CREAT. MATH. INFORM.
Volume **33** (2024), No. 2,
Pages 231 - 237

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2024.02.08

Some considerations about l'Hôpital-type rules for the monotonicity

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ABSTRACT. The aim of this note is to present some results associated to the l'Hôpital-type rules for the monotonicity (LMR). We will prove the monotonicity of the ratio $\frac{f(x)-f(c)}{g(x)-g(c)}$, were c is an arbitrary point from the interval (a, b). Also, we extend LMR to the ratio of two higher-order differentiable functions. We complete with some applications of our results.

1. INTRODUCTION

In [1], Anderson et al. proposed a rule by l'Hôpital type for the monotonicity of the ratio of two differentiable functions. Later, Pinelis [3] developed this topic and proposed the following result:

Theorem 1. *Suppose $-\infty \leq a < b \leq \infty$ and let $f, g : (a, b) \to \mathbb{R}$ be two differentiable functions such that $g' \neq 0$ and $\frac{f'}{g'}$ is increasing (decreasing) on the interval (a, b).*

a) *Assume that there are the finite limits $f(a+)$ and $g(a+)$. Then the function*

$$h : (a, b) \to \mathbb{R}, h(x) = \frac{f(x) - f(a+)}{g(x) - g(a+)}$$

is increasing (decreasing) on the interval (a, b).

b) *Assume that there are the finite limits $f(b-)$ and $g(b-)$. Then the function*

$$h : (a, b) \to \mathbb{R}, h(x) = \frac{f(x) - f(b-)}{g(x) - g(b-)}$$

is increasing (decreasing) on the interval (a, b).

For example, let us consider the functions $f, g : (1, \infty) \to \mathbb{R}$, defined by $f(x) = e^x - e$, $g(x) = \ln x$. The function $\frac{f'}{g'} : (1, \infty) \to \mathbb{R}$, $\frac{f'(x)}{g'(x)} = xe^x$, is increasing on $(1, \infty)$. Then the function

$$h : (1, \infty) \to \mathbb{R}, h(x) = \frac{f(x) - f(1+)}{g(x) - g(1+)} = \frac{e^x - e}{\ln x},$$

is also increasing on $(1, \infty)$.

Theorem 1 is called l'Hôpital's rule for the monotonicity (LMR). The readers can find more papers dedicated to this topic as [4], [5] or [6], where many applications of LMR are proposed. Also, in [2], we find a counterexample for the reverse of LMR, that is a case where the function $\frac{f(x)-f(a+)}{g(x)-g(a+)}$ is monotone but $\frac{f'}{g'}$ is not monotone (see Example 13 from the mentioned paper).

Received: 29.04.2023. In revised form: 05.02.2024. Accepted: 12.02.2024
2000 *Mathematics Subject Classification.* 26A48.
Key words and phrases. *l'Hôpital-type rules, monotonicity, differentiable function, higher-order differentiable function.*
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The aim of this note is to present some results associated to LMR. We extend Theorem 1 and we will prove the monotonicity of the ratio $\frac{f(x)-f(c)}{g(x)-g(c)}$, for every point c of the interval (a,b). We will show and prove some versions of LMR involving high order differentiable functions. We also include some applications of our results.

We mention that in this paper, a non-decreasing function will be called an increasing function, and a non-increasing function will be called a decreasing function.

2. THE MAIN RESULTS

Firstly, we present a version of LMR for the points of an open interval.

Theorem 2. *Let $-\infty \leq a < b \leq \infty$ and $f, g : (a,b) \to \mathbb{R}$ two differentiable functions such that $g'(x) \neq 0$, for any $x \in (a,b)$, and $\frac{f'}{g'}$ is increasing (decreasing) on (a,b). Then the function*

$$h : (a,b) \to \mathbb{R}, h(x) = \begin{cases} \frac{f(x)-f(c)}{g(x)-g(c)}, & \text{if } x \neq c \\ \frac{f'(c)}{g'(c)}, & \text{if } x = c \end{cases}$$

is increasing (decreasing) on (a,b), for any $c \in (a,b)$.

