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Countably Many Positive Symmetric Solutions For Sturm Liouville Type Boundary Conditions Of Second Order Iterative System

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ABSTRACT. In this paper we consider second order iterative boundary value problem with Sturm Liouville type boundary conditions and establish the existence of countably many positive symmetric solutions by using Krasnoselskii's fixed point theorem for operator on a cone.

1. INTRODUCTION

Most of the real-world problems in the universe include complex systems with several degrees of freedom, requiring a set of differential equations under specific assumptions. Developing a model for complex systems is the first challenge step, followed by investigating the potential solutions. In recent years, much attention has been focused on the iterative system of nonlinear boundary value problems (BVPs) associated with ordinary and fractional differential equations, see [1, 3, 11, 14, 20].

Eloe, Henderson and Kosmatov [7] demonstrated the existence of countably positive solutions for the following boundary value problem

$$\begin{aligned}
&(-1)^{n-k}\mathtt{u}^{(n)}(t) = f(\mathtt{u}(t)),\ 0 < t < 1,\\
&\mathtt{u}^{(i)}(0) = 0, i = 0, 1, \cdots, k-1,\\
&\mathtt{u}^{(j)}(1) = 0, j = 0, 1, \cdots, n-k-1,
\end{aligned}$$

where $n \geq 2$, and $k \in \{1, \cdots, n-1\}$.

In 2011, Sun [19] considered second order two-point BVP

$$\begin{aligned}
&\mathtt{u}''(t) + g(t)f(t, \mathtt{u}(t)) = 0,\ t \in [0, 1],\\
&\mathtt{u}(0) = \mathtt{u}(1) = \int_0^1 \mathtt{m}(s)\mathtt{u}(s)ds,
\end{aligned}$$

and established the existence of three symmetric positive solutions by using Leggett-Williams fixed point theorem.

Following that, the researchers have studied the existence of positive symmetric solutions, see [2, 6, 13, 15, 17], and countably many positive solutions, see [10, 18, 21, 22].

Inspired by the aforementioned works, in this paper, we consider the second order iterative BVP with Sturm Liouville type boundary conditions,

$$\text{(1.1)}\qquad \begin{cases} \mathrm{y}_\mathrm{k}''(x) + \psi(x)h_\mathrm{k}(\mathrm{y}_{\mathrm{k}+1}(x)) = 0,\ 1 \leqslant \mathrm{k} \leqslant \tau,\ 0 \leqslant x \leqslant \rho,\\ \mathrm{y}_{\tau+1}(x) = \mathrm{y}_1(x),\ 0 \leqslant x \leqslant \rho, \end{cases}$$

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$$\begin{cases} \mu_1 \mathrm{y_k}(0) = \mu_2 \mathrm{y}'_\mathrm{k}(0), \\ \mu_1 \mathrm{y_k}(\rho) = -\mu_2 \mathrm{y}'_\mathrm{k}(\rho), \end{cases} \tag{1.2}$$

where $\tau \in \mathbb{N}$, $\psi(x) = \prod_{s=1}^{z} \psi_s(x) \in L^{\mathrm{p}_s}[0,\rho]$, $(\mathrm{p}_s \geqslant 1)$, μ_1, μ_2 are positive constants, $\Delta = \mu_1(\mu_1\rho + 2\mu_2)$, and establish the existence of countably many positive symmetric solutions by employing the Krasnoselskii's fixed point theorem.

Definition 1.1. [19] A function $\mathrm{y}(x) : [0,\rho] \to \mathbb{R}$ is said to be symmetric on $[0,\rho]$ if for any $x \in [0,\rho]$, $\mathrm{y}(x) = \mathrm{y}(\rho - x)$.

Definition 1.2. [19] Let B be a real Banach space. A nonempty closed convex $\mathcal{P} \subset$ B is called a cone if it satisfies the following conditions:

(i) $\mathtt{r} \in \mathcal{P},\ \alpha > 0 \implies \alpha\mathtt{r} \in \mathcal{P}$.
(ii) $\mathtt{r} \in \mathcal{P},\ -\mathtt{r} \in \mathcal{P} \implies \mathtt{r} = 0$.

Definition 1.3. [19] Suppose $\mathcal{P}$ is a cone in a Banach space B. The map $\mathcal{F}$ is a non negative continuous concave functional on $\mathcal{P}$ provided $\mathcal{F} : \mathcal{P} \to [0,\infty)$ is continuous and

$$\mathcal{F}(\mathtt{r}\vartheta_1 + (1-\mathtt{r})\vartheta_2) \geq \mathtt{r}\,\mathcal{F}(\vartheta_1) + (1-\mathtt{r})\mathcal{F}(\vartheta_2)$$

for all $\vartheta_1, \vartheta_2 \in \mathcal{P}$ and $\mathtt{r} \in [0,1]$.

Definition 1.4. [19] An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Theorem 1.1 (Arzela-Ascoli [5]). *A subset A of* $\mathbf{C}([\mathrm{a},\mathrm{b}],\mathbb{R})$ *is relatively compact if and only if it is bounded and equicontinuous.*

We assume the following conditions are true in the entire paper

Z1) $h_\mathrm{k} : [0,\infty) \to [0,\infty)$ is continuous, $1 \leqslant \mathrm{k} \leqslant \tau$.
Z2) $\psi_s \in L^{\mathrm{p}_s}[0,\rho]$ for $1 \leqslant \mathrm{p}_s < \infty$, ψ_s is symmetric on $[0,\rho]$, and each ψ_s does not vanish identically on any sub interval of $[0,\rho]$. Further, $\exists\, \alpha_s > 0 \ni \alpha_s < \psi_s(x) < \infty$, a.e. on $[0,\rho]$, $s = 1, 2, \cdots, z$.

The remaining part of the paper is arranged as follows. In section 2, we construct Green's function for the homogeneous BVP corresponding to (1.1)-(1.2) and obtain bounds for the Green's function. In section 3, we develop criteria for the existence of countably many positive symmetric solutions of (1.1)-(1.2) by using Krasnoselskii's fixed point theorem. Finally, we give an example to illustrate our results in section 4.

2. Preliminary findings

In this section, we determine Green's function for the homogeneous BVP corresponding to (1.1)-(1.2) and certain lemmas on Green's function are established. These lemmas are useful in demonstrating our main findings.

