u_m\}$. Then observe that $\delta_2(v_i) = 2r(m+n)$ for $1 \le i \le n$ and $\delta_2(u_j) = 2r(m+2n)$ for $1 \le j \le m$. Therefore, we obtain

$$\min\{2r(m+n),\ 2r(m+2n)\} = 2r(m+n). \tag{3.6}$$

Thus, it is clear from (3.6), that $M_S(T_2(G)) = 2r(m+n)A(T_2(G))$, where $A(T_2(G))$ is the adjacency matrix of $T_2(G)$. Hence $E_{SND}(T_2(G)) = 2r(m+n)\mathcal{E}(T_2(G))$. □

The total graph [9] $T(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices of $T(G)$ are adjacent if and only if the corresponding elements of $G$ are adjacent or incident.

**Theorem 3.5.** *Let $G$ be an $r-$regular graph of order $n$, and $T(G)$ be the total graph of $G$. Then $P_{SND}(T(G);\ \mu) = [\mu - 2r(n-1)]\,[\mu + 2r]^{(n-1)}$ and $E_{SND}(T(G)) = 4r(n-1)$.*

*Proof.* Observe that if $G$ is an $r-$regular graph, then $T(G)$ is a $2r-$regular graph. Thus, by Proposition 3.1, we have

$$P_{SND}(T(G);\ \mu) = [\mu - 2r(n-1)]\,[\mu + 2r]^{(n-1)}.$$

Hence $E_{SND}(T(G)) = 4r(n-1)$. □

**Corollary 3.1.** *For every integer $n \ge 3$, there exists a graph $G$ with regularity $2n$ such that $E_{SND}(G)$ is a perfect square.*

*Proof.* Consider the total graph of the complete graph $K_n$ on $n$ vertices. Then it is clear from Theorem 3.5, that $E_{SND}(T(K_n)) = (2n-2)^2$. Thus the proof. □

The graph $G^+$ is a graph obtained from the graph $G$ by attaching $k \geq 1$ pendant edges to each vertex of $G$. If $G$ is a graph of order $n$ and size $m$, then $G^+$ is a graph of order $n + nk$ and size $m + nk$.

**Theorem 3.6.** *Let $G$ be an $r-$regular graph of order $n$, then $M_S(G^+) = nr(r+k)A(G^+)$, where $A(G^+)$ is the adjacency matrix of $G^+$. Hence $E_{SND}(G^+) = nr(r+k)\mathcal{E}(G^+)$.*

*Proof.* Let the vertices of $G$ be $V(G) = \{v_1,\ v_2, \cdots,\ v_n\}$ and the vertices of $G^+$ be $V(G^+) = V(G) \cup \{u_1,\ u_2, \cdots,\ u_{nk}\}$. Then observe that $\delta_2(v_i) = nr(r+k)$, for $1 \leq i \leq n$ and $\delta_2(u_j) = nrk(r+k)$, for $1 \leq j \leq nk$. Therefore, we have the following

$$\min\{nr(r+k),\ nrk(r+k)\} = nr(r+k). \tag{3.7}$$

Consequently, from (3.7), we get $M_S(G) = nr(r+k)A(G^+)$, where $A(G^+)$ is the adjacency matrix of $G^+$. Hence $E_{SND}(G^+) = nr(r+k)\mathcal{E}(G^+)$. □

The union [9] of the graphs $G$ and $H$ is a graph denoted by $G \cup H$, whose vertex set is $V(G \cup H) = V(G) \cup V(H)$ and edge set is $E(G \cup H) = E(G) \cup E(H)$.

**Theorem 3.7.** *Let $G$ be an $r_1-$regular graph of order $n_1$ and $H$ be an $r_2-$regular graph of order $n_2$. Then*

$$P_{SND}(\ G \cup H;\ \mu) = P_{SND}(G;\ \mu)P_{SND}(H,\ \mu) - n_1 n_2 k^2 (\mu + a)^{(n_1-1)}(\mu + b)^{(n_2-1)},$$

*and*

$$E_{SND}(G \cup H) = a(2n_1 - 1) + b(2n_2 - 1) + \sqrt{an_1(an_1 - 2bn_2) + n_2(4n_1k^2 + b^2)},$$