Proof. We have $\lim_{x\to c} h(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot \frac{x-c}{g(x)-g(c)} = \frac{f'(c)}{g'(c)} = h(c)$, so h is continuous in c and also h is continuous on (a,b). We assume that $\frac{f'}{g'}$ is increasing. Due to Theorem 1, the function h is increasing on (a,c). Then, for any $x \in (a,c)$, we have $h(x) \leq \lim_{x\nearrow c} h(x) = h(c)$ and h is increasing on $(a,c]$. Similarly, we obtain that h is increasing on $[c,b)$. Now we conclude that h is increasing on (a,b). □

As a simple application of the previous result we obtain that the function

$$h : (0,\infty) \to \mathbb{R},\ h(x) = \begin{cases} \frac{e^x - e}{\ln x}, & \text{if } x \neq 1 \\ e, & \text{if } x = 1 \end{cases}$$

is increasing on $(0,\infty)$.

An interesting example is related to the *weighted power mean*. Let n be a positive integer and $a_1, a_2, ..., a_n \in (0,\infty)$. Let $p_1, p_2, ..., p_n \geq 0$ such that $p_1 + p_2 + ... + p_n = 1$. We consider the function $M : \mathbb{R} \to \mathbb{R}$ defined by

$$M(x) = \begin{cases} \left(p_1 a_1^x + p_2 a_2^x + ... + p_n a_n^x\right)^{\frac{1}{x}}, & \text{if } x \neq 0 \\ a_1^{p_1} a_2^{p_2} ... a_n^{p_n} & \text{if } x = 0 \end{cases}.$$

To evaluate the the monotonicity of M we consider the function $\ln M$. Denote

$$f(x) = \ln\left(p_1 a_1^x + p_2 a_2^x + ... + p_n a_n^x\right) \text{ and } g(x) = x.$$

We have

$$\ln M(x) = \begin{cases} \frac{f(x)}{g(x)}, & x \neq 0 \\ \sum_{k=1}^n p_k \ln a_k, & x = 0 \end{cases}.$$

Note that the function $\ln M$ is continuous on $\mathbb{R}$. Let us define the function

$$r(x) := \frac{f'(x)}{g'(x)} = \frac{\sum_{k=1}^n p_k a_k^x \ln a_k}{\sum_{k=1}^n p_k a_k^x},\ x \in \mathbb{R}.$$

We have

$$r'(x) = \frac{\sum_{k=1}^n p_k a_k^x \ln^2 a_k \cdot \sum_{k=1}^n p_k a_k^x - \left(\sum_{k=1}^n p_k a_k^x \ln a_k\right)^2}{\left(\sum_{k=1}^n p_k a_k^x\right)^2}.$$

From the Cauchy-Schwarz inequality, we obtain $r'(x) \geq 0$, for all $x \in \mathbb{R}$. This means that $r = \frac{f'}{g'}$ is an increasing function on $\mathbb{R}$. Theorem 2 ensures the increasing monotony on $\mathbb{R}$ of the function

$$\ln M(x) = \begin{cases} \frac{f(x)-f(0)}{g(x)-g(0)}, & x \neq 0 \\ \frac{f'(0)}{g'(0)}, & x = 0 \end{cases}.$$

Therefore, the weighted power mean M is increasing on $\mathbb{R}$.

It is well known that the classic l'Hôpital's rule can be applies successively. For example, we have

$$\lim_{x\to 1} \frac{x^3-3x+2}{x^4-4x+3} = \lim_{x\to 1} \frac{3x^2-3}{4x^3-4} = \lim_{x\to 1} \frac{6x}{12x^2} = \frac{1}{2}.$$

Then we raise the question if a similar situation holds in a case of monotonicity. The answer is positive as we will prove in Theorem 3. We will continue with some useful lemmas.