Lemma 2.1. *Let* $\omega(x) \in C([0,\rho],\mathbb{R})$. *Then the unique solution of BVP*

$$\mathrm{y}''_1(x) + \omega(x) = 0,\ 0 \leqslant x \leqslant \rho, \tag{2.3}$$

$$\begin{cases} \mu_1 \mathrm{y}_1(0) = \mu_2 \mathrm{y}'_1(0), \\ \mu_1 \mathrm{y}_1(\rho) = -\mu_2 \mathrm{y}'_1(\rho), \end{cases} \tag{2.4}$$

is

$$\mathrm{y}_1(x) = \int_0^\rho G(x,r)\omega(r)dr, \tag{2.5}$$

in which

$$G(x,r)=\frac{1}{\Delta}\begin{cases}(\mu_1 r+\mu_2)(\mu_1(\rho-x)+\mu_2), \ 0\leqslant r\leqslant x\leqslant\rho,\\ (\mu_1 x+\mu_2)(\mu_1(\rho-r)+\mu_2), \ 0\leqslant x\leqslant r\leqslant\rho.\end{cases} \tag{2.6}$$

Proof. The corresponding integral equation for (2.3) is

$$\begin{aligned}\mathrm{y}_1(x)&=-\int_0^x\int_0^r\omega(r_1)dr_1dr+\mathfrak{a}_1x+\mathfrak{a}_2\\ &=-\int_0^x(x-r)\omega(r)dr+\mathfrak{a}_1x+\mathfrak{a}_2,\end{aligned}$$

where $\mathfrak{a}_1, \ \mathfrak{a}_2$ are constants. By using boundary conditions (2.4), we get

$$\begin{aligned}\mathfrak{a}_1&=\frac{1}{(\mu_1\rho+2\mu_2)}\int_0^\rho(\mu_1(\rho-r)+\mu_2)\omega(r)dr,\\ \mathfrak{a}_2&=\frac{\mu_2}{\mu_1(\mu_1\rho+2\mu_2)}\int_0^\rho(\mu_1(\rho-r)+\mu_2)\omega(r)dr.\end{aligned}$$

So, we have,

$$\begin{aligned}\mathrm{y}_1(x)=&\int_0^x(x-r)\omega(r)dr+\frac{x}{(\mu_1\rho+2\mu_2)}\int_0^\rho(\mu_1(\rho-r)+\mu_2)\omega(r)dr+\\ &\frac{\mu_2}{\mu_1(\mu_1\rho+2\mu_2)}\int_0^\rho(\mu_1(\rho-r)+\mu_2)\omega(r)dr\\ =&\frac{1}{\Delta}\bigg[\int_0^x(\mu_1r+\mu_2)(\mu_1(\rho-x)+\mu_2)\omega(r)dr+\int_x^\rho(\mu_1x+\mu_2)(\mu_1(\rho-r)+\mu_2)\omega(r)dr\bigg]\\ =&\int_0^\rho G(x,r)\omega(r)dr,\end{aligned}$$

where $G(x,r)$ is in (2.6). □

Lemma 2.2. *For $\eta\in(0,\rho/2)$, let $\sigma(\eta)=\frac{\mu_1\eta+\mu_2}{\mu_1\rho+\mu_2}$, then $G(x,r)$ has the following properties:*

i) $0\leqslant G(x,r)\leqslant G(r,r), \ \forall\, x,r\in[0,\rho]$.

ii) $G(x,r)\geqslant\sigma(\eta)G(r,r), \ \forall\, x\in[\eta,\rho-\eta]$ *and* $r\in[0,\rho]$.

iii) $G(\rho-x,\rho-r)=G(x,r), \ \forall\, x,r\in[0,\rho]$.

Proof. We can easily establish the inequality (i). For inequality (ii), let $x\in[\eta,\rho-\eta]$, then for $0\leqslant r\leqslant x$,

$$\frac{G(x,r)}{G(r,r)}=\frac{\mu_1(\rho-x)+\mu_2}{\mu_1(\rho-r)+\mu_2}\geqslant\sigma(\eta),$$

and for $x\leqslant r\leqslant\rho$,

$$\frac{G(x,r)}{G(r,r)}=\frac{\mu_1x+\mu_2}{\mu_1r+\mu_2}\geqslant\sigma(\eta).$$

Hence, the inequality (ii).

For the inequality (iii), consider

$$\begin{aligned}G(\rho-x,\rho-r)&=\frac{1}{\Delta}\begin{cases}(\mu_1(\rho-r)+\mu_2)(\mu_1(\rho-(\rho-x))+\mu_2), \ 0\leqslant\rho-r\leqslant\rho-x,\\ (\mu_1(\rho-x)+\mu_2)(\mu_1(\rho-(\rho-r))+\mu_2), \ \rho-x\leqslant\rho-r\leqslant\rho,\end{cases}\\ &=\frac{1}{\Delta}\begin{cases}(\mu_1(\rho-r)+\mu_2)(\mu_1(x)+\mu_2), \ x\leqslant r\leqslant\rho,\\ (\mu_1(\rho-x)+\mu_2)(\mu_1(r)+\mu_2), \ 0\leqslant r\leqslant x,\end{cases}\\ &=G(x,r).\end{aligned}$$

This completes the proof. $\square$

Lemma 2.3. *Let $\omega(x)$ be continuous and symmetric on $[0,\rho]$. Then the solution $\mathrm{y}_1(x)$ of (2.3)-(2.4) is symmetric on $[0,\rho]$.*

Proof. Consider

$$\begin{aligned}\mathrm{y}_1(\rho-x) &= \int_0^\rho G(\rho-x,r)\omega(r)dr\\ &= \int_\rho^0 G(\rho-x,\rho-r)\omega(\rho-r)d(\rho-r)\\ &= \int_0^\rho G(x,r)\omega(r)dr\\ &= \mathrm{y}_1(x).\end{aligned}$$

Therefore $\mathrm{y}_1(x)$ is symmetric on $[0,\rho]$. $\square$

Notice that an τ-tuple $\big(\mathrm{y}_1(x),\mathrm{y}_2(x),\cdots,\mathrm{y}_\tau(x)\big)$ is a solution of (1.1)-(1.2) if and only if

$$\mathrm{y}_\mathrm{k}(x) = \int_0^\rho G(x,r)\psi(r)h_\mathrm{k}(\mathrm{y}_{\mathrm{k}+1}(r))dr,\ x\in[0,\rho],\ 1\leqslant \mathrm{k}\leqslant\tau,$$
$$\mathrm{y}_{\tau+1}(x) = \mathrm{y}_1(x),\ x\in[0,\rho],$$

Equivalently,

$$\begin{aligned}\mathrm{y}_1(x) = \int_0^\rho G(x,r_1)\psi(r_1)h_1\bigg(\int_0^\rho G(r_1,r_2)\psi(r_2)h_2\bigg(\int_0^\rho G(r_2,r_3)\psi(r_3)\cdots\\ h_{\tau-1}\bigg[\int_0^\rho G(r_{\tau-1},r_\tau)\psi(r_\tau)h_\tau(\mathrm{y}_1(r_\tau)dr_\tau\bigg]\cdots dr_3\bigg)dr_2\bigg)dr_1.\end{aligned}$$

3. Main results

In this section, we establish the existence of countably many positive symmetric solutions for (1.1)-(1.2) by using Krasnoselskii's theorem and Hölder's inequality. Let $\mathrm{B}\in C([0,\rho],\mathbb{R})$ be a Banach space with norm $\|\mathrm{y}\| = \max_{x\in[0,\rho]}|\mathrm{y}(x)|$. For $\eta\in(0,\frac{\rho}{2})$, define the cone $\mathcal{P}\subset\mathrm{B}$ as $\mathcal{P}=\big\{\mathrm{y}\in\mathrm{B} : \mathrm{y}(x)$ is non negative, concave, symmetric and $\min_{x\in[\eta,\rho-\eta]}\mathrm{y}(x)\geqslant \sigma(\eta)\|\mathrm{y}\|\big\}$.

