CREAT. MATH. INFORM.
Volume **33** (2024), No. 2,
Pages 281 - 287

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2024.02.13

Fixed point theorem for composition of multivalued and single-valued mappings in Banach spaces

T. M. M. SOW

ABSTRACT. In this paper, we present a new characterization for finding a solution of common fixed points problem involving multivalued and single-valued mappings in Banach spaces by applying the properties of composition without commuting assumption. We equally obtain a convergence result to a solution of system of inclusion problems in Banach spaces. Our results unify, generalize and complement various known comparable results from the current literature.

1. INTRODUCTION

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to be *smooth*, if the limit

$$\lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S$. Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\right\},$$

where $\langle ., .\rangle$ denotes the generalized duality pairing. J_2 is called the *normalized duality mapping* and is denoted by J. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by:

$$\delta_E(\epsilon) := \inf\Big\{1 - \frac{1}{2}\|x+y\| \ : \|x\| = \|y\| = 1, \ \|x-y\| \geq \epsilon\Big\}.$$

A normed linear space E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. For $p > 1$, E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \leq c\epsilon^p$ for all $\epsilon \in (0, 2]$. Observe that every p-uniformly convex space is uniformly convex.

Typical examples of such spaces are the L_p, l_p and ${W_p}^m$ spaces for $1 < p < \infty$ where, L_p (or l_p) or ${W_p}^m$ is

(1) 2-uniformly smooth and and p-uniformly convex if $2 \leq p < \infty$;
(2) 2-uniformly convex and and p-uniformly smooth if $2 < p < 2$.

It is well known that E is smooth if and only if J is single valued.

Received: 24.09.2023. In revised form: 19.02.2024. Accepted: 08.04.2024
2000 *Mathematics Subject Classification.* 47H06,49J20,49J53.
Key words and phrases. *Common fixed points, Composition operators, Multivalued mppapping, Single-valued mappins.*

Let K be a nonempty subset of a normed space E. The Pompeiu *Hausdorff metric* on $CB(K)$ is defined by:

$$H(A,B)=\max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

for all $A,B\in CB(K)$ (see, Berinde [3]). A multi-valued mapping $T:D(T)\subseteq E\to CB(E)$ is called *β-Lipschitzian* if there exists $\beta>0$ such that

$$H(Tx,Ty)\leq \beta\|x-y\|\ \ \forall x,y\in D(T). \tag{1.1}$$

When $\beta\in(0,1)$, we say that T is a *contraction*, and T is called *nonexpansive* if $\beta=1$. An element $x\in K$ is called a fixed point of T if $x\in Tx$. For single valued mapping, this reduces to $Tx=x$. The fixed point set of T is denoted by $Fix(T):=\{x\in D(T): x\in Tx\}$. For several years, the study of common fixed point problems involving multivalued and singlevalued mappings has attracted, and continues to attract, the interest of several well known mathematicians (see, for example, [7, 9, 8, 11]). Interest in such studies stems, perhaps, mainly from the usefulness of such fixed point theory in real-world applications, such as in Game Theory and Market Economy and in other areas of mathematics, such as in Non-Smooth Differential Equations and Differential Inclusions, Optimization theory. We describe briefly the connection of fixed point theory for multi-valued mappings with these applications.

1.1. **Optimization problems with constraints.** Let $f:H\to\mathbb{R}\cup\{+\infty\}$ be a proper convex lower semicontinuous function and $A:H\to H$ be a single-valued mapping. Consider the following optimization problem:

$$(P)\quad \begin{cases} \min f(x) \\ 0=Ax. \end{cases}$$

It is known that the multivalued map, ∂f the subdifferential of f, is maximal monotone, where for $x,w\in\ H$,

$$w\in\ \partial f(x)\Leftrightarrow f(y)-f(x)\geq\langle y-x,w\rangle,\ \ \forall\ \ y\in\ H$$
$$\Leftrightarrow x\in\ \text{argmin}(f-\langle\cdot,w\rangle).$$

It is easily seen that, for $x\in\ H$ with x is a solution of (P) if and only if

$$x\in\ Fix(T_1)\cap Fix(T_2),$$

with $T_1:=I-\partial f$ and $T_2:=I-A$, where I where I is the identity map of H. Therefore, x is a solution of (P) if and only if x is a solution of common fixed point problem involving multivalued and single-valued mappings.