*where $a = r_1^2(r_1 - 1),\ b = r_2^2(r_2 - 1)$ and $k = \min\{a,\ b\}$.*

*Proof.* The matrix $M_S(G \cup H)$ is given by

$$M_S(G \cup H) = \begin{bmatrix} M_S(G) & kJ_{n_1 \times n_2} \\ \\ kJ_{n_2 \times n_1} & M_S(H) \end{bmatrix},$$

where $k = \min\{r_1^2(r_1 - 1),\ r_2^2(r_2 - 1)\}$. Therefore,

$$\begin{aligned} P_{SND}(G \cup H,\ \mu) &= |\mu I - M_s(G \cup H)| \\ &= \begin{vmatrix} (\mu + a)I_{n_1 \times n_1} - aJ_{n_1 \times n_1} & -kJ_{n_1 \times n_2} \\ -kJ_{n_2 \times n_1} & (\mu + b)I_{n_2 \times n_2} - bJ_{n_2 \times n_2} \end{vmatrix}, \end{aligned}$$

where $a = r_1^2(r_1 - 1)$ and $b = r_2^2(r_2 - 1)$. Now from Lemma 3.7, we have

$$P_{SND}(G \cup H,\ \mu) = (\mu + a)^{(n_1-1)}(\mu + b)^{(n_2-1)} \left[(\mu - (n_1 - 1)a)(\mu - (n_2 - 1)b) - n_1 n_2 k^2\right]. \tag{3.8}$$

Moreover, using Proposition 3.1, we can further simplify (3.8) into

$$P_{SND}(G \cup H;\ \mu) = P_{SND}(G;\ \mu)P_{SND}(H,\ \mu) - (\mu + a)^{(n_1-1)}(\mu + b)^{(n_2-1)} n_1 n_2 k^2.$$

Therefore, from (3.8), we obtain

$$E_{SND}(G \cup H) = a(2n_1 - 1) + b(2n_2 - 1) + \sqrt{an_1(an_1 - 2bn_2) + n_2(4n_1k^2 + b^2)},$$

where $a = r_1^2(r_1 - 1),\ b = r_2^2(r_2 - 1)$ and $k = \min\{a,\ b\}$ as required. □

We shall conclude this section by obtaining the minimum second neighborhood degree polynomial and minimum second neighborhood degree energy of the cross product of two graphs [9] and the join of two graphs [9].

**Theorem 3.8.** *Let $G$ be an $r_1-$regular graph of order $n_1$ and $H$ be an $r_2-$regular graph of order $n_2$. Then $P_{SND}(G \times H;\ \mu) = [\mu - R(p-1)]\,[\mu + R]^{(p-1)}$ and $E_{SND}(G \times H) = R\mathcal{E}(K_p)$, where $p = n_1 n_2$ and $R = r_1 + r_2$.*

*Proof.* Since $G$ and $H$ are $r_1$ and $r_2$ regular graphs, respectively, the cross product graph $G \times H$ is $R-$regular, where $R = r_1 + r_2$. If we let $p = n_1 n_2$, then by Proposition 3.1, we have $P_{SND}(G \times H;\ \mu) = [\mu - R(p-1)]\,[\mu + R]^{(p-1)}$. Consequently, it follows that

$$\begin{aligned} E_{SND}(G \times H) &= 2R\,[p-1] \\ &= r_1\,[2(p-1)] + r_2\,[2(p-1)] \\ &= R\mathcal{E}(K_p). \end{aligned}$$

Hence the theorem. □

**Theorem 3.9.** *Let $G$ be an $r_1-$regular graph of order $n_1$ and $H$ be an $r_2-$regular graph of order $n_2$. If $R = r_1 + r_2$, then*

$$P_{SND}(G\nabla H;\ \mu) = (\mu + a)^{(n_1-1)}\,(\mu + b)^{(n_2-1)}\,\left[(\mu - a(n_1-1))(\mu - b(n_2-1)) - n_1 n_2 c^2\right],$$

*and*

$$E_{SND}(G\nabla H) = 2a(n_1-1) + 2b(n_2-1) + \sqrt{4n_1n_2c^2 + (an_1 - bn_2 + b - a)^2},$$

*where $a = n_2(R + n_1 - 1) + r_1(r_1 - 1)$, $b = n_1(R + n_2 - 1) + r_2(r_2 - 1)$ and $c = \min\{a,\ b\}$.*

