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# Coincidence and common fixed point theorems for nonself $G$-almost contractions in Banach spaces endowed with a graph

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ABSTRACT. In this paper, we establish some coincidence point theorems and common fixed point theorems for nonself $G$-almost contractions in Banach spaces endowed with a directed graph and display some examples to confirm our main results. Our main theorems extend and generalize many known theorems in this area.

## 1. INTRODUCTION

A large number of the important nonlinear problems of applied mathematics reduce to seeking a solution of an equation which in turn may be reduced to looking for the fixed points of a certain mapping or the coincidence points or common fixed points of two or more mappings. This is the reason why the study of the coincidence points and common fixed points of some mappings satisfying some contractive type mappings attracted many mathematicians.

Many of the researches in metric fixed point theory deals with single-valued self mappings $T : X \to X$ and multi-valued self mappings $T : X \to P(X)$ satisfying a contraction type condition, where $X$ is a set endowed with a certain metric structure. These outputs are mainly generalizations of the Banach contraction principle [10], which can be stated as follows.

**Theorem 1.1.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ a contraction, i.e., a map satisfying*

$$d(Tx, Ty) \le \alpha \cdot d(x, y), \text{ for all } x, y \in X, \tag{1.1}$$

*where $0 < \alpha < 1$ is a constant. Then $T$ has a unique fixed point in $X$, say $x^*$, and the Picard iteration $\{T^n x_0\}$ converges to $x^*$ for all $x_0 \in X$ (that is, $T$ is a Picard operator, see* [44]*).*

The Banach fixed point theorem is one of the most useful researches in nonlinear analysis, and has many applications in solving nonlinear functional equations, optimization problems, variational inequalities etc., by transforming them into a fixed point problem. However, under this form it has at least two drawbacks: first, the contraction condition (1.1) compels $T$ to be continuous and, secondly, the condition $T(X) \subset X$ makes it not applicable to most of the nonlinear problems where the relevant operator $T$ is actually a nonself operator.

Therefore, there are many important reasons for the study of the fixed points not only for self mappings, but also to tackle a great and challenging research topic to obtain fixed point theorems for nonself mappings.

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In 1972, Assad and Kirk [5] extended Banach contraction mapping principle to nonself multi-valued contraction mappings $T : K \to P(X)$ in the case $(X, d)$ is a convex metric space in the sense of Menger and $K$ is a nonempty closed subset of $X$ such that $T$ maps $\partial K$ (the boundary of $K$) into $K$. Later, there are many researches devoted to nonself mappings, see [22, 32, 33, 41, 44].

Moreover, in 2004, Berinde [15], introduced a new class of self mappings (usually called weak contractions, almost contractions or Berinde operators) that satisfy a simple but more general contraction condition. In 2010, Berinde [18] studied about approximating common fixed points of noncommuting self almost contractions in metric spaces. Next, in 2013, Berinde and Păcurar [24] introduced fixed point theorems for nonself single-valued almost contractions. Later, in 2019, Berinde, Sridarat and Suantai [27] were interested in establishing coincidence point theorems and common fixed point theorems for nonself single-valued almost contractions, to extended the previous results in [18] and [24].

On the other hand, a very interesting idea came out by Jachymski [39], in 2008, who combined the concepts of fixed point theory and graph theory to study fixed point theorems in a metric space endowed with a directed graph. Jachymski introduced the concept of $G$-contraction and thus obtained a generalization of Banach's contraction principle.

Subsequently, there appeared many results on the fixed point theory in metric spaces endowed with a directed graph, which extended many researches in this area, see [3, 8, 9, 12, 25, 30, 47, 48, 49]. Next, in 2016, Tiammee, Cho and Suantai [50] studied fixed point theorems for nonself $G$-almost contractive mappings in Banach spaces endowed with graphs.

Inspired by these researches, especially the ones in [27] and [50], the aim of this work is to prove the existence of coincidence points and common fixed points of a nonself $G$-almost contraction in Banach spaces endowed with graphs.

Our theorems and corollaries extend the outputs of [24] and [27] and generalize many known theorems in this field. Furthermore, we also display some examples to confirm our main results.

## 2. Preliminaries

In this section, we give some fundamental and beneficial definitions and state some known results that are useful for the proofs of the main theorems in this research.

