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On the derived numbers of a real function

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ABSTRACT. The derived numbers of a real function at a point of its domain provide useful information about the variation of the function around that point. In this note, we introduce the concept of one-sided derived numbers and we obtain specific characterizations of some functional properties. We exemplify the usefulness of our results through some interesting consequences and applications.

1. INTRODUCTION

The derived numbers of a real function are closely related to the notion of differentiability. The definition of these numbers can be found, for example, in Natanson [3]. We introduce below the concept of one-sided derived number. In what follows, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \{1, 2, \ldots\}$.

Definition 1.1. Let I be a real open interval. Assume $f : I \to \mathbb{R}$.

(i) The number $\lambda \in \overline{\mathbb{R}}$ is said to be a derived number from the right of f at the point $x_0 \in I$ if there is a sequence $\{h_n\}_{n\geq 0}$, with $h_n > 0$, for all $n \in \mathbb{N}$, such that $\lim_{n\to\infty} h_n = 0$ and $\lim_{n\to\infty} \dfrac{f(x_0 + h_n) - f(x_0)}{h_n} = \lambda$.

(ii) The number $\lambda \in \overline{\mathbb{R}}$ is said to be a derived number from the left of f at the point $x_0 \in I$ if there is a sequence $\{h_n\}_{n\geq 0}$, with $h_n < 0$, for all $n \in \mathbb{N}$, such that $\lim_{n\to\infty} h_n = 0$ and $\lim_{n\to\infty} \dfrac{f(x_0 + h_n) - f(x_0)}{h_n} = \lambda$.

For $f : I \to \mathbb{R}$ and $x_0 \in I$, let us denote:

$$D_+f(x_0) = \{\lambda \in \overline{\mathbb{R}} : \ \lambda \text{ is a derived number from the right of } f \text{ at the point } x_0\}$$

and

$$D_-f(x_0) = \{\lambda \in \overline{\mathbb{R}} : \ \lambda \text{ is a derived number from the left of } f \text{ at the point } x_0\}.$$

The set $D_+f(x_0)$ is not empty. Indeed, assume a sequence $\{h_n\}_{n\geq 0}$, with $h_n > 0$, for all $n \in \mathbb{N}$, such that $\lim_{n\to\infty} h_n = 0$. The sequence $r_n = \dfrac{f(x_0 + h_n) - f(x_0)}{h_n}$, $n \in \mathbb{N}$, has at least one limit point $\lambda \in \overline{\mathbb{R}}$. We deduce $\lambda \in D_+f(x_0)$. Similarly, we get $D_-f(x_0) \neq \emptyset$.

Clearly, if f is differentiable at $x_0 \in I$, then $D_+f(x_0) = D_-f(x_0) = \{f'(x_0)\}$.

If f is convex on I, then $D_+f(x_0) = \{f'_+(x_0)\}$, where

$$f'_+(x_0) = \lim_{x\to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \inf_{x\in I,\ x>x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \mathbb{R}$$

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and $D_-f(x_0) = \{f'_-(x_0)\}$, where

$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = \sup_{x \in I,\ x < x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \mathbb{R}.$$

We have $f'_-(x_0) \leq f'_+(x_0)$, for all $x_0 \in I$ (see, for example, [4] for details).

If f is monotonically increasing on I, then $D_+f(x_0)$ and $D_-f(x_0)$ are subsets of the set $[0, \infty]$ (see [3]).

It should be noted that the problem of generalized antiderivatives of functions is a topic related to the derived numbers. Among relevant classical results from the literature, we mention the following theorem.

Theorem 1.1. ([2]) *For a function $f : \mathbb{R} \to \mathbb{R}$ and an arbitrary sequence $\{h_n\}_{n \geq 0}$ of non-zero numbers converging to 0, there exists a function F such that $\lim_{n \to \infty} \frac{F(x + h_n) - F(x)}{h_n} = f(x)$, for all $x \in \mathbb{R}$.*

2. Main results

We previously mentioned that the one-sided derivative numbers of a monotonically increasing function are non-negative. The converse implication is not true. For example, for the fractional part function $f : \mathbb{R} \to [0, 1)$, we have $D_+f(x) = \{1\}$, $\forall\, x \in \mathbb{R}$, but f is not a monotonically increasing function. However, under the assumption of continuity, the following property holds.

Proposition 2.1. *If $f : I \to \mathbb{R}$ is a continuous function on the open interval I and $D_+f(x) \cap [0, \infty] \neq \emptyset$, for any $x \in I$, then f is monotonically increasing on I.*