For any $\mathrm{y}_1\in\mathcal{P}$, define an operator $\mathcal{F}:\mathcal{P}\to\mathrm{B}$ by

$$\begin{aligned}\mathcal{F}\mathrm{y}_1(x) = \int_0^\rho G(x,r_1)\psi(r_1)h_1\bigg(\int_0^\rho G(r_1,r_2)\psi(r_2)h_2\bigg(\int_0^\rho G(r_2,r_3)\psi(r_3)\cdots\\ h_{\tau-1}\bigg[\int_0^\rho G(r_{\tau-1},r_\tau)\psi(r_\tau)h_\tau(\mathrm{y}_1(r_\tau)dr_\tau\bigg]\cdots dr_3\bigg)dr_2\bigg)dr_1.\end{aligned}$$

Lemma 3.4. *Assume that (Z1)-(Z2) hold. Then for each $\eta\in(0,\frac{\rho}{2})$, $\mathcal{F}(\mathcal{P})\subset\mathcal{P}$ and $\mathcal{F}:\mathcal{P}\to\mathcal{P}$ is completely continuous.*

Proof. From lemma (2.2), $G(x,r)\geqslant 0,\ \forall\, x,r\in[0,\rho]$, then $(\mathcal{F}\mathtt{y}_1)(x)\geqslant 0$. Let $\mathtt{y}_1\in\mathcal{P}$, then

$$\begin{aligned}(\mathcal{F}\mathtt{y}_1)(1-x)=&\int_0^\rho G(1-x,r_1)\psi(r_1)h_1\bigg(\int_0^\rho G(r_1,r_2)\psi(r_2)h_2\bigg(\int_0^\rho G(r_2,r_3)\psi(r_3)\cdots\\ &\qquad h_{\tau-1}\bigg[\int_0^\rho G(r_{\tau-1},r_\tau)\psi(r_\tau)h_\tau(\mathtt{y}_1(r_\tau))dr_\tau\bigg]\cdots dr_3\bigg)dr_2\bigg)dr_1\\ =&\int_\rho^0 G(1-x,1-r_1)\psi(1-r_1)h_1\bigg(\int_0^\rho G(1-r_1,r_2)\psi(r_2)\\ &\qquad h_2\bigg(\int_0^\rho G(r_2,r_3)\psi(r_3)\cdots h_{\tau-1}\bigg[\int_0^\rho G(r_{\tau-1},r_\tau)\psi(r_\tau)h_\tau(\mathtt{y}_1(r_\tau))dr_\tau\bigg]\\ &\qquad\cdots dr_3\bigg)dr_2\bigg)d(1-r_1)\\ =&\int_0^\rho G(x,r_1)\psi(r_1)h_1\bigg(\int_0^\rho G(r_1,r_2)\psi(r_2)h_2\bigg(\int_0^\rho G(r_2,r_3)\\ &\qquad\psi(r_3)\cdots h_{\tau-1}\bigg[\int_0^\rho G(r_{\tau-1},r_\tau)\psi(r_\tau)h_\tau(\mathtt{y}_1(r_\tau))dr_\tau\bigg]\cdots dr_3\bigg)dr_2\bigg)dr_1\\ =&(\mathcal{F}\mathtt{y}_1)(x).\end{aligned}$$

Hence $\mathcal{F}\mathtt{y}_1$ is symmetric on $[0,\rho]$.

Similarly by Lemma 2.2, we obtain

$$\begin{aligned}(\mathcal{F}\mathtt{y}_1)(x)\leqslant&\int_0^\rho G(r_1,r_1)\psi(r_1)h_1\bigg(\int_0^\rho G(r_1,r_2)\psi(r_2)h_2\bigg(\int_0^\rho G(r_2,r_3)\\ &\psi(r_3)\cdots h_{\tau-1}\bigg[\int_0^\rho G(r_{\tau-1},r_\tau)\psi(r_\tau)h_\tau(\mathtt{y}_1(r_\tau))dr_\tau\bigg]\cdots dr_3\bigg)dr_2\bigg)dr_1.\end{aligned}$$

So,

$$\begin{aligned}\|(\mathcal{F}\mathtt{y}_1)\|\leqslant&\int_0^\rho G(r_1,r_1)\psi(r_1)h_1\bigg(\int_0^\rho G(r_1,r_2)\psi(r_2)h_2\bigg(\int_0^\rho G(r_2,r_3)\\ &\psi(r_3)\cdots h_{\tau-1}\bigg[\int_0^\rho G(r_{\tau-1},r_\tau)\psi(r_\tau)h_\tau(\mathtt{y}_1(r_\tau))dr_\tau\bigg]\cdots dr_3\bigg)dr_2\bigg)dr_1.\end{aligned}$$

Again from Lemma 2.2, we get

$$\begin{aligned}\min_{x\in[\eta,\rho-\eta]}\{(\mathcal{F}\mathtt{y}_1)(x)\}\geqslant\sigma(\eta)&\int_0^\rho G(r_1,r_1)\psi(r_1)h_1\bigg(\int_0^\rho G(r_1,r_2)\psi(r_2)h_2\bigg(\int_0^\rho G(r_2,r_3)\psi(r_3)\\ &\cdots h_{\tau-1}\bigg[\int_0^\rho G(r_{\tau-1},r_\tau)\psi(r_\tau)h_\tau(\mathtt{y}_1(r_\tau))dr_\tau\bigg]\cdots dr_3\bigg)dr_2\bigg)dr_1.\end{aligned}$$

By using above two inequalities one can write,

$$\min_{x\in[\eta,\rho-\eta]}\{(\mathcal{F}\mathtt{y}_1)(x)\}\geqslant\sigma(\eta)\|(\mathcal{F}\mathtt{y}_1)\|.$$

So, $\mathcal{F}\mathtt{y}_1\in\mathcal{P}$ and $\mathcal{F}(\mathcal{P})\subset\mathcal{P}$. By using Arzela-Ascoli theorem and stranded methods it can be prove easily $\mathcal{F}$ is completely continuous. □

Theorem 3.2. (Hölder's[11]) . *Let $f\in\mathtt{L}^{\mathtt{p}_s}[0,1]$ with $\mathtt{p}_s>1$, for $s=1,2,\cdots,z$ and $\sum_{s=1}^{z}\frac{1}{\mathtt{p}_s}=1$. Then $\prod_{s=1}^{z}f_s\in\mathtt{L}^1[0,1]$ and $\left\|\prod_{s=1}^{z}f_s\right\|_1\leqslant\prod_{s=1}^{z}\|f_s\|_{\mathtt{p}_s}$. Further, if $f\in\mathtt{L}^1[0,1]$ and $g\in\mathtt{L}^\infty[0,1]$, then $fg\in\mathtt{L}^1[0,1]$ and $\|fg\|_1\leqslant\|f\|_1\|g\|_\infty$.*

Theorem 3.3. (Krasnoselskii's[9]). *Let $\mathcal{P}$ be a cone in a Banach space* B *and* λ_1, λ_2 *are open sets with* $0 \in \lambda_1, \bar{\lambda}_1 \subset \lambda_2$. *Let* $\mathcal{F} : \mathcal{P} \cap (\bar{\lambda}_2 \setminus \lambda_1) \to \mathcal{P}$ *be completely continuous operator such that*

a) $\|\mathcal{F}x\| \leqslant \|x\|,\ x \in \mathcal{P} \cap \partial\lambda_1$, *and* $\|\mathcal{F}x\| \geqslant \|x\|,\ x \in \mathcal{P} \cap \partial\lambda_2$, *or*
b) $\|\mathcal{F}x\| \geqslant \|x\|,\ x \in \mathcal{P} \cap \partial\lambda_1$, *and* $\|\mathcal{F}x\| \leqslant \|x\|,\ x \in \mathcal{P} \cap \partial\lambda_2$.
Then $\mathcal{F}$ *has a fixed point in* $\mathcal{P} \cap (\bar{\lambda}_2 \setminus \lambda_1)$.