Let $G = (V(G), E(G))$ be a directed 1-graph, where $V(G)$ is a set of vertices of the graph and $E(G)$ is the set of its edges. We denote by $G^{-1}$ the directed graph obtained from $G$ by reversing the direction of edges, that is,

$$E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}.$$

If $x$ and $y$ are vertices in $G$, then a *path* in $G$ from $x$ to $y$ of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$, $(x_{i-1}, x_i) \in E(G)$ for each $i = 1, 2, ..., n$. A closed directed path of length $N > 1$ from $x$ to $y$, that is $x = y$, is called a *directed cycle*. A directed graph without directed cycles is called *a directed acyclic graph*. A directed graph $G$ is *symmetric* if, whenever $(x, y) \in E(G)$, then $(y, x) \in E(G)$.

**Definition 2.1** ([39]). *Let $(X, d)$ be a metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, i.e., $\Delta = \{(x, x) : x \in X\} \subseteq E(G)$. A mapping $f : X \to X$ is said to be G-contractive if $f$ is edge-preserving, i.e.,*

$$x, y \in X, (x, y) \in E(G) \Longrightarrow (f(x), f(y)) \in E(G)$$

*and there exists $\alpha \in [0, 1)$ such that, for all $x, y \in X$,*

$$(x, y) \in E(G) \Longrightarrow d(f(x), f(y)) \leq \alpha d(x, y).$$

**Property (A).** ([12]). For any sequence $\{x_n\}_{n\in\mathbb{N}}$, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$ for each $n \in \mathbb{N}$.

In order to obtain our main outputs, we use the following definition of domination in graphs ([11, 42]).

Let $G = (V(G), E(G))$ be a directed graph. A set $X \subseteq V(G)$ is called a *dominating set* if, for any $v \in V(G) \setminus X$, there exists $x \in X$ such that $(x, v) \in E(G)$ and we call that $x$ dominates $v$ or $v$ is dominated by $x$. For all $v \in V$, a set $X \subseteq V$ is *dominated* by $v$ if $(v, x) \in E(G)$ for each $x \in X$ and we say that $X$ *dominates* $v$ if $(x, v) \in E(G)$ for each $x \in X$.

A graph $G$ is said to be *transitive* if for every $x, y, z \in V(G)$ such that $(x, y)$ and $(y, z)$ are in $E(G)$, then $(x, z) \in E(G)$.

Next, we give some definitions about coincidence points and weakly compatible mappings and present a useful proposition.

**Definition 2.2** ([27]). *Let $X$ be a metric space, $K$ a nonempty closed subset of $X$ and let $T, S : K \to X$ be two nonself mappings. If there exists $x \in K$ such that $Tx = Sx$, then $x$ is called a coincidence point of $T$ and $S$, and $y = Tx = Sx$ is called a point of coincidence of $T$ and $S$. If $Tx = Sx = x$, then $x$ is called a common fixed point of $T$ and $S$.*

**Definition 2.3** ([27]). *Let $X$ be a metric space, $K$ a nonempty closed subset of $X$ and let $T, S : K \to X$ be two nonself mappings. The pair of mappings $T$ and $S$ is said to be weakly compatible if they commute at their coincidence points.*

**Proposition 2.1** ([27]). *Let $X$ be a metric space, $K$ a nonempty closed subset of $X$ and let $T$ and $S : K \to X$ be weakly compatible nonself mappings. If $T$ and $S$ have a unique point of coincidence $y \in K$, then $y$ is the unique common fixed point of $T$ and $S$.*

Let $X$ be a Banach space, $K$ a nonempty closed subset of $X$ and $T, S : K \to X$ two nonself mappings. Let $S(K)$ be a closed subset of $X$. Let $X_{ST} = \{x \in K \mid Tx \notin S(K)\}$. For $x \in X_{ST}$, we assume that one can always to choose $y \in \partial(S(K))$ such that

$$y = (1 - \lambda)Sx + \lambda Tx, (0 < \lambda < 1) \tag{2.2}$$

and denote by $Y_x$ the set of all points $y \in \partial(S(K))$ satisfying (2.2). We see that

$$\| Sx - Tx \| = \| Sx - y \| + \| y - Tx \| .$$

In general, the set $Y_x$ of points satisfying condition (2.2) may contain more than one element. In this situation we will need the following property.