Mustafa and Sims [6] introduced the G-metric spaces as a generalization of the notion of metric spaces.

Definition 1.1. [6] Let X be a non-empty set, $G:X\times X\times X\to\mathbb{R}^+$ be a function satisfying the following properties

(1) $G(x,y,z)=0$ if $x=y=z$,
(2) $0<G(x,x,y)$ for all $x,y\in X$ with $x\neq y$,
(3) $G(x,x,y)\leq G(x,y,z)$ for all $x,y,z\in X$ with $y\neq z$,
(4) $G(x,y,z)=G(x,z,y)=G(y,z,x)$ (symmetry in all three variables),
(5) $G(x,y,z)\leq G(x,a,a)+G(a,y,z)$ for all $x,y,z,a\in X$ (rectangle inequality).
Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X,G) is called a G-metric space.

Recently, N. Tahat et al. [12], proved the following theorem for common fixed points problem involving single-valued and multivalued maps in G-metric spaces.

Theorem 1.1. [12] *Let (X, G) be a G-metric space. Set $g : X \to X$ and $T : X \to CB(X)$. Assume that there exists a function a $\alpha[0, +\infty) \to [0, 1)$ satisfying $\limsup_{r\to t} \alpha(r) < 1$ for every $t \geq 0$ such that*

$$H_G(Tx, Ty, Tz) \leq \alpha(G(gx, gy, gz))G(gx, gy, gz),$$

for all $x, y, z \in X$. If for any $x \in X, Tx \subseteq g(X)$ and $g(X)$ is a G-complete subspace of X, then g and T have a point of coincidence in X. Furthermore, if we assume that $gp \in Tp$ and $gq \in Tq$ implies $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$, then

(1) g and T have a unique point of coincidence.
(2) If in addition g and T are weakly compatible, then g and T have a unique common fixed point.

Remark 1.1. Most existing results for solving common fixed points problem require that the operators of underlying operators must be commuting and also, the intersection of the fixed point sets $Fix(T_1) \cap Fix(T_2)$ must be nonempty.

Motivated and inspired by the above work, we propose a new appraoch for solving common fixed points problem involving multivalued and single-valued mappings by using composition properties in Banach spaces. As application, we use our new results and a modified Mann algorithme for solving system of inclusion problems in Banach space.

2. Preliminaries

Let E be a smooth real Banach space with dual space E^*. We introduce the the Lyapunov functional $\phi : E \times E \to \mathbb{R}$, defined by,

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \ \ \forall\, x, y \in E. \tag{2.2}$$

It was introduced by Alber in [1] and has been studied by Alber and Guerre-Delabriere [2], Kamimura and Takahashi [5] and a host of other authors. Note that if $E = H$, a real Hilbert space, then the normalized duality map J is the identity map. Hence, equation (2.2) reduces to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$.
In the sequel, the following result will be useful.

Lemma 2.1. [1] *For $p > 1$, let E be a p-uniformly convex real Banach space and let S be a bounded subset of E. Then, there exists $\alpha > 0$ such that :*

$$\alpha\|x - y\|^p \leq \phi(x, y) \ \ \forall\, x, y \in S.$$

The following definition contains the nonlinear mapping we are working with and that will appear throughout the entire paper.

Definition 2.2. Let E be a smooth real Banach space and $T : D(T) \subset E \to E$, then T is said to be firmly nonexpansive if for all $x, y \in D(T)$, we have

$$\|Tx - Ty\|^2 \leq \langle x - y, J(Tx - Ty)\rangle.$$

The resolvent operator has the following properties:

Lemma 2.2. [4] *For any $r > 0$,*
(i) A is accretive if and only if the resolvent J_r^A of A is single-valued and firmly nonexpansive;
(ii) A is m-accretive if and only if J_r^A of A is single-valued and firmly nonexpansive and its domain is the entire E;
(iii) $0 \in A(x^*)$ if and only if $x^* \in F(J_r^A)$, where $F(J_r^A)$ denotes the fixed-point set of J_r^A.