Proof. Assume $x, y \in I$, with $x < y$. For an arbitrary number $\varepsilon > 0$, let us consider the set $M(\varepsilon) = \{z \in [x, y] :\ f(z) - f(x) \geq -\varepsilon(z - x)\}$. We have $x \in M(\varepsilon) \subset [x, y]$. Then there is $s = \sup M(\varepsilon) \in [x, y] \subset I$. From the continuity of f, we obtain $s \in M(\varepsilon)$, that is

$$f(s) - f(x) \geq -\varepsilon(s - x). \tag{2.1}$$

Assume $s < y$. The hypothesis ensures the existence of a number $\lambda \in D_+f(s) \cap [0, \infty]$, that is there exists a sequence $\{h_n\}_{n \geq 0}$, with $h_n > 0$, for all $n \in \mathbb{N}$, such that $h_n \to 0$ and $\lim_{n \to \infty} \frac{f(s + h_n) - f(s)}{h_n} = \lambda$. So, we can find $p \in \mathbb{N}$ such that $s + h_p \leq y$ and $\frac{f(s + h_p) - f(s)}{h_p} \geq -\varepsilon$. From (2.1) we get

$$f(s + h_p) - f(x) = [f(s + h_p) - f(s)] + [f(s) - f(x)] \geq -\varepsilon h_p - \varepsilon(s - x) = -\varepsilon[(s + h_p) - x].$$

As a consequence, $s + h_p \in M(\varepsilon)$, with $s < s + h_p \leq y$, in contradiction with the definition of s. Thus, $y = s \in M(\varepsilon)$ and have $f(y) - f(x) \geq -\varepsilon(y - x)$. Since $\varepsilon > 0$ is arbitrary, we obtain $f(y) - f(x) \geq 0$.

In conclusion, f is a monotonically increasing function on I. □

Remark 2.1. Consider the famous everywhere continuous, nowhere monotonic, Takagi-Van der Waerden function (see, for example, [1])

$$w(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} d\left(2^n x\right),\ x \in \mathbb{R},$$

where $d(x)$ is the distance of x to the nearest integer. It follows from Proposition 2.1 that in any open interval $(a, b) \subset \mathbb{R}$ there is a point c such that $D_+(w)(c) \subset (-\infty, 0)$, that is the set $\{x \in \mathbb{R} :\ D_+(w)(x) \subset (-\infty, 0)\}$ is dense in $\mathbb{R}$.

Also remark that the assumption of continuity cannot be eliminated in the above proposition. As an elementary counterexample, let us consider the discontinuous at 0 function $f(x) = \begin{cases} x, & x < 0 \\ x-1, & x \geq 1 \end{cases}$. We have $D_+(f)(x) = \{1\}$, for all $x \in \mathbb{R}$, but the function f is not increasing on $\mathbb{R}$.

The following consequences refer to a continuous function $f : I \to \mathbb{R}$, where I is an open real interval.

Corollary 2.1. *If $D_+f(x) \cap (0, \infty] \neq \emptyset$, for all $x \in I$, then f is strictly increasing on I.*

Proof. From Proposition 2.1, f is monotonically increasing on I. Assume that there are $x, y \in I$, with $x < y$ and $f(x) = f(y)$. Then f is constant on $[x, y]$. Hence $D_+f(z) = \{0\}$, for all $z \in [x, y)$, in contradiction with the assumption. As follows f is strictly increasing on the interval I. □

Corollary 2.2.

(i) If $D_+f(x) \cap [-\infty, 0] \neq \emptyset,\ \forall\, x \in I$, then f is monotonically decreasing on I.
(ii) If $D_+f(x) \cap [-\infty, 0) \neq \emptyset,\ \forall\, x \in I$, then f is strictly decreasing on I.

Proof. We apply Proposition 2.1 and Corollary 2.1 for the function $-f$. □

Corollary 2.3. *If $0 \in D_+f(x),\ \forall\, x \in I$, then f is constant on I.*

Proof. From Proposition 2.1 and Corollary 2.2 (i) we deduce that f is is simultaneous monotonically increasing and decreasing. Therefore, f is constant. □

A derivability criterion is formulated below in terms of one-sided from the right derived numbers.

Proposition 2.2. *Let $f : I \to \mathbb{R}$ be a continuous function on the open real interval I. Assume that there is a differentiable function $g : I \to \mathbb{R}$, such that $g'(x) \in D_+f(x)$, for all $x \in I$. Then f is differentiable on I.*

Proof. Consider the continuous function $h = g - f : I \to \mathbb{R}$. From the assumption, $0 \in D_+h(x),\ \forall\, x \in I$. Hence h is constant (Corollary 2.3). Therefore $f = g - h$ is differentiable on I. □

Remark 2.2. The above result extends a contest problem recently proposed by Săvescu [7], pp. 43-44. In this problem it is assumed that there is a strictly positive sequence $\{h_n\}_{n \geq 0}$ converging to 0, such that $g'(x) = \lim\limits_{n \to \infty} \dfrac{f(x + h_n) - f(x)}{h_n}$, for all $x \in I$. Thus, the function f is a generalized antiderivative of g'. Note that, Theorem 1.1 ensures the existence of generalized antiderivatives of an arbitrary function defined on an open interval, given a strictly positive sequence which converges to 0. Such a continuous generalized antiderivative of the derivative of a function is therefore differentiable.

In what follows, we characterize the Lipschitz property in our context.

Proposition 2.3. *Let $f : I \to \mathbb{R}$ be a function defined on the real interval I. Assume $L > 0$. The following properties are equivalent:*

(i) f is a L-Lipschitz function on I.
(ii) $D_+f(x) \subset [-L, L]$, for all $x \in I$.