*Proof.* Consider the graph $G\nabla H$, then observe that $\delta_2(v) = a$ for $v \in V(G)$ and $\delta_2(u) = b$ for $u \in V(H)$, where $a = n_2(R + n_1 - 1) + r_1(r_1 - 1)$ and $b = n_1(R + n_2 - 1) + r_2(r_2 - 1)$. If we let $c = \min\{a,\ b\}$, then the matrix $M_S(G\nabla H)$ is given by,

$$M_S(G\nabla H) = \begin{bmatrix} aJ_{n_1\times n_1} - aI_{n_1\times n_1} & cJ_{n_1\times n_2} \\ cJ_{n_2\times n_1} & bJ_{n_2\times n_2} - bI_{n_2\times n_2} \end{bmatrix}$$

Thus,

$$\begin{aligned} P_{SND}(G\nabla H;\ \mu) &= |\mu I - M_S(G\nabla H)| \\ &= \begin{vmatrix} (\mu + a)I_{n_1\times n_1} - aJ_{n_1\times n_1} & -cJ_{n_1\times n_2} \\ -cJ_{n_2\times n_1} & (\mu + b)I_{n_2\times n_2} - bJ_{n_2\times n_2} \end{vmatrix} \end{aligned}$$

Hence by Lemma 3.7 and Proposition 3.1, we have

(3.9)
$$P_{SND}(G\nabla H;\ \mu) = (\mu + a)^{(n_1-1)}\,(\mu + b)^{(n_2-1)}\,\left[(\mu - a(n_1-1))(\mu - b(n_2-1)) - n_1 n_2 c^2\right].$$

Therefore, from (3.9), we get

$$E_{SND}(G\nabla H) = 2a(n_1-1) + 2b(n_2-1) + \sqrt{4n_1n_2c^2 + (an_1 - bn_2 + b - a)^2}.$$

Hence the theorem. □

### 3.2. **Bounds for the Spectral Radius and Second neighborhood Energy of Graph $G$.**

In this section, we obtain some bounds for the spectral radius $\mu_1$ for the second neighborhood matrix $M_S(G)$. Furthermore, we deduce some bounds for the minimum second neighborhood degree energy $E_{SND}(G)$ of a graph $G$.

**Theorem 3.10.** *Let $G$ be an $r-$regular graph of order $n$, then the spectral radius $\mu_1$ of $M_S(G)$ is $r^2(r-1)(n-1)$.*

*Proof.* The theorem follows from Proposition 3.1. □

**Theorem 3.11.** *Let $G$ be a connected graph of order $n$ and size $m$, then*

$$0 \le \mu_1 \le 2m\sqrt{\frac{n-1}{n}}. \tag{3.10}$$

*Equality holds if and only if $G$ is the disjoint union of finite number of complete graphs $K_1$.*

*Proof.* Using Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=2}^{n}\mu_i\right)^2 \leq (n-1)\sum_{i=2}^{n}(\mu_i)^2. \tag{3.11}$$

Since $Trace(M_S(G)) = 0$, it is clear that $\sum_{i=2}^{n}\mu_i = -\mu_1$. Moreover, we get

$$\begin{aligned}\sum_{i=1}^{n}\mu_i^2 &= Trace(M_S(G)^2)\\ &= \sum_{i=i}^{n}\sum_{j=1}^{n}\min\{\delta_2(v_i),\ \delta_2(v_j)\}\\ &= 2\sum_{i<j}\left[\min\{\delta_2(v_i),\ \delta_2(v_j)\}\right]^2 = \mathcal{K}\ (say)\end{aligned}$$

Therefore, (3.11), can be modified as

$$\begin{aligned}(-\mu_1)^2 &\leq (n-1)(\mathcal{K}-\mu_1^2)\\ &\leq (n-1)(4m^2-\mu_1^2).\end{aligned}$$

Since $\mathcal{K} \leq 2\left(\sum_{i=1}^{n}d(v_i)\right)^2 = 4m^2$, we have (3.11), further reduces to $0 \leq \mu_1 \leq 2m\sqrt{\frac{n-1}{n}}$.

Now assume that the equality holds in (3.10). Then all the inequalities in the proof must be equalities. That is, we have $\mathcal{K} = 4m^2$ and $\mu_1^2 = \frac{4m^2(n-1)}{n}$. Hence $G$ is the disjoint union of finite number of complete graphs $K_1$.