Lemma 2.3 (Song and Cho [10]). *Let K be a nonempty subset of a real Banach space and $T : K \to P(K)$ be a multi-valued map. Then the following are equivalent:*
(i) $x^* \in Fix(T)$;
(ii) $P_T(x^*) = \{x^*\}$;
(iii) $x^* \in Fix(P_T)$. *Moreover,* $Fix(T) = Fix(P_T)$.

3. Main results

We now state and prove the following theorem.

Theorem 3.2. *For $p > 1$, let E be a p-uniformly convex smooth real Banach space and K be a closed, bounded, convex set in E. Let $T_1 : K \to K$ be a firmly nonexpansive mapping and $T_2 : K \to CB(K)$ be a multivalued nonexpansive mapping such that $T_2p = \{p\}\, \forall\, p \in Fix(T_2)$ and $Fix(T_2) \cap Fix(T_1) \neq \emptyset$. Then, $Fix(T_2) \cap Fix(T_1) = Fix(T_2 \circ T_1)$ and $T_2 \circ T_1$ is a multivalued nonexpansive mapping on K.*

Proof. First, we observe that $Fix(T_2) \cap Fix(T_1) \subseteq Fix(T_2 \circ T_1)$. Let $p \in Fix(T_2) \cap Fix(T_1)$ and $q \in Fix(T_2 \circ T_1)$. By using properties of T_2, we have

$$\begin{aligned} \|q-p\|^2 &\leq H(T_2 \circ T_1 q, T_2 p)^2 \\ &\leq \|T_1 q - p\|^2. \end{aligned} \tag{3.3}$$

Using the fact that T_1 is firmly nonexpansive, we have

$$\|T_1q - p\|^2 \leq \langle q-p, J(T_1q-p)\rangle. \tag{3.4}$$

Furthermore, using properties of Lyapunov function, we have

$$\phi(q-p, T_1q-p) = \|q-p\|^2 - 2\langle q-p, J(T_1q-p)\rangle + \|T_1q-p\|^2.$$

Hence,

$$\langle q-p, J(T_1q-p)\rangle = \frac{1}{2}\Big(\|q-p\|^2 + \|T_1q-p\|^2 - \phi(q-p, T_1q-p)\Big). \tag{3.5}$$

Using (3.4) and (3.5), we obtain

$$\|T_1q-p\|^2 \leq \|q-p\|^2 - \phi(q-p, T_1q-p). \tag{3.6}$$

From (3.3), we have

$$\phi(q-p, T_1q-p) \leq 0.$$

By lemma 2.1, we have $\|T_1q - q\| = 0$ which implies that

$$q = T_1q.$$

We obtain,

$$q = T_1q \in T_2 \circ T_1 q = T_2 q.$$

Thus, $q \in Fix(T_2) \cap Fix(T_1)$. Hence, $Fix(T_2) \cap Fix(T_1) = Fix(T_2 \circ T_1)$.
Next, we show $T_1 \circ T_1$ is a nonexpansive mapping on K. Let $x \in K$ and $y \in K$. We observe that,

$$\begin{aligned} H(T_2 \circ T_1 x, T_2 \circ T_1 y) &\leq \|T_1x - T_1y\| \\ &\leq \|x-y\|. \end{aligned}$$

This completes the proof. □

Corollary 3.1. *Let $E = L_p$, $2 \leq p < \infty$ and K be a closed, bounded, convex set in E. Let $T_1 : K \to K$ be a firmly nonexpansive mapping and $T_2 : K \to CB(K)$ be a multivalued nonexpansive mapping such that $T_2p = \{p\}\, \forall\, p \in Fix(T_2)$ and $Fix(T_2) \cap Fix(T_1) \neq \emptyset$. Then, $Fix(T_2) \cap Fix(T_1) = Fix(T_2 \circ T_1)$ and $T_2 \circ T_1$ is a multivalued nonexpansive mapping on K.*

Proof. Since L_p-spaces, $2 \leq p < \infty$ are p-uniformly convex smooth, then the proof follows from Theorem 3.2. □