Lemma 1. *Let $-\infty \leq a < b \leq +\infty$ and $f : (a,b) \to \mathbb{R}$ a differentiable function. If there are the limits $f(a+)$ and $f(b-)$, with $f(a+) = f(b-)$, then there exists a point $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. If we assume that $f'(x) \neq 0$, for any $x \in (a,b)$ then f' is positive or negative. This means that f is strictly monotone. As consequence we obtain $f(a+) \neq f(b-)$ that contradicts the hypothesis. □

The next two lemmas are useful to explain that the function from Theorems 4-6 are correct defined. We will present the proof only for the first, the second being similarly.

Lemma 2. *Assume $-\infty; a < b \leq +\infty$ and n a positive integer. Let $h : (a,b) \to \mathbb{R}$ a $(n+1)$-times differentiable function on (a,b) such that $h^{(n+1)}(x) \neq 0$, for any $x \in (a,b)$. If there exist finite limits $h^{(k)}(a+)$, for any $k \in \{0,1,2,...,n\}$, then*

$$h(x) - \sum_{k=0}^{n} \frac{h^{(k)}(a+)}{k!}(x-a)^k \neq 0,$$

for any $x \in (a,b)$.

Proof. Denote

$$H(x) = h(x) - \sum_{k=0}^{n} \frac{h^{(k)}(a+)}{k!}(x-a)^k.$$

It is clear that H is $(n+1)$-times differentiable on (a,b) and $H^{(k)}(a+) = 0$, for any $k \in \{0,1,2,...,n\}$.

We assume by contradiction that there exists $c_0 \in (a,b)$ such that $H(c_0) = 0$. Hence $H(a+) = 0$, we can apply the previous lemma and we find $c_1 \in (a, c_0)$ such that $H'(c_1) = 0$. In the same mode, we find $c_2 \in (a, c_1)$ such that $H''(c_2) = 0$. We repeat this reasoning and obtain $a < c_n < c_{n-1} < ... < c_2 < c_1$ and $H^{(k)}(c_k) = 0$, for any $k \in \{1,2,...,n\}$. The previous lemma give us another point $c_{n+1} \in (a, c_n)$ such that $H^{(n+1)}(c_{n+1}) = 0$. Hence $H^{(n+1)}(c_{n+1}) = h^{(n+1)}(c_{n+1})$ then $h^{(n+1)}(c_{n+1}) = 0$ that it contradicts the hypothesis and concludes the lemma proof. □

Lemma 3. *Let $-\infty \leq a < b < +\infty$ and n a positive integer. Let $h : (a, b) \to \mathbb{R}$ a $(n+1)$-times differentiable function on (a, b) such that $h^{(n+1)}(x) \neq 0$, for any $x \in (a, b)$. If there exist $h^{(k)}(b-)$ exist and are finite, for any $k \in \{0, 1, 2, ..., n\}$, then*

$$h(x) - \sum_{k=0}^{n} \frac{h^{(k)}(b-)}{k!}(x-b)^k \neq 0,$$

for any $x \in (a, b)$.

Now we are in position to present the main results of this paper. Hence the results from Theorem 4 and 5 are similar, we will present the proof only for the first theorem.

Theorem 4. *Let $-\infty < a < b \leq +\infty$ and n a positive integer. Let $f, g : (a, b) \to \mathbb{R}$ two $(n+1)$-times differentiable functions on (a, b) such that $g^{(n+1)}(x) \neq 0$, for any $x \in (a, b)$, and $\frac{f^{(n+1)}}{g^{(n+1)}}$ is increasing (decreasing) on (a, b). If there exist finite limits $f^{(k)}(a+)$ and $g^{(k)}(a+)$, for any $k \in \{0, 1, 2, ..., n\}$, then the function*

$$h : (a, b) \to \mathbb{R}, h(x) = \frac{f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k}{g(x) - \sum_{k=0}^{n} \frac{g^{(k)}(a+)}{k!}(x-a)^k}$$

is increasing (decreasing) on (a, b).