Conversely, if $G$ is the disjoint union of finite number of complete graphs $K_1$, then obviously equality holds in (3.10). □

**Theorem 3.12.** *Let $G$ be a connected graph of order $n$ and size $m$, then*

$$E_{SND}(G) \geq \sqrt{2n\mathcal{K} - \frac{n^2}{4}\left(|\mu_n| - |\mu_1|\right)^2}. \tag{3.12}$$

*Equality holds if and only if $G$ is the disjoint union of finite number of complete graphs $K_1$ or $K_2$.*

*Proof.* Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the eigenvalues of $M_S(G)$. Then by letting $a_i = 1$ and $b_i = |\mu_i|$ in Lemma 2.1, we get

$$\sum_{i=1}^{n}(1)\sum_{i=1}^{n}|\mu_i|^2 - \left(\sum_{i=1}^{n}\mu_i\right)^2 \leq \frac{n^2}{4}\left(|\mu_n| - |\mu_1|\right)^2. \tag{3.13}$$

Consequently, (3.13), reduces to

$$2n\mathcal{K} - (E_{SND}(G))^2 \leq \frac{n^2}{4}\left(|\mu_n| - |\mu_1|\right)^2. \tag{3.14}$$

Rearranging (3.14), yields to $E_{SND}(G) \geq \sqrt{2n\mathcal{K} - \frac{n^2}{4}\left(|\mu_n| - |\mu_1|\right)^2}$.

Now assume that the equality holds in (3.12). Then all the inequalities in the proof must be equalities. This is possible only if $\mathcal{K} = 0$ and $|\mu_1| = |\mu_n| = 0$. Thus $G$ is the disjoint union of finite number of complete graphs $K_1$ or $K_2$.

Conversely, if $G$ is the disjoint union of finite number of complete graphs $K_1$ or $K_2$, then obviously equality holds in (3.12). □

**Theorem 3.13.** *Let $G$ be a connected graph of order $n$ and size $m$, then*

$$\sqrt{2\mathcal{K}} \leq E_{SND}(G) \leq 2m\sqrt{n}. \tag{3.15}$$

*Equality holds if and only if $G$ is the disjoint union of finite number of complete graphs $K_1$ or $K_2$.*

*Proof.* For the upper bound, we may consider the Cauchy Schwartz inequality. It follows that,

$$\left(\sum_{i=1}^{n} |\mu_i|\right)^2 \leq n \sum_{i=1}^{n} |\mu_i|^2.$$

This yields to

$$E_{SND}(G) \leq \sqrt{n\mathcal{K}} \leq 2m\sqrt{n}.$$

Hence the upper bound follows. For the lower bound, we have

$$\left(\sum_{i=1}^{n} |\mu_i|\right)^2 \geq \sum_{i=1}^{n} |\mu_i|^2 = 2\mathcal{K}.$$

As a result, we obtain $E_{SND}(G) \geq \sqrt{2\mathcal{K}}$. Hence the lower bound follows.

Now assume that the equality holds in (3.15). It is possible if and only if $\sqrt{2\mathcal{K}} = 2m\sqrt{n}$, which holds if and only if $G$ is the disjoint union of finite number of complete graphs $K_1$ or $K_2$. □

**Theorem 3.14.** *Let $G$ be a connected graph of order $n \geq 3$ and size $m$. Furthermore, let $\delta$ be the minimum degree of $G$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the eigenvalues of $M_S(G)$. Then*

$$E_{SND}(G) \geq \frac{1}{p}(nq + 2\delta),$$

*where $p = \mu_1 + \mu_n$ and $q = \mu_1\mu_n$.*

*Proof.* Letting $a_i = 1,\ b_i = |\mu_i|,\ p = |\mu_n|$ and $P = |\mu_1|$ in Lemma 2.2, we get

$$\sum_{i=1}^{n} |\mu_i|^2 + n|\mu_1||\mu_n| \leq (\mu_1 + \mu_n)\sum_{i=1}^{n} |\mu_i|. \tag{3.16}$$

On simplifying (3.16), we get

$$2\mathcal{K} + nq \leq p\sum_{i=1}^{n} |\mu_i|. \tag{3.17}$$

Observe that $\mathcal{K} \geq \delta$. Thus (3.17), can be further reduced to $E_{SND}(G) \geq \dfrac{1}{p}(nq + 2\delta)$, where $p = \mu_1 + \mu_n$ and $q = \mu_1\mu_n$. □

We shall wrap up this section by obtaining the following theorem, which establishes a sharp upper and lower bound for $E_{SND}(G)$ in terms of $\mathcal{K}$ and the size $m$ of a graph $G$.