**Definition 2.4** ([27]). *Let $X$ be a Banach space, $K$ a nonempty closed subset of $X$ and $T, S : K \to X$ two nonself mappings. Let $S(K)$ be a closed subset of $X$. Let $X_{ST} = \{x \in K \mid Tx \notin S(K)\}$. For $x \in X_{ST}$, let $y \in \partial(S(K))$ be the corresponding elements given by* (2.2). *If, for any $x \in X_{ST}$, the inequality*

$$\| Sy' - Ty' \| \leq \| Sx - Tx \| \tag{2.3}$$

*is satisfied for at least one point $y \in Y_x$, where $y = Sy'$, with $y' \in K$, then we say that the pair $(T, S)$ has property $(M')$.*

**Definition 2.5** ([50]). *Let $K$ be a nonempty subset of a normed space $X$ and $G = (V(G), E(G))$ be a directed graph such that $V(G) = K$.*

*(1) A mapping $T : K \to X$ is said to be $G$-almost contraction if there exist $\delta \in (0, 1)$ and $L \geq 0$ with $\delta(1 + L) < 1$ such that, for all $x, y \in K$,*

$$\| Tx - Ty \| \leq \delta \| x - y \| + L \| y - Tx \|$$

*whenever $(x, y) \in E(G)$;*

*(2) A mapping $T : K \to X$ is said to be G-contraction if there exists $k \in (0,1)$ such that, for all $x, y \in K$,*

$$\| Tx - Ty \| \leq k \| x - y \|$$

*whenever $(x, y) \in E(G)$.*

## 3. Main result

In this section, we prove the existence of coincidence points and common fixed points of two nonself $G$-almost contractions $T$ and $S$.

**Theorem 3.2.** *Let $X$ be a Banach space, $K$ a nonempty closed subset of $X$ and $G = (V(G), E(G))$ a directed graph such that transitive. Let $T, S : K \to X$ be two nonself mappings for which there exist two constants $\delta \in (0,1)$ and $L \geq 0$ such that, for any $x, y \in K$,*

$$\| Tx - Ty \| \leq \delta \cdot \| Sx - Sy \| + L \| Sy - Tx \|, \text{ for all } (x, y) \in E(G). \tag{3.4}$$

*Assume that $V(G) = K \cup T(K) \cup S(K)$, $T$ is edge-preserving, $S$ satisfies the condition:*

$$\text{for all } x, y \in K, (Sx, Sy) \in E(G) \Longrightarrow (x, y) \in E(G), \tag{3.5}$$

*$S(K)$ is closed and has property (A), the pair $(T, S)$ has property $(M')$ and satisfies the condition: for any $x \in K$,*

$$\text{if } Sx \in \partial(S(K)), \text{ then } Tx \in S(K). \tag{3.6}$$

*Suppose also that $Y_x$ is dominated by $Sx$ and $Y_x$ dominates $Tx$ for every $x \in X_{ST}$. If there exists $y \in \partial(S(K))$ such that $y = Sx_0$ for some $x_0 \in K$ and $(Sx_0, Tx_0) \in E(G)$, then $T$ and $S$ have a point of coincidence in $X$.*

*Proof.* Assume that $y \in \partial(S(K))$ such that $y = Sx_0$ for some $x_0 \in K$ and $(Sx_0, Tx_0) \in E(G)$. By (3.6) we have $Tx_0 \in S(K)$. Thus there exists $x_1 \in K$ such that $Sx_1 = Tx_0$. So $(Sx_0, Sx_1) \in E(G)$. We have $(x_0, x_1) \in E(G)$ and hence $(Tx_0, Tx_1) \in E(G)$.

Next, if $Tx_1 \in S(K)$, then there is $x_2 \in K$ such that $Sx_2 = Tx_1$. We obtain $(Sx_1, Sx_2) = (Tx_0, Tx_1) \in E(G)$. Thus $(x_1, x_2) \in E(G)$.