Proof. For any $s \in \{0, 1, 2, ..., n, n+1\}$ and $x \in (a, b)$ denote

$$F_s(x) = f^{(s)}(x) - \sum_{k=0}^{n-s} \frac{f^{(k+s)}(a+)}{k!}(x-a)^{k+s}.$$

In particular $F_0(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k$ and $F_{n+1}(x) = f^{(n+1)}(x)$. Moreover, we have $F_{s+1}(x) = F_s'(x)$ and $F_s(a+) = 0$, for any $s \in \{0, 1, 2, ..., n\}$.

Also, for any $s \in \{0, 1, 2, ..., n, n+1\}$ and $x \in (a, b)$ denote

$$G_s(x) = g^{(s)}(x) - \sum_{k=0}^{n-s} \frac{g^{(k+s)}(a+)}{k!}(x-a)^{k+s}.$$

From previous lemma we obtain that $G_s(x) \neq 0$, for any $s \in \{0, 1, 2...., n, n+1\}$ and $x \in (a, b)$. This means that the ratio $\frac{F_s(x)}{G_s(x)}$ is correct defined for any $s \in \{0, 1, 2...., n, n+1\}$ and $x \in (a, b)$.

Let $s \in \{0, 1, 2, ..., n\}$. If $\frac{F_{s+1}}{G_{s+1}}$ is increasing (decreasing) then $\frac{F_s'}{G_s'}$is is increasing (decreasing) and Theorem 1 give as that $\frac{F_s}{G_s}$ is increasing (decreasing). From the hypothesis we obtain that $\frac{f^{(n+1)}}{g^{(n+1)}} = \frac{F_{n+1}}{G_{n+1}}$ is increasing (decreasing). We repeat the previous reasoning and we find that $\frac{F_n}{G_n}, \frac{F_{n-1}}{G_{n-1}}, ..., \frac{F_1}{G_1}$ are simultaneous increasing (decreasing). Finally, $\frac{F_0}{G_0}$ is increasing (decreasing) and the proof is complete. □

Theorem 5. *Let $-\infty \leq a < b < +\infty$ and n a positive integer. Let $f, g : (a, b) \to \mathbb{R}$ two $(n+1)$-times differentiable functions on (a, b) such that $g^{(n+1)}(x) \neq 0$, for any $x \in (a, b)$, and $\frac{f^{(n+1)}}{g^{(n+1)}}$ is increasing (decreasing) on (a, b). If there exist finite limits $f^{(k)}(b-)$ and $g^{(k)}(b-)$, for any $k \in \{0, 1, 2, ..., n\}$, then the function*

$$h : (a, b) \to \mathbb{R}, h(x) = \frac{f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(b-)}{k!}(x-b)^k}{g(x) - \sum_{k=0}^{n} \frac{g^{(k)}(-b)}{k!}(x-b)^k}$$

is increasing (decreasing) on (a,b) .

As an application of the previous results we obtain that the function

$$h:(0,\infty)\to\mathbb{R}, h(x)=\frac{e^x-1-x-\frac{x}{2}}{x^3}$$

is increasing. Indeed, if we denote $f(x)=e^x-1-x-\frac{x^2}{2}$ and $g(x)=x^3$ then f,g is satisfying the hypothesis of Theorem 4. More, we have $\frac{f^{(3)}(x)}{g^3(x)}=\frac{1}{6}e^x$, also an increasing function. Hence

$$h(x)=\frac{f(x)-\sum_{k=0}^{2}\frac{f^{(k)}(0+)}{k!}\cdot x^k}{g(x)-\sum_{k=0}^{2}\frac{g^{(k)}(0+)}{k!}\cdot x^k},$$

we obtain that h is increasing.

The following result represents the generalization of Theorem 2.