Proof. Let us define $g, h : I \to \mathbb{R}$, by $g(x) = Lx - f(x)$ and $h(x) = Lx + f(x)$, for all $x \in I$. We easily notice that (i) holds if and only if g and h are monotonically increasing functions, hence $D_+g(x),\ D_+h(x) \subset [0, \infty],\ \forall\, x \in I$. But the last conditions are equivalent with $D_+f(x) \subset [-\infty, L] \cap [-L, \infty] = [-L, L]$. Thus (i) and (ii) are equivalent. □

Corollary 2.4. *Let f be a real convex function on an open interval I. Then f is a Lipschitz function on any compact interval $[a,b] \subset I$.*

Proof. The convex function $f : I \to \mathbb{R}$ is continuous on the open interval I. Assume $[a,b] \subset I$, with $a < b$. Let us denote $L = \max\{|f'_+(a)|,\ |f'_+(b)|\} \in [0,\infty)$. We have

$$-L \le f'_+(a) \le f'_+(x) \le f'_+(b) \le L, \text{ for all } x \in [a,b].$$

Thus, $D_+f(x) = \{f'_+(x)\} \subset [-L, L]$, for all $x \in [a,b]$. From Proposition 2.3, we get the conclusion. □

Corollary 2.5. *Let f be a real differentiable function on an open interval I. The following statements are equivalent:*

(i) f is convex on I;

(ii) f' is continuous on I and $D_+(f')(x) \cap [0,\infty) \neq \emptyset$, for all $x \in I$.

Proof. (i)⇒(ii). Since the differentiable function f is convex on I, its derivative f' is monotonically increasing on I. Then $D_+(f')(x) \cap [0,\infty) \neq \emptyset$, for all $x \in I$. The continuity of f' results from the monotony and Darboux's theorem.
(ii)⇒(i). From Proposition 2.1 and the assumption (ii), we conclude that f' is monotonically increasing on I. Hence f is convex on I. □

Remark 2.3. The function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \displaystyle\int_0^x w(t)\,\mathrm{d}t$, for $x \in \mathbb{R}$, has a continuous derivative, but it is nowhere convex (see Remark 2.1). On the other hand, if f' has discontinuities on I, then f is not convex on I and $\{x \in I :\ D_+(f)(x) \subset (-\infty, 0)\} \neq \emptyset$.

The applications below show the usefulness of our results in providing natural solutions to some interesting contest problems.

Example 2.1. (Andrica and Piticari [5], pp. 14, 72)
Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function on $\mathbb{R}$. Assume that the function $F : \mathbb{R} \to \mathbb{R},\ F(x) = \displaystyle\int_0^x f(t)\,\mathrm{d}t,\ x \in \mathbb{R}$, is differentiable. Then f is continuous.

Proof. Since f is monotonically increasing, F is convex. Indeed, for $x_1, x_2, x_3 \in \mathbb{R}$, such that $x_1 < x_2 < x_3$, we have:

$$\frac{F(x_2) - F(x_1)}{x_2 - x_1} = \frac{\int_{x_1}^{x_2} f(t)\,\mathrm{d}t}{x_2 - x_1} \le f(x_2) \le \frac{\int_{x_2}^{x_3} f(t)\,\mathrm{d}t}{x_3 - x_2} = \frac{F(x_3) - F(x_2)}{x_3 - x_2}.$$

Then F' is continuous (Corollary 2.5). Clearly, $F'(x) = f(x)$ at any point x of continuity of f. Taking into account that f has finite lateral limits that frame its value at any point $x \in \mathbb{R}$, and the set of continuity points of f is dense in $\mathbb{R}$, we deduce that $F'(x) = f(x),\ \forall\, x \in \mathbb{R}$. Thus, f is continuous on $\mathbb{R}$. □

Example 2.2. (Bălună [6], pp. 25-26)
Let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function and $F : \mathbb{R} \to \mathbb{R}$ a function having right and left finite derivatives at any point in $\mathbb{R}$ and $F(0) = 0$. Suppose $f(x_0 - 0) \le F'_-(x_0)$ and $f(x_0 + 0) \ge F'_+(x_0)$, for any $x_0 \in \mathbb{R}$. Then $F(x) = \displaystyle\int_0^x f(t)\,\mathrm{d}t,\ x \in \mathbb{R}$.

Proof. Define the functions $G, H : \mathbb{R} \to \mathbb{R}$ by $G(x) = \int_0^x f(t)\,\mathrm{d}t$ and $H(x) = G(x) - F(x)$, for all $x \in \mathbb{R}$. The function F, having lateral finite derivatives at any point, is continuous on $\mathbb{R}$. Also, the function G is continuous on $\mathbb{R}$. Hence H is continuous on $\mathbb{R}$. We have $D_+(H)(x_0) = \{f(x_0+0) - F'_+(x_0)\} \subset [0,\infty)$. Then H is monotonically increasing on $\mathbb{R}$ (Proposition 2.1). Similarly, from Corollary 2.2, it follows that H is monotonically decreasing on $\mathbb{R}$. Therefore H is a constant function. Since $H(0) = 0$, we obtain the conclusion. □

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