If $Tx_1 \notin S(K)$, by property $(M')$ we can choose $y_1 \in Y_{x_1}$ such that $y_1 \in \partial(S(K))$, $y_1 = Sx_2$ for some $x_2 \in K$ which satisfies

$$\| Sx_2 - Tx_2 \| \leq \| Sx_1 - Tx_1 \|$$

and

$$y_1 = Sx_2 = (1 - \lambda_1) Sx_1 + \lambda_1 Tx_1, \text{ for some } \lambda_1 \in (0,1).$$

Note that $Sx_2 \neq Tx_1$. Since $Y_{x_1}$ is dominated by $Sx_1$, we have $(Sx_1, Sx_2) = (Sx_1, y_1) \in E(G)$. So $(x_1, x_2) \in E(G)$. Continuing in this manner, we get a sequence $\{Sx_n\}$ such that

$(i)$ $Sx_n = Tx_{n-1}$, if $Tx_{n-1} \in S(K)$;

$(ii)$ $Sx_n = (1 - \lambda_{n-1}) Sx_{n-1} + \lambda_{n-1} Tx_{n-1} \in \partial(S(K))$ $(0 < \lambda_{n-1} < 1)$, if $Tx_{n-1} \notin S(K)$.

Let us denote

$$P = \{Sx_k \in \{Sx_n\} \mid Sx_k = Tx_{k-1}\}$$

and

$$Q = \{Sx_k \in \{Sx_n\} \mid Sx_k \neq Tx_{k-1}\}.$$

We see that $\{Sx_n\} \subset S(K)$ and that, if $Sx_k \in Q$, then both $Sx_{k-1}$ and $Sx_{k+1}$ belong to the set $P$. Moreover, by virtue of (3.6), we cannot have two consecutive terms of $\{Sx_n\}$ in the set $Q$ (but we can have two consecutive terms of $\{Sx_n\}$ in the set $P$).

We claim that $\{Sx_n\}$ is a Cauchy sequence. To prove this, we have to discuss the following three distinct cases:

- Case I. $Sx_n,\ Sx_{n+1} \in P$;
- Case II. $Sx_n \in P,\ Sx_{n+1} \in Q$;
- Case III. $Sx_n \in Q,\ Sx_{n+1} \in P$.

We see that $(Sx_n, Sx_{n+1}) \in E(G)$ and $(x_n, x_{n+1}) \in E(G)$ when $n \in \mathbb{N}$ for Case I and Case II.

Now, we consider Case III. $Sx_n \in Q,\ Sx_{n+1} \in P$.

Since $Sx_n \in Q$, $Sx_n \neq Tx_{n-1}$ and $Tx_{n-1} \notin S(K)$. Thus $Sx_n \in \partial(S(K))$ and $Sx_n \in Y_{x_{n-1}}$. Since $Y_{x_{n-1}}$ is dominated by $Sx_{n-1}$, $(Sx_{n-1}, Sx_n) \in E(G)$.

Then $(x_{n-1}, x_n) \in E(G)$. Therefore $(Tx_{n-1}, Tx_n) \in E(G)$. We get $(Sx_n, Tx_{n-1}) \in E(G)$ because $Y_{x_{n-1}}$ dominates $Tx_{n-1}$.

From $(Sx_n, Tx_{n-1}) \in E(G)$, $(Tx_{n-1}, Tx_n) \in E(G)$ and since $G$ is transitive, $(Sx_n, Sx_{n+1}) = (Sx_n, Tx_n) \in E(G)$. We obtain $(x_n, x_{n+1}) \in E(G)$, too.

Therefore $(Sx_n, Sx_{n+1}) \in E(G)$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$.

Next, we prove that $\{Sx_n\}$ is a Cauchy sequence.

*Case I.* $Sx_n,\ Sx_{n+1} \in P$.

In this case we get $Sx_n = Tx_{n-1}$, $Sx_{n+1} = Tx_n$ and by (3.4) we have

$$\| Sx_n - Sx_{n+1} \| = \| Tx_{n-1} - Tx_n \| \leq \ \delta \cdot \| Sx_{n-1} - Sx_n \| + L \| Sx_n - Tx_{n-1} \|$$

so,

$$\| Sx_n - Sx_{n+1} \| \leq \ \delta \cdot \| Sx_{n-1} - Sx_n \| . \tag{3.7}$$

*Case II.* $Sx_n \in P,\ Sx_{n+1} \in Q$.