We consider the cases for $\psi_s \in \mathtt{L}^{\mathtt{p}_s}[0,1]$:
(i) $\displaystyle\sum_{s=1}^{z} \frac{1}{\mathtt{p}_s} < 1$, (ii) $\displaystyle\sum_{s=1}^{z} \frac{1}{\mathtt{p}_s} = 1$, (iii) $\displaystyle\sum_{s=1}^{z} \frac{1}{\mathtt{p}_s} > 1$.

Firstly, we seek countably many positive symmetric solutions for the case $\displaystyle\sum_{s=1}^{z} \frac{1}{\mathtt{p}_s} < 1$.

Theorem 3.4. *Suppose that (Z1)-(Z2) hold. Let* $\{x_\ell\}_{\ell=1}^{\infty}$ *be decreasing sequence with upper bound* $\rho/2$ *and* $\{\eta_\ell\}_{\ell=1}^{\infty}$ *be a sequence with* $\eta_\ell \in (x_{\ell+1}, x_\ell)$. *Let* $\{\mathrm{R}_\ell\}_{\ell=1}^{\infty}$ *and* $\{\mathrm{M}_\ell\}_{\ell=1}^{\infty}$ *be such that*

$$\mathrm{R}_{\ell+1} < \sigma(\eta_\ell)\mathrm{M}_\ell < \mathrm{M}_\ell < \mathrm{Q}\,\mathrm{M}_\ell < \mathrm{R}_\ell,\ \ell \in \mathbb{N},$$

where

$$\mathrm{Q} = \max\left\{\left[\sigma(\eta_1)\prod_{s=1}^{z}\alpha_{\mathrm{k}}\int_{\eta_1}^{\rho-\eta_1} G(r,r)dr\right]^{-1},\ 1\right\}.$$

Further assume that h_{k} *satisfies*

(C1) $h_{\mathrm{k}}(\mathrm{y}) \leqslant \mathrm{O}_1\mathrm{R}_\ell,\ \forall\, x \in [0,\rho],\ 0 \leqslant \mathrm{y} \leqslant \mathrm{R}_\ell$,
where

$$\mathrm{O}_1 < \left[\|G\|_{\mathtt{L}^{\mathtt{q}}}\prod_{s=1}^{z}\|\psi_s\|_{\mathtt{L}^{\mathtt{p}_s}}\right]^{-1},$$

(C2) $h_{\mathrm{k}}(\mathrm{y}) \geqslant \mathrm{Q}\mathrm{M}_\ell,\ \forall\, x \in [\eta_\ell, \rho-\eta_\ell],\ \eta_\ell\mathrm{M}_\ell \leqslant \mathrm{y} \leqslant \mathrm{M}_\ell$.

Then (1.1)-(1.2) has countably many positive symmetric solutions $\big\{(\mathrm{y}_1^{[\ell]}, \mathrm{y}_2^{[\ell]}, \cdots, \mathrm{y}_\tau^{[\ell]})\big\}_{\ell=1}^{\infty}$ *such that* $\mathrm{y}_{\mathrm{k}}^{[\ell]}(x) \geqslant 0$ *on* $[0,\rho],\ \mathrm{k} = 1,2,\cdots,\tau$.

Proof. Let

$$\lambda_{1,\ell} = \big\{\mathrm{y} \in \mathrm{B} : \|\mathrm{y}\| \leqslant \mathrm{R}_\ell\big\},$$
$$\lambda_{2,\ell} = \big\{\mathrm{y} \in \mathrm{B} : \|\mathrm{y}\| \leqslant \mathrm{M}_\ell\big\},$$

be open subsets of B. From the hypothesis we can write

$$x^* < x_{\ell+1} < \eta_\ell < x_\ell < \frac{\rho}{2},\ \forall\, \ell \in \mathbb{N},$$

where $x^* = \lim_{\ell\to\infty} x_\ell$. For each $\ell \in \mathbb{N}$, define the cone $\mathcal{P}_\ell$ by

$$\mathcal{P}_\ell = \big\{\mathrm{y} \in \mathrm{B} : \mathrm{y}(x) \text{ is non negative, concave, symmetric and } \min_{x\in[\eta_\ell,\rho-\eta_\ell]} \mathrm{y}(x) \geqslant \sigma(\eta_\ell)\|\mathrm{y}\|\big\}.$$

Let $\mathrm{y}_1 \in \mathcal{P}_\ell \cap \partial\lambda_{1,\ell}$. Then, $\mathrm{y}_1(r) \leqslant \mathrm{R}_\ell = \|\mathrm{y}_1\|$ for all $r \in [0,\rho]$. By (C1) and for $r_{\tau-1} \in [0,\rho]$, we have

$$\begin{aligned}\int_0^\rho G(r_{\tau-1}, r_\tau)\psi(r_\tau)h_\tau(\mathrm{y}_1(r_\tau))dr_\tau &\leqslant \int_0^\rho G(r_\tau, r_\tau)\psi(r_\tau)h_\tau(\mathrm{y}_1(r_\tau))dr_\tau\\ &\leqslant \mathrm{O}_1\mathrm{R}_\ell\int_0^\rho G(r_\tau, r_\tau)\psi(r_\tau)dr_\tau\\ &\leqslant \mathrm{O}_1\mathrm{R}_\ell\int_0^\rho G(r_\tau, r_\tau)\prod_{s=1}^{z}\psi_s(r_\tau)dr_\tau.\end{aligned}$$

From (i), $\exists\, \mathtt{q} > 1$ such that $\dfrac{1}{\mathtt{q}} + \displaystyle\sum_{s=1}^{z} \dfrac{1}{\mathtt{p}_s} = 1.$

So,

$$\begin{aligned}\int_0^{\rho} G(r_{\tau-1}, r_\tau)\psi(r_\tau)h_\tau(\mathrm{y}_1(r_\tau))dr_\tau &\leqslant \mathrm{O}_1\mathrm{R}_\ell \|G\|_{\mathrm{L}^{\mathtt{q}}} \left\| \prod_{s=1}^{z} \psi_s \right\|_{\mathrm{L}^{\mathtt{p}_s}} \\ &\leqslant \mathrm{O}_1\mathrm{R}_\ell \|G\|_{\mathrm{L}^{\mathtt{q}}} \prod_{s=1}^{z} \|\psi_s\|_{\mathrm{L}^{\mathtt{p}_s}} \\ &\leqslant \mathrm{R}_\ell.\end{aligned}$$