Theorem 6. *Let* $-\infty\le a<b\le+\infty$ *and* n *a positive integer. Let* $f,g:(a,b)\to\mathbb{R}$ *two* $(n+1)$*-times differentiable functions on* (a,b) *with* $g^{(n+1)}(x)\neq 0,$ *for any* $x\in(a,b)$, *such that* $\frac{f^{(n+1)}}{g^{(n+1)}}$ *is increasing (decreasing) on* (a,b) . *Then, for any* $c\in(a,b)$, *the function*

$$h:(a,b)\to\mathbb{R}, h(x)=\begin{cases}\frac{f(x)-\sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k}{g(x)-\sum_{k=0}^{n}\frac{g^{(k)}(c)}{k!}(x-c)^k}, & \text{if } x\neq c\\ \frac{f^{(n+1)}(c)}{g^{(n+1)}(c)}, & \text{if } x=c\end{cases}$$

is increasing (decreasing) on (a,b) .

Proof. We observe that the function h is continuous in c, so continuous on (a,b) . Indeed, applying successively the l'Hôpital rule, we obtain

$$\begin{aligned}\lim_{x\to c}h(x)&=\lim_{x\to c}\frac{f^{(n)}(x)-f^{(n)}(c)}{g^{(n)}(x)-g^{(n)}(c)}=\lim_{x\to c}\frac{f^{(n)}(x)-f^{(n)}(c)}{x-c}\cdot\frac{x-c}{g^{(n)}(x)-g^{(n)}(c)}\\&=\frac{f^{(n+1)}(c)}{g^{(n+1)}(c)}=h(c).\end{aligned}$$

We assume that $\frac{f^{(n+1)}}{g^{(n+1)}}$ is increasing. Due to Theorem 3, the function h is increasing on (a,c) . Then, for any $x\in(a,c)$, we have $h(x)\le\lim_{x\nearrow c}h(x)=h(c)$ and h is increasing on $(a,c]$. Similarly, we obtain that h is increasing on $[c,b)$ .Hence h is increasing on (a,b) . $\square$

As a consequence of the previous result we obtain the monotonicity of the function

$$h:(-1,\infty)\to\mathbb{R}, h(x)=\begin{cases}\frac{e^x-1-\sum_{k=1}^{n}\frac{x^k}{k!}}{\ln(x+1)-\sum_{k=1}^{n}\frac{(-1)^{k-1}x^k}{k!}}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases},$$

where $n\ge 2$ is a positive integer. If we denote $f(x)=e^x-1-\sum_{k=1}^{n}\frac{x^k}{k!}$ and $g(x)=\ln(x+1)-\sum_{k=1}^{n}\frac{(-1)^{k-1}x^k}{k!}$, we have

$$\frac{f^{(n+1)}}{g^{(n+1)}}(x)=\frac{e^x(x+1)^{n+1}}{(-1)^n\cdot n!},$$

for any $x\in(-1,\infty)$. This means that $\frac{f^{(n+1)}}{g(n+1)}$ is increasing if n is even and decreasing if n is odd. Now, the monotonicity of h follows.

The last result of this paper represents a generalization of Theorem 2 from [6].

Corollary 1. *Assume $-\infty < a < b < +\infty$ and n a positive integer. Let $f, g : (a, b) \to \mathbb{R}$ two $(n+1)$-times differentiable functions on (a, b) such that $g^{(n+1)}(x) \neq 0$, for any $x \in (a, b)$. We assume that $\frac{f^{(n+1)}}{g^{(n+1)}}$ is increasing on (a, b) and there are the finite limits $f(a+), f(b-), g(a+)$ and $g(b-)$.*

a) *If there are the finite limits $f^{(k)}(b-)$ and $g^{(k)}(b-)$, finite for any $k \in \{1, 2, ..., n\}$, then*

$$\frac{f(a+) - \sum_{k=0}^{n} \frac{f^{(k)}(b-)}{k!}(a-b)^k}{g(a+) - \sum_{k=0}^{n} \frac{g^{(k)}(b-)}{k!}(a-b)^k} \leq \frac{f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(b-)}{k!}(x-b)^k}{g(x) - \sum_{k=0}^{n} \frac{g^{(k)}(b-)}{k!}(x-b)^k} \leq \frac{f^{(n+1)}}{g^{(n+1)}}(b-), \tag{2.1}$$

for any $x \in (a, b)$.