Corollary 3.2. *For $p > 1$, let E be a p-uniformly convex smooth real Banach space and K be a closed, bounded, convex set in E. Let $T_1 : K \to K$ be a firmly nonexpansive mapping and $T_2 : K \to K$ be a nonexpansive mapping such that $Fix(T_2) \cap Fix(T_1) \neq \emptyset$. Then, $Fix(T_2) \cap Fix(T_1) = Fix(T_2 \circ T_1)$ and $T_2 \circ T_1$ is a nonexpansive mapping on K.*

Proof. Since single-valued nonexpansive mappings is a particular case of multivalued nonexpansive mappings, then the proof follows from Theorem 3.2. □

Now, using the similar arguments as in the proof of Theorem 3.2 and Lemma 2.3, we obtain the following result by replacing $T_2 \circ T_1$ by $P_{T_2} \circ T_1$ and removing the rigid restriction on $Fix(T_2)$ $(T_2 p = \{p\},\ \forall\, p \in Fix(T_2))$.

Theorem 3.3. *For $p > 1$, let E be a p-uniformly convex smooth real Banach space and K be a closed, bounded, convex set in E. Let $T_1 : K \to K$ be a firmly nonexpansive mapping and $T_2 : K \to CB(K)$ be a multivalued mapping such that P_{T_2} is nonexpansive and $Fix(T_2) \cap Fix(T_1) \neq \emptyset$. Then, $Fix(T_2) \cap Fix(T_1) = Fix(P_{T_2} \circ T_1)$ and $P_{T_2} \circ T_1$ is a multivalued nonexpansive mapping on K.*

4. Application

Solving system of inclusion nonlinear problems in any Banach space (real or complex nonlinear equations, nonlinear systems, and nonlinear matrix equations, among others) is a non-trivial task that involves many areas of science and technology. Usually the solution is not directly affordable and requires an approach utilizing iterative algorithms. This is an area of research that has grown exponentially over the last few years.

Problem 4.1. *Let K be a nonempty, closed convex subset of a real Banach space E. We consider the following monotone inclusion problem :*

$$\text{find } x \in K \ \text{ such that } 0 \in Ax, \tag{4.7}$$

where A be a set-valued mapping.

We denote the set of solutions of Problem 4.1 by Ω_1.

Problem 4.2. *We also consider the following inclusion fixed point problem :*

$$\text{find } x \in K \ \text{ such that } x \in Tx, \tag{4.8}$$

where $T : K \to CB(K)$ be a multivalued mapping.

We denote the set of solutions of Problem 4.1 by Ω_2. Recently, Sow [11] motivated by the fact that Mann algorithm method is remarkably useful for finding fixed points of nonlinear mappings, proved the following theorem.

Theorem 4.4. *Let E be a uniformly convex real Banach space having a weakly continuous duality map J_φ and K be a nonempty, closed and convex cone of E. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping nonexpansive mapping with convex-values such that $Fix(T) \neq \emptyset$ and $Tp = \{p\}, \forall p \in Fix(T)$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) v_n, \ v_n \in Tx_n, \\ \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n) y_n \end{cases} \tag{4.9}$$

$\{\beta_n\}$, $\{\lambda_n\}$ and $\{\alpha_n\}$ be sequences in $(0,1)$ satisfying:
(i) $\lim\limits_{n\to\infty}\alpha_n = 0$; (ii) $\beta_n \in [a,b] \subset (0,1)$,
(iii) $\lim\limits_{n\to\infty}\lambda_n = 1$ *and* $\sum\limits_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.
Then, the sequence $\{x_n\}$ generated by (4.10) *converges strongly to $x^* \in Fix(T)$.*

We now state and prove the following theorem.

Theorem 4.5. *Assume that $E = l_p$, $2 \le p < \infty$. Let K be a closed, bounded, convex cone set in E. Let $A : D(A) \subset K \to 2^E$ be a accretive operator such that $\overline{D(A)} \subset K \subset \bigcap\limits_{r>0} R(I + rA)$ and $T : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values such that $Tp = \{p\}\,\forall\, p \in Fix(T)$ and Such that $\Omega_1 \cap \Omega_2 \ne \emptyset$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)v_n, \;\; v_n \in T \circ J_r^A x_n, \\ \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n \end{cases} \tag{4.10}$$