Similarly for $r_{\tau-2} \in [0, \rho]$

$$\begin{aligned}&\int_0^{\rho} G(r_{\tau-2}, r_{\tau-1})\psi(r_{\tau-1})h_{\tau-1}\left[\int_0^{\rho} G(r_{\tau-1}, r_\tau)\psi(r_\tau)h_\tau(\mathrm{y}_1(r_n))dr_\tau\right]dr_{\tau-1} \\ &\qquad \leqslant \int_0^{\rho} G(r_{\tau-2}, r_{\tau-1})\psi(r_{\tau-1})h_{\tau-1}(\mathrm{R}_\ell)dr_{\tau-1} \\ &\qquad \leqslant \int_0^{\rho} G(r_{\tau-1}, r_{\tau-1})\psi(r_{\tau-1})h_{\tau-1}(\mathrm{R}_\ell)dr_{\tau-1} \\ &\qquad \leqslant \mathrm{O}_1\mathrm{R}_\ell \int_0^{\rho} G(r_{\tau-1}, r_{\tau-1})\psi(r_{\tau-1})dr_{\tau-1} \\ &\qquad \leqslant \mathrm{O}_1\mathrm{R}_\ell \int_0^{\rho} G(r_{\tau-1}, r_{\tau-1}) \prod_{s=1}^{z} \psi_s(r_{\tau-1})dr_{\tau-1} \\ &\qquad \leqslant \mathrm{O}_1\mathrm{R}_\ell \|G\|_{\mathrm{L}^{q}} \prod_{s=1}^{z} \|\psi_s\|_{\mathrm{L}^{\mathtt{p}_s}} \\ &\qquad \leqslant \mathrm{R}_\ell.\end{aligned}$$

Continue, we get

$$\begin{aligned}(\mathcal{F}\mathrm{y}_1)(x) &= \int_0^{\rho} G(x, r_1)\psi(r_1)h_1\bigg(\int_0^{\rho} G(r_1, r_2)\psi(r_2)h_2\bigg(\int_0^{\rho} G(r_2, r_3)\psi(r_3)\cdots \\ &\qquad h_{\tau-1}\bigg[\int_0^{\rho} G(r_{\tau-1}, r_\tau)\psi(r_\tau)h_\tau(\mathrm{y}_1(r_\tau))dr_\tau\bigg]\cdots dr_3\bigg)dr_2\bigg)dr_1 \\ &\leqslant \mathrm{R}_\ell.\end{aligned}$$

Since $\mathrm{R}_\ell = \|\mathrm{y}_1\|$ for $\mathrm{y}_1 \in \mathcal{P}_\ell \cap \partial\lambda_{1,\ell}$, we get

$$\|\mathcal{F}\mathrm{y}_1\| \leqslant \|\mathrm{y}_1\|. \tag{3.7}$$

Let $x \in [\eta_\ell, \rho - \eta_\ell]$. Then

$$\mathrm{M}_\ell = \|\mathrm{y}_1\| \geqslant \mathrm{y}_1(x) \geqslant \min_{x \in [\eta_\ell, \rho-\eta_\ell]} \mathrm{y}_1(x) \geqslant \sigma(\eta_\ell)\|\mathrm{y}_1\| \geqslant \sigma(\eta_\ell)\mathrm{M}_\ell.$$

By (C2) and for $r_{\tau-1} \in [\eta_\ell, \rho-\eta_\ell]$, we have

$$\begin{aligned}\int_0^\rho G(r_{\tau-1}, r_\tau)\psi(r_\tau)h_\tau(\mathrm{y}_1(r_\tau))dr_\tau &\geqslant \sigma(\eta_\ell)\int_{\eta_\ell}^{\rho-\eta_\ell} G(r_\tau, r_\tau)\psi(r_\tau)h_\tau(\mathrm{y}_1(r_\tau))dr_\tau \\ &\geqslant \sigma(\eta_\ell)\mathrm{Q}\,\mathrm{M}_\ell\int_{\eta_\ell}^{\rho-\eta_\ell} G(r_\tau, r_\tau)\psi(r_\tau)dr_\tau \\ &\geqslant \sigma(\eta_\ell)\mathrm{Q}\,\mathrm{M}_\ell\int_{\eta_\ell}^{\rho-\eta_\ell} G(r_\tau, r_\tau)\prod_{s=1}^{z}\psi_s(r_\tau)dr_\tau \\ &\geqslant \sigma(\eta_1)\mathrm{Q}\,\mathrm{M}_\ell\prod_{s=1}^{z}\alpha_s\int_{\eta_1}^{\rho-\eta_1} G(r_\tau, r_\tau)dr_\tau \\ &\geqslant \mathrm{M}_\ell.\end{aligned}$$

Then,

$$\begin{aligned}(\mathcal{F}\mathrm{y}_1)(x) = \int_0^\rho G(x, r_1)\psi(r_1)h_1\bigg(\int_0^\rho G(r_1, r_2)\psi(r_2)h_2\bigg(\int_0^\rho G(r_2, r_3)\psi(r_3)\cdots \\ h_{\tau-1}\bigg[\int_0^\rho G(r_{\tau-1}, r_\tau)\psi(r_\tau)h_\tau(\mathrm{y}_1(r_\tau))dr_\tau\bigg]\cdots dr_3\bigg)dr_2\bigg)dr_1 \\ \geqslant \mathrm{M}_\ell.\end{aligned}$$

Thus if $\mathrm{y}_1 \in \mathcal{P}_\ell \cap \partial\lambda_{2,\ell}$, then

$$\|\mathcal{F}\mathrm{y}_1\| \geqslant \|\mathrm{y}_1\|. \tag{3.8}$$

It is evident that $0 \in \partial\lambda_{2,\ell} \subset \partial\bar{\lambda}_{2,\ell} \subset \partial\lambda_{1,\ell}$. From (3.7), (3.8), by Theorem 3.3, $\mathcal{F}$ has a fixed point $\mathrm{y}_1^{[\ell]} \in \mathcal{P}_\ell \cap (\bar{\lambda}_{1,\ell} \setminus \lambda_{2,\ell}) \ni \mathrm{y}_1^{[\ell]}(x) \geqslant 0$ on $[0, \rho]$. Next setting $\mathrm{y}_{\tau+1} = \mathrm{y}_1$, we obtain countably many positive symmetric solutions $\big\{(\mathrm{y}_1^{[\ell]}, \mathrm{y}_2^{[\ell]}, \cdots, \mathrm{y}_\tau^{[\ell]})\big\}_{\ell=1}^{\infty}$ of (1.1)-(1.2) given by

$$\mathrm{y}_\mathrm{k}(x) = \int_0^\rho G(x, r)\psi(r)h_\mathrm{k}(\mathrm{y}_{\mathrm{k}+1}(r))dr,\ x \in [0, \rho],\ 1 \leqslant \mathrm{k} \leqslant \tau.$$

The proof is completed. □

For $\displaystyle\sum_{s=1}^{z}\frac{1}{\mathrm{p}_s} = 1$, we have the following theorem.