b) *If there are the finite limits $f^{(k)}(a+)$ and $g^{(k)}(a+)$, for any $k \in \{1, 2, ..., n\}$, then*

$$\frac{f^{(n+1)}}{g^{(n+1)}}(a+) \leq \frac{f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k}{g(x) - \sum_{k=0}^{n} \frac{g^{(k)}(a+)}{k!}(x-a)^k} \leq \frac{f(b-) - \sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(b-a)^k}{g(b-) - \sum_{k=0}^{n} \frac{g^{(k)}(a+)}{k!}(b-a)^k}, \tag{2.2}$$

for any $x \in (a, b)$.

Proof. For the assertion a) we denote $h(x) = \frac{f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-b)^k}{g(x) - \sum_{k=0}^{n} \frac{g^{(k)}(a+)}{k!}(x-b)^k}$. From the hypothesis and Theorem 5 we obtain that h is increasing. Now, the conclusion follows due to the inequality

$$\lim_{x \searrow a} h(x) \leq h(x) \leq \lim_{x \nearrow b} h(x).$$

A similar argument for the function $u(x) = \frac{f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k}{g(x) - \sum_{k=0}^{n} \frac{g^{(k)}(a+)}{k!}(x-a)^k}$ concludes the assertion b) too. ☐

It clear that the inequalities from Corollary 1 will be reversed if the ratio $\frac{f^{(n+1)}}{g^{(n+1)}}$ is decreasing.

Also, if we choose $g(x) = (b-x)^{n+1}$, we obtain that $g^{(n+1)}(x) = (-1)^{n+1} \cdot (n+1)!$. The function $\frac{f^{(n+1)}}{g^{(n+1)}} = (-1)^{n+1} \cdot \frac{f^{(n+1)}}{(n+1)!}$ is increasing, so $(-1)^{n+1} \cdot f^{(n+1)}$ is increasing and we obtain the hypothesis of the first part of Theorem 2 from [6]. Then (2.1) becomes

$$\frac{f(a+) - \sum_{k=0}^{n} \frac{f^{(k)}(b-)}{k!}(a-b)^k}{(b-a)^{n+1}} \leq \frac{f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(b-)}{k!}(x-b)^k}{(b-x)^{n+1}} \leq \frac{f^{(n+1)}(b-)}{(-1)^{n+1}(n+1)!},$$

also

$$\frac{(b-x)^{n+1}}{(b-a)^{n+1}} \left(f(a+) - \sum_{k=0}^{n} \frac{f^{(k)}(b-)}{k!}(a-b)^k \right) \leq f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(b-)}{k!}(x-b)^k$$
$$\leq \frac{(-1)^{n+1} f^{(n+1)}(b-)}{(n+1)!}(b-x)^{n+1},$$

equivalent with

$$\sum_{k=0}^{n} \frac{f^{(k)}(b-)}{k!}(x-b)^k + \frac{(b-x)^{n+1}}{(b-a)^{n+1}} \left(f(a+) - \sum_{k=0}^{n} \frac{f^{(k)}(b-)}{k!}(a-b)^k \right) \leq$$
$$\leq f(x) \leq -\sum_{k=0}^{n+1} \frac{f^{(k)}(b-)}{k!}(x-b)^k.$$

and we obtain the first inequality from mention theorem. Some similar argument, but choosing $g(x) = (x-a)^{n+1}$, led us to the second inequality of the same theorem and concludes our paper.

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