$\{\beta_n\}$, $\{\lambda_n\}$ and $\{\alpha_n\}$ be sequences in $(0,1)$ satisfying:
(i) $\lim\limits_{n\to\infty}\alpha_n = 0$; (ii) $\beta_n \in [a,b] \subset (0,1)$,
(iii) $\lim\limits_{n\to\infty}\lambda_n = 1$ *and* $\sum\limits_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.
Then, the sequence $\{x_n\}$ generated by (4.10) *converges strongly to a common solution of Problems 4.1 and 4.2.*

Proof. From Theorem 3.2 and Lemma 2.2, $T \circ J_r^A$ is nonexpansive on K and $\Omega_1 \cap \Omega_2 = Fix(T \circ J_r^A) = Fix(T) \cap Fix(J_r^A)$. Since l_p spaces, $2 \le p < \infty$ have weakly continuous duality map, it follows Theorem 4.4 that $\{x_n\}$ converges strongly to some point $x^* \in Fix(T \circ J_r^A) \iff x^* \in \Omega_1 \cap \Omega_2$, completing the proof. □

We now give example of mappings T_1 and T_2 satisfying the assumptions of Theorem 3.2. Let $H = \mathbb{R}$ and $K = [1,7]$. For each $x \in K$ we define $F : K \to (-\infty,\infty]$ by $F(x) := \frac{1}{2}\|x-1\|^2$ and define a mapping $T : K \to CB(K)$ by

$$T_2 x = \begin{cases} \{1\}, \;\; x \in [1,4], \\ \\ \left[1, \dfrac{2x^2+1}{x^2+1}\right], \;\; x \in (4,7]. \end{cases} \tag{4.11}$$

It can easily be seen that $T_1 := J_\lambda^F x$ and T_2 are satisfied the conditions in Theorem 3.2 and $Fix(T_2) \cap \operatorname{argmin}_{u\in K} F(u) = Fix(T_2 \circ T_1) = \{1\}$.

References

[1] Alber, Ya. *Metric and generalized projection operator in Banach space:properties and applications*, In Theory and Applications of Nonlinear Operators of Accretive Monotone Type(A. G Kartsatos, Ed.), Marcel Dekker, New York (1996), 15–50.

[2] Alber, Ya.; Guerre-Delabiere, S. On the projection methods for fixed point problems. *Analysis (Munich)* **21** (2001), no. 1, 17–39.

[3] Berinde, V.; Păcurar, M. The role of the Pompeiu-Hausdorff metric in fixed point theory. *Creat. Math. Inform.* **22** (2013), no. 2, 143–150.

[4] Goebel, K.; Reich, S. *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings* Marcel Dekker, New York, 1984.

[5] Kamimura, S.; Takahashi, W. Strong convergence of proximal-type algorithm in Banach space. *SIAM J. Optim.* **13** (2002), no. 3, 938–945.

[6] Mustafa, Z.; Sims, B. A new approach to generalized metric spaces. *J Nonlinear Convex Anal.* **7** (2006), 289–297.

[7] Nadler Jr., S. B. Multivalued contraction mappings. *Pacific J. Math.* **30** (1969), 475–488.

[8] Panyanak, B. Endpoints of multivalued nonexpansive mappings in geodesic spaces. *Fixed Point Theory Appl.* 2015, 2015:147, 11 pp.

[9] Sastry, K. P. R.; Babu, G. V. R. Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point. *Czechoslovak Math. J.* **55** (2005), 817–826.

[10] Song, Y.; Cho, Y. J. Some notes on Ishikawa iteration for multi-valued mappings. *Bull. Korean Math. Soc.* **48** (2011), no. 3, 575–584.

[11] Sow, T. M. M. A modified Krasnosel'skii-Mann algorithms for computing fixed points of multivalued quasi-nonexpansive mappings in Banach spaces. *Creat. Math. Inform.* **28** (2019), no. 2, 191–198.

[12] Tahat, N.; Aydi, H.; Karapinar, E.; Shatanawi, W. Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in G-metric spaces. *Fixed Point Theory and Applications.* 2012, 1–9.

UNIVERSITÉ AMADOU MAHTAR MBOW, SENEGAL
Email address: thierno.sow@uam.edu.sn