Theorem 3.5. *Suppose that (Z1)-(Z2) hold. Let $\{x_\ell\}_{\ell=1}^{\infty}$ be a decreasing sequence with upper bound $\rho/2$ and $\{\eta_\ell\}_{\ell=1}^{\infty}$ be a sequence with $\eta_\ell \in (x_{\ell+1}, x_\ell)$. Let $\{\mathrm{R}_\ell\}_{\ell=1}^{\infty}$ and $\{\mathrm{M}_\ell\}_{\ell=1}^{\infty}$ be such that*

$$\mathrm{R}_{\ell+1} < \sigma(\eta_\ell)\mathrm{M}_\ell < \mathrm{M}_\ell < \mathrm{Q}\,\mathrm{M}_\ell < \mathrm{R}_\ell,\ \ell \in \mathbb{N},$$

where

$$\mathrm{Q} = \max\bigg\{\bigg[\sigma(\eta_1)\prod_{s=1}^{z}\alpha_s\int_{\eta_1}^{\rho-\eta_1} G(r, r)dr\bigg]^{-1},\ 1\bigg\}.$$

Further assume that h_k satisfies

(C3) $h_\mathrm{k}(\mathrm{y}) \leqslant \mathrm{O}_2\mathrm{R}_\ell,\ \forall\, x \in [0, \rho],\ 0 \leqslant \mathrm{y} \leqslant \mathrm{R}_\ell,$
where
$$\mathrm{O}_2 < \min\bigg\{\bigg[\|G\|_{\mathrm{L}^\infty}\prod_{s=1}^{z}\|\psi_s\|_{\mathrm{L}^{\mathrm{p}_s}}\bigg]^{-1}, \mathrm{Q}\bigg\},$$

(C4) $h_\mathrm{k}(\mathrm{y}) \geqslant \mathrm{Q}\mathrm{M}_\ell,\ \forall\, x \in [\eta_\ell, \rho-\eta_\ell],\ \eta_\ell\mathrm{M}_\ell \leqslant \mathrm{y} \leqslant \mathrm{M}_\ell.$

Then (1.1)-(1.2) has countably many positive symmetric solutions $\big\{(\mathrm{y}_1^{[\ell]}, \mathrm{y}_2^{[\ell]}, \cdots, \mathrm{y}_\tau^{[\ell]})\big\}_{\ell=1}^{\infty}$ *such that* $\mathrm{y}_{\mathrm{k}}^{[\ell]}(x) \geqslant 0$ *on* $[0,\rho],\ 1 \leqslant \mathrm{k} \leqslant \tau.$

Proof. From the hypothesis we can write

$$x^* < x_{\ell+1} < \eta_\ell < x_\ell < \frac{\rho}{2},\ \forall\, \ell \in \mathbb{N},$$

where $x^* = \lim_{\ell\to\infty} x_\ell$. For a fixed ℓ, let $\lambda_{1,\ell}$ be as in the proof of Theorem 3.4, and let $\mathrm{y}_1 \in \mathcal{P}_\ell \cap \partial\lambda_{1,\ell}$. Again

$$\mathrm{y}_1(x) \leqslant \mathrm{R}_\ell = \|\mathrm{y}_1\|$$

for all $x \in [0,\rho]$. By (C3) and for $r_{\tau-1} \in [0,\rho]$, we have

$$\begin{aligned}
\int_0^\rho G(r_{\tau-1}, r_\tau)\psi(r_\tau)h_\tau(\mathrm{y}_1(r_\tau))dr_\tau &\leqslant \int_0^\rho G(r_\tau, r_\tau)\psi(r_\tau)h_\tau(\mathrm{y}_1(r_\tau))dr_\tau \\
&\leqslant \mathrm{Q}_2\mathrm{R}_\ell \int_0^\rho G(r_\tau, r_\tau)\psi(r_\tau)dr_\tau \\
&\leqslant \mathrm{Q}_2\mathrm{R}_\ell \int_0^\rho G(r_\tau, r_\tau)\prod_{s=1}^{z}\psi_s(r_\tau)dr_\tau \\
&\leqslant \mathrm{Q}_2\mathrm{R}_\ell\|G\|_{\mathrm{L}^\infty}\prod_{s=1}^{z}\|\psi_s\|_{\mathrm{L}^{\mathrm{p}_s}} \\
&\leqslant \mathrm{R}_\ell.
\end{aligned}$$

Similarly for $r_{\tau-2} \in [0,\rho]$

$$\begin{aligned}
&\int_0^\rho G(r_{\tau-2}, r_{\tau-1})\psi(r_{\tau-1})h_{\tau-1}\Big[\int_0^\rho G(r_{\tau-1}, r_\tau)\psi(r_\tau)h_\tau(\mathrm{y}_1(r_\eta))dr_\tau\Big]dr_{\tau-1} \\
&\qquad\leqslant \int_0^\rho G(r_{\tau-2}, r_{\tau-1})\psi(r_{\tau-1})h_{\tau-1}(\mathrm{R}_\ell)dr_{\tau-1} \\
&\qquad\leqslant \int_0^\rho G(r_{\tau-1}, r_{\tau-1})\psi(r_{\tau-1})h_{\tau-1}(\mathrm{R}_\ell)dr_{\tau-1} \\
&\qquad\leqslant \mathrm{Q}_2\mathrm{R}_\ell\int_0^\rho G(r_{\tau-1}, r_{\tau-1})\psi(r_{\tau-1})dr_{\tau-1} \\
&\qquad\leqslant \mathrm{Q}_2\mathrm{R}_\ell\int_0^\rho G(r_{\tau-1}, r_{\tau-1})\prod_{s=1}^{z}\psi_s(r_{\tau-1})dr_{\tau-1} \\
&\qquad\leqslant \mathrm{Q}_2\mathrm{R}_\ell\|G\|_{\mathrm{L}^\infty}\prod_{s=1}^{z}\|\psi_s\|_{\mathrm{L}^{\mathrm{p}_s}} \\
&\qquad\leqslant \mathrm{R}_\ell.
\end{aligned}$$

Continue, we get

$$\begin{aligned}
(\mathcal{F}\mathrm{y}_1)(x) = &\int_0^\rho G(x, r_1)\psi(r_1)h_1\bigg(\int_0^\rho G(r_1, r_2)\psi(r_2)h_2\bigg(\int_0^\rho G(r_2, r_3)\psi(r_3)\cdots \\
&h_{\tau-1}\Big[\int_0^\rho G(r_{\tau-1}, r_\tau)\psi(r_\tau)h_\tau(\mathrm{y}_1(r_\tau))dr_\tau\Big]\cdots dr_3\bigg)dr_2\bigg)dr_1 \\
\leqslant &\mathrm{R}_\ell.
\end{aligned}$$

Since $\mathrm{R}_\ell = \|\mathrm{y}_1\|$ for $\mathrm{y}_1 \in \mathcal{P}_\ell \cap \partial\lambda_{1,\ell}$, we get

$$\|\mathcal{F}\mathrm{y}_1\| \leqslant \|\mathrm{y}_1\|. \tag{3.9}$$

Now define $\lambda_{2,\ell} = \{y_1 \in B : \|y_1\| < M_\ell\}$. Let $y_1 \in \mathcal{P}_\ell \cap \partial\lambda_{2,\ell}$, and let $x \in [\eta_\ell, \rho - \eta_\ell]$. Then, the argument leading to (3.8) can be followed to the present case. Hence, the result. □

Lastly, consider the case $\displaystyle\sum_{s=1}^{z} \frac{1}{p_s} > 1$.