**Theorem 3.15.** *Let $G$ be a connected graph of order $n$ and size $m$, then*

$$e^{-\sqrt{\mathcal{K}}} < E_{SND}(G) < e^{2\sqrt{m}}.$$

*Proof.* Since $x < e^x$ for any real number $x$, we have

$$
\begin{aligned}
E_{SND}(G) &= \sum_{i=1}^{n} |\mu_i| < \sum_{i=1}^{n} e^{|\mu_i|} \\
&\leq \sum_{i=1}^{n} \sum_{k\geq 0} \frac{|\mu_i|^k}{k!} = \sum_{k\geq 0} \frac{1}{k!} \sum_{i=1}^{n} |\mu_i|^k.
\end{aligned} \tag{3.18}
$$

From Lemma 2.6 and (3.18), we obtain

$$
\begin{aligned}
\sum_{k\geq 0} \frac{1}{k!} \sum_{i=1}^{n} |\mu_i|^k &\leq \sum_{k\geq 0} \frac{1}{k!} \left( \sum_{i=1}^{n} |\mu_i|^2 \right)^{\frac{k}{2}} \\
&\leq \sum_{k\geq 0} \frac{1}{k!} (4m)^{\frac{k}{2}} = \sum_{k\geq 0} \frac{1}{k!} (2\sqrt{m})^k = e^{2\sqrt{m}}.
\end{aligned} \tag{3.19}
$$

Thus, (3.18) and (3.19) together yields to

$$
E_{SND}(G) < e^{2\sqrt{m}}. \tag{3.20}
$$

Hence the upper bound follows.

Now for the lower bound, we have from Lemma 2.3,

$$
\mu_n = \mu_n(M_S(G)) \leq \mu_2([M_S(G)]) \leq \mu_2(M_S(G)),
$$

where $[M_S(G)]$ is the leading $2 \times 2$ submatrix of $M_S(G)$. Thus we get,

$$
\mu_n \leq -\sqrt{\delta_2(v_i)\delta_2(v_j)} \leq -1, \ and \ \sum_{i=1}^{n} \mu_i \geq |\mu_n| > 1.
$$

Let $\rho_1, \ \rho_2, \cdots, \ \rho_p$ be the absolute values of the non-zero eigenvalues of $M_S(G)$. Again, using AM-GM inequality and Lemmas 2.3, 2.4, 2.5 in the above equation, we get

$$
\begin{aligned}
E_{SND}(G) &= \sum_{i=1}^{n} |\mu_i| = \sum_{i=1}^{p} \rho_i \\
&= p \left( \sum_{i=1}^{p} \frac{1}{p} \rho_i \right) \geq p \left[ \prod_{i=1}^{p} \rho_i \right]^{\frac{1}{p}} \\
&\geq p \left[ \frac{p}{\sum_{i=1}^{p} \frac{1}{\rho_i} \sum_{i=1}^{p} \rho_i} \right]
\end{aligned}
$$

$$\geq p\left[\frac{p}{p^2\sum\limits_{i=1}^{p}\rho_i}\right]$$

$$> \frac{1}{\sum\limits_{i=1}^{p} e^{\rho_i}} = \frac{1}{\sum\limits_{k\geq 0}\sum\limits_{i=1}^{p}\frac{\rho^k}{k!}}$$

$$\geq \sum_{k\geq 0}\frac{1}{k!}\left(\sqrt{\mathcal{K}}\right)^{-k} = e^{-\sqrt{\mathcal{K}}} \tag{3.21}$$

Consequently, from (3.20) and (3.21), the theorem follows. □

## 4. CONCLUSIONS

After performing various graph operations on the regular graphs, the characteristic polynomials of the minimum second neighborhood degree matrix of graphs are evaluated. In this work, we also calculated the minimum second neighborhood degree energy of regular graphs. Furthermore, the bounds for the largest minimum second neighborhood degree eigenvalue and minimum second neighborhood degree energy of graphs are established. Theorem 3.15 provides the sharp lower and upper bound for minimum second neighborhood degree energy.

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