In this case we get $Sx_n = Tx_{n-1}$ but $Sx_{n+1} \neq Tx_n$ and

$$\| Sx_n - Sx_{n+1} \| + \| Sx_{n+1} - Tx_n \| = \| Sx_n - Tx_n \| .$$

Then

$$\| Sx_n - Sx_{n+1} \| \leq \| Sx_n - Tx_n \| = \| Tx_{n-1} - Tx_n \|$$

and therefore by using (3.4) we have

$$\begin{aligned} \| Sx_n - Sx_{n+1} \| \leq \| Tx_{n-1} - Tx_n \| &\leq \delta \cdot \| Sx_{n-1} - Sx_n \| + L \| Sx_n - Tx_{n-1} \| \\ &= \delta \cdot \| Sx_{n-1} - Sx_n \|, \end{aligned}$$

which satisfies again inequality (3.7).

*Case III.* $Sx_n \in Q,\ Sx_{n+1} \in P$.

In this case, we obtain $Sx_{n-1} \in P$. Since the pair $(T, S)$ has property $(M')$, it follows that

$$\| Sx_n - Sx_{n+1} \| = \| Sx_n - Tx_n \| \leq \| Sx_{n-1} - Tx_{n-1} \| .$$

Since $Sx_{n-1} \in P$, we have $Sx_{n-1} = Tx_{n-2}$ and by (3.4) we get

$$\begin{aligned} \| Tx_{n-2} - Tx_{n-1} \| &\leq \delta \cdot \| Sx_{n-2} - Sx_{n-1} \| + L \| Sx_{n-1} - Tx_{n-2} \| \\ &= \delta \cdot \| Sx_{n-2} - Sx_{n-1} \|, \end{aligned}$$

which yields that

$$\| Sx_n - Sx_{n+1} \| \leq \ \delta \cdot \| Sx_{n-2} - Sx_{n-1} \| . \tag{3.8}$$

From the above three cases, and (3.7) and (3.8), we have that the sequence $\{Sx_n\}$ satisfies the inequality

$$\| Sx_n - Sx_{n+1} \| \leq \ \delta \ \max\{\| Sx_{n-2} - Sx_{n-1} \|, \| Sx_{n-1} - Sx_n \|\}, \tag{3.9}$$

for any $n \geq 2$. From (3.9), by induction, we obtain that

$$\| Sx_n - Sx_{n+1} \| \leq \ \delta^{[n/2]} \ \max\{\| Sx_0 - Sx_1 \|, \| Sx_1 - Sx_2 \|\},$$

for any $n \geq 2$, where $[n/2]$ denotes the greatest integer not exceeding $n/2$.

Furthermore, for $m > n > N$,

$$\| Sx_n - Sx_m \| \leq \sum_{i=N}^{\infty} \| Sx_i - Sx_{i+1} \| \leq 2\frac{\delta^{[N/2]}}{1-\delta} \cdot \max\{\| Sx_0 - Sx_1 \|, \| Sx_1 - Sx_2 \|\},$$

which shows that indeed $\{Sx_n\}$ is a Cauchy sequence.

Since $\{Sx_n\} \subset S(K)$ and $S(K)$ is closed, $\{Sx_n\}$ converges to some point $x^*$ in $S(K)$.

Let $\{Sx_{n_k}\} \subset P$ be an infinite subsequence of $\{Sx_n\}$ (such a subsequence always exists). Since $x^* \in S(K)$, there exists a $p \in K$ such that $x^* = Sp$. Since $S(K)$ has property (A), $(Sx_n, Sp) = (Sx_n, x^*) \in E(G)$, for all $n \in \mathbb{N}$. So $(x_n, p) \in E(G)$ for all $n \in \mathbb{N}$. Therefore we have

$$\begin{aligned}
\| Sp - Tp \| &\leq \| Sp - Sx_{n_k+1} \| + \| Sx_{n_k+1} - Tp \| \\
&= \| Sp - Sx_{n_k+1} \| + \| Tx_{n_k} - Tp \| \\
&\leq \| Sp - Sx_{n_k+1} \| + \delta\cdot \| Sx_{n_k} - Sp \| + L \| Sp - Tx_{n_k} \| \\
&= (1+L) \| Sp - Sx_{n_k+1} \| + \delta\cdot \| Sx_{n_k} - Sp \|,
\end{aligned}$$

from which we get

$$\| Sp - Tp \| \leq (1+L) \| Sp - Sx_{n_k+1} \| + \delta\cdot \| Sx_{n_k} - Sp \|, \tag{3.10}$$

for all $k \geq 0$. Taking $k \to \infty$ in (3.10), we obtain

$$\| Sp - Tp \| = 0,$$

which shows that $Sp = Tp$, that is, $p$ is a coincidence point of $T$ and $S$ and $x^*$ is a point of coincidence of $T$ and $S$. □