Theorem 3.6. *Suppose that (Z1)-(Z2) hold. Let $\{x_\ell\}_{\ell=1}^{\infty}$ be a decreasing sequence with upper bound $\rho/2$ and $\{\eta_\ell\}_{\ell=1}^{\infty}$ be a sequence with $\eta_\ell \in (x_{\ell+1}, x_\ell)$. Let $\{R_\ell\}_{\ell=1}^{\infty}$ and $\{M_\ell\}_{\ell=1}^{\infty}$ be such that*

$$R_{\ell+1} < \sigma(\eta_\ell)M_\ell < M_\ell < Q\, M_\ell < R_\ell,\ \ell \in \mathbb{N},$$

where

$$Q = \max\left\{ \left[\sigma(\eta_1) \prod_{s=1}^{z} \alpha_s \int_{\eta_1}^{\rho-\eta_1} G(r,r)dr \right]^{-1},\ 1 \right\}.$$

Further assume that h_k satisfies

(C5) $h_k(y) \leqslant O_3 R_\ell,\ \forall\, x \in [0,\rho],\ 0 \leqslant y \leqslant R_\ell$,
where

$$O_3 < \min\left\{ \left[\|G\|_{L^\infty} \prod_{s=1}^{z} \|\psi_s\|_{L^1} \right]^{-1}, Q \right\},$$

(C6) $h_k(y) \geqslant QM_\ell,\ \forall\, x \in [\eta_\ell, \rho - \eta_\ell],\ \eta_\ell M_\ell \leqslant y \leqslant M_\ell$.

Then (1.1)-(1.2) has countably many positive symmetric solutions $\{(y_1^{[\ell]}, y_2^{[\ell]}, \cdots, y_\eta^{[\ell]})\}_{\ell=1}^{\infty}$ such that $y_k^{[\ell]}(x) \geqslant 0$ on $[0,\rho]$, $1 \leqslant k \leqslant \tau$ and $l \in \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 3.5. Therefore, we omit the details here.

□

4. Example

In this section, we give an example to illustrate our main results.
Consider the following BVP

$$\begin{cases} y_k''(x) + \psi(x)h_k(y_{k+1}(x)) = 0,\ 0 \leqslant x \leqslant 1, k = 1,2, \\ y_3(x) = y_1(x),\ 0 \leqslant x \leqslant 1, \end{cases} \tag{4.10}$$

$$\begin{cases} y_k(0) = 2\, y_k'(0), \\ y_k(1) = -2\, y_k'(1), \end{cases} \tag{4.11}$$

where

$$\psi(x) = \psi_1(x)\psi_2(x)$$

in which $\psi_1(x) = \dfrac{1}{[|x - \frac{1}{2}| + \frac{2}{3}]^{\frac{1}{2}}}$ and $\psi_2(x) = \dfrac{1}{[|x - \frac{1}{2}| + \frac{1}{11}]^{\frac{1}{2}}}$

$$h_1(\mathtt{y}) = h_2(\mathtt{y}) = \begin{cases} \dfrac{2}{21} \times 10^{-11}, \ \mathtt{y} \in (10^{-11}, \infty), \\ \dfrac{37 \times 10^{-(11\ell+5)} - \frac{2}{21} \times 10^{-(11\ell)}}{10^{-(11\ell+5)} - 10^{-(11\ell)}}(\mathtt{y} - 10^{-(11\ell)}) + \dfrac{2}{21} \times 10^{-(11\ell)}, \\ \qquad \mathtt{y} \in [10^{-(11\ell+5)}, 10^{-(11\ell)}], \\ 37 \times 10^{-(11\ell+5)}, \ \mathtt{y} \in (\dfrac{1}{5} \times 10^{-(11\ell+5)}, 10^{-(11\ell+5)}), \\ \dfrac{37 \times 10^{-(11\ell+5)} - \frac{2}{21} \times 10^{-(11\ell+7)}}{\frac{1}{5} \times 10^{-(11\ell+5)} - 10^{-(11\ell+7)}}(\mathtt{y} - 10^{-(11\ell+7)}) + \dfrac{2}{21} \\ \times 10^{-(11\ell+7)}, \ \mathtt{y} \in [10^{-(11\ell+7)}, \dfrac{1}{5} \times 10^{-(11\ell+5)}], \\ 0, \ \mathtt{y} = 0. \end{cases}$$

$$\text{Let, } x_\ell = \frac{23}{57} - \sum_{\mathtt{i}=1}^{\ell} \frac{1}{3(\mathtt{i}+1)^6}, \ \eta_\ell = \frac{1}{2}(x_\ell + x_{\ell+1}), \ \ell = 1, 2, 3, \cdots,$$

then,

$$\eta_1 = \frac{1453}{3648} - \frac{1}{4374} < \frac{1453}{3648}$$

and

$$x_{\ell+1} < \eta_\ell < x_\ell, \ \eta_\ell > \frac{1}{5}, \ \ell = 1, 2, 3, \cdots.$$

It is clear that,

$$x_1 = \frac{1453}{3648} < \frac{1}{2}, \ x_\ell - x_{\ell+1} = \frac{1}{3(\ell+2)^6}, \ \ell = 1, 2, 3, \cdots.$$

Since $\displaystyle\sum_{\mathtt{i}=1}^{\infty} \frac{1}{\mathtt{i}^2} = \frac{\pi^2}{6}$ and $\displaystyle\sum_{\mathtt{i}=1}^{\infty} \frac{1}{\mathtt{i}^6} = \frac{\pi^6}{945}$, it follows that

$$x^* = \lim_{\ell\to\infty} x_\ell = \frac{23}{57} - \sum_{\mathtt{i}=1}^{\infty} \frac{1}{3(\mathtt{i}+1)^6} = \frac{14}{19} - \frac{\pi^6}{2835} > \frac{1}{5}.$$

Let $\psi_1, \psi_2 \in \mathtt{L}^{\mathtt{p}}[0,1]$, for $1.1 \leqslant \mathtt{p} \leqslant 200$. Since $0.9 \leqslant \psi_1(x) \leqslant 1.3, 1.3 \leqslant \psi_2(x) \leqslant 3.4$, $0 \leqslant x \leqslant 1$. So $\exists\, \alpha_s \in \mathbb{R} \ni \alpha_s < \psi_s < \infty$. Let $\alpha_1 = \alpha_2 = 1/2$, then

$$\int_{\eta_1}^{1-\eta_1} G(r,r)dr = \int_{\frac{1058629}{2659392}}^{1-\frac{1058629}{2659392}} G(r,r)dr = 0.2546792670.$$

We get

$$\begin{aligned} \mathrm{Q} &= \max\left\{ \left[\sigma(\eta_1) \prod_{s=1}^{z} \alpha_s \int_{\eta_1}^{1-\eta_1} G(r,r)dr \right]^{-1}, \ 1 \right\} \\ &= \max\{19.64832215, 1\} \\ &= 19.64832215. \end{aligned}$$

Case 1:

$$\|G\|_{\mathtt{L}^{\mathtt{q}}} = \left[\int_0^1 |G(r,r)|^{\mathtt{q}} dr \right]^{\frac{1}{\mathtt{q}}} < 1.3, \text{ for any } \mathtt{q} \in (1,2].$$

Since $\psi_1, \psi_2 \in \mathtt{L}^{\mathtt{p}}[0,1]$, it follows that

$$0.2104647218 \leqslant \left[\|G\|_{\mathtt{L}^{\mathtt{q}}} \prod_{s=1}^{z} \|\psi_s\|_{\mathtt{L}^{\mathtt{p}_s}} \right]^{-1} \leqslant 0.4114451085.$$

Taking $\mathrm{O}_1 = 0.1022008$, in addition if we take

$$\mathrm{R}_\ell = 10^{-11\ell},\ \mathrm{M}_\ell = 10^{-(11\ell+5)},$$

then,

$$\mathrm{R}_{\ell+1} = 10^{-(11\ell+11)} < \frac{1}{5} \times 10^{-(11\ell+5)} < \eta_\ell \mathrm{M}_\ell < \mathrm{M}_\ell = 10^{-(11\ell+5)} < \mathrm{R}_\ell = 10^{-11\ell},$$

$$\mathrm{Q}\ \mathrm{M}_\ell = 19.64832215 \times 10^{-(11\ell+5)} < 0.1022008 \times 10^{-11\ell} = \mathrm{O}_1\mathrm{R}_\ell,\ \ell = 1, 2, 3, \cdots,$$

and h_1, h_2 satisfies the following growth conditions,

$$h_1(\mathrm{y}) = h_2(\mathrm{y}) \leqslant \mathrm{O}_1\ \mathrm{R}_\ell = 0.1022008 \times 10^{-11\ell},\ \mathrm{y} \in [0, 10^{-11\ell}]$$

$$h_1(\mathrm{y}) = h_2(\mathrm{y}) \geqslant \mathrm{Q}\ \mathrm{M}_\ell = 19.64832215 \times 10^{-(11\ell+5)},\ \mathrm{y} \in \left[\frac{1}{5} \times 10^{-(11\ell+5)}, 10^{-(11\ell+5)}\right]$$

for $\ell \in \mathbb{N}$. Hence all the conditions in Theorem 3.4, are satisfied. The BVP (4.10)-(4.11) has countably many positive symmetric solutions $\{(\mathrm{y}_1^{[\ell]}, \mathrm{y}_2^{[\ell]})\}_{\ell=1}^{\infty}$ such that $\mathrm{y}_k^{[\ell]}(x) \geqslant 0$ on $[0, 1],\ k = 1, 2$ and $\ell \in \mathbb{N}$.

Case 2:

$$0.2341835419 \leqslant \left[\|G\|_{\mathrm{L}^\infty} \prod_{s=1}^{z} \|\psi_s\|_{\mathrm{L}^{\mathrm{p}_s}}\right]^{-1} \leqslant 0.3889529640.$$

Taking $\mathrm{O}_2 = 0.1589454$ then,

$$\mathrm{R}_{\ell+1} = 10^{-(11\ell+11)} < \frac{1}{5} \times 10^{-(11\ell+5)} < \eta_\ell \mathrm{M}_\ell < \mathrm{M}_\ell = 10^{-(11\ell+5)} < \mathrm{R}_\ell = 10^{-11\ell},$$

$$\mathrm{Q}\ \mathrm{M}_\ell = 19.64832215 \times 10^{-(11\ell+5)} < 0.1022008 \times 10^{-11\ell} = \mathrm{O}_2\mathrm{R}_\ell,\ \ell = 1, 2, 3, \cdots,$$

and h_1, h_2 satisfies the following growth conditions,

$$h_1(\mathrm{y}) = h_2(\mathrm{y}) \leqslant \mathrm{O}_2\ \mathrm{R}_\ell = 0.1022008 \times 10^{-11\ell},\ \mathrm{y} \in [0, 10^{-11\ell}]$$

$$h_1(\mathrm{y}) = h_2(\mathrm{y}) \geqslant \mathrm{Q}\ \mathrm{M}_\ell = 19.64832215 \times 10^{-(11\ell+5)},\ \mathrm{y} \in \left[\frac{1}{5} \times 10^{-(11\ell+5)}, 10^{-(11\ell+5)}\right]$$

for $\ell \in \mathbb{N}$. Hence all the conditions in Theorem 3.5, are satisfied. The BVP (4.10)-(4.11) has countably many positive symmetric solutions $\{(\mathrm{y}_1^{[\ell]}, \mathrm{y}_2^{[\ell]})\}_{\ell=1}^{\infty}$ such that $\mathrm{y}_k^{[\ell]}(x) \geqslant 0$ on $[0, 1],\ k = 1, 2$ and $\ell \in \mathbb{N}$.

Case 3:

$$\left[\|G\|_{\mathrm{L}^\infty} \prod_{s=1}^{z} \|\psi_s\|_{\mathrm{L}^1}\right]^{-1} \leqslant 0.4059591736.$$

Taking $\mathrm{O}_3 = 0.1845561$ then,

$$\mathrm{R}_{\ell+1} = 10^{-(11\ell+11)} < \frac{1}{5} \times 10^{-(11\ell+5)} < \eta_\ell \mathrm{M}_\ell < \mathrm{M}_\ell = 10^{-(11\ell+5)} < \mathrm{R}_\ell = 10^{-11\ell},$$

$$\mathrm{Q}\ \mathrm{M}_\ell = 19.64832215 \times 10^{-(11\ell+5)} < 0.184556 \times 10^{-11\ell} = \mathrm{O}_3\mathrm{R}_\ell,\ \ell = 1, 2, 3, \cdots,$$

and h_1, h_2 satisfies the following growth conditions,

$$h_1(\mathrm{y}) = h_2(\mathrm{y}) \leqslant \mathrm{O}_3\ \mathrm{R}_\ell = 0.184556 \times 10^{-11\ell},\ \mathrm{y} \in [0, 10^{-11\ell}]$$

$$h_1(\mathrm{y}) = h_2(\mathrm{y}) \geqslant \mathrm{Q}\ \mathrm{M}_\ell = 19.64832215 \times 10^{-(11\ell+5)},\ \mathrm{y} \in \left[\frac{1}{5} \times 10^{-(11\ell+5)}, 10^{-(11\ell+5)}\right]$$

for $\ell \in \mathbb{N}$. Hence all the conditions in Theorem 3.6, are satisfied. The BVP (4.10)-(4.11) has countably many positive symmetric solutions $\{(\mathrm{y}_1^{[\ell]}, \mathrm{y}_2^{[\ell]})\}_{\ell=1}^{\infty}$ such that $\mathrm{y}_k^{[\ell]}(x) \geqslant 0$ on

$[0,1],\ k=1,2$ and $\ell \in \mathbb{N}$.

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