**Theorem 3.3.** *Let $X$ be a Banach space, $K$ a nonempty closed subset of $X$ and $G = (V(G), E(G))$ a directed graph such that transitive. Let $T, S : K \to X$ be two nonself mappings satisfying* (3.4) *for which there exist two constants $\theta \in (0,1)$ and $L_1 \geq 0$ such that, for any $x, y \in K$,*

$$\| Tx - Ty \| \leq \theta\cdot \| Sx - Sy \| + L_1 \| Sx - Tx \|, \text{ for all } (x,y) \in E(G). \tag{3.11}$$

*Assume that*

*(i) $V(G) = K \cup T(K) \cup S(K)$;*

*(ii) $T$ is edge-preserving;*

*(iii) $S$ satisfies the condition* (3.5)*;*

*(iv) $S(K)$ is closed and has property (A);*

*(v) the pair $(T,S)$ has property $(M')$ and satisfies the condition* (3.6).

*Suppose also that $Y_x$ is dominated by $Sx$ and $Y_x$ dominates $Tx$ for every $x \in X_{ST}$.*

*If there exists $y \in \partial(S(K))$ such that $y = Sx_0$ for some $x_0 \in K$ and $(Sx_0, Tx_0) \in E(G)$, $(x^*, u^*) \in E(G)$ for each $x^*, u^*$ which is a point of coincidence of $T$ and $S$, then $T$ and $S$ have a unique point of coincidence in $X$.*

*Moreover, if $T$ and $S$ are weakly compatible and have a unique point of coincidence of $T$ and $S$ is in $K$, then $T$ and $S$ have a unique common fixed point in $K$.*

*Proof.* By Theorem 3.2, $T$ and $S$ have a point of coincidence, say $x^* = Tp = Sp$, for some $p \in K$. Now, let us show that $T$ and $S$ actually have a unique point of coincidence.

Assume that there exists $q \in K$ such that $Tq = Sq$. Since $Sq, Sp$ are the points of coincidences of $T$ and $S$, $(Sq, Sp) \in E(G)$. So $(q, p) \in E(G)$. Thus by (3.11) we get

$$\| Sq - Sp \| = \| Tq - Tp \| \leq \theta \| Sq - Sp \| + L_1 \| Sq - Tq \| = \theta \| Sq - Sp \|,$$

that is, $(1-\theta) \| Sq - Sp \| \leq 0$, which implies $\| Sq - Sp \| = 0$, that is $Sq = Sp = x^*$. Therefore $T$ and $S$ have a unique point of coincidence, $x^*$.

Next, suppose that $T$ and $S$ are weakly compatible and $x^* \in K$. By Proposition 2.1, we have $x^*$ is a unique common fixed point of $T$ and $S$. □

As a consequence of Theorem 3.2, by setting $S = I$, where $I : K \to X$ is the identity mapping, we get the following:

**Corollary 3.1.** *Let $X$ be a Banach space, $K$ a nonempty closed subset of $X$ and $G = (V(G), E(G))$ a directed graph such that transitive. Let $T : K \to X$ be a nonself mapping for which there exist two constants $\delta \in (0, 1)$ and $L \geq 0$ such that, for any $x, y \in K$,*

(3.12) $$\| Tx - Ty \| \leq \delta \cdot \| x - y \| + L \| y - Tx \|, \text{ for all } (x, y) \in E(G).$$

*Assume that*
*(i) $V(G) = K \cup T(K)$;*
*(ii) $T$ is edge-preserving;*
*(iii) $K$ has property (A);*
*(iv) $T$ has property $(M)$ and satisfies Rothe's boundary condition*

(3.13) $$T(\partial K) \subset K.$$

*Suppose also that $Y_x$ is dominated by $x$ and $Y_x$ dominates $Tx$ for all $x \in K$ with $Tx \notin K$. If there exists $x_0 \in \partial K$ such that $(x_0, Tx_0) \in E(G)$, then $T$ has a fixed point in $K$.*

**Remark 3.1.** *Property $(M)$ (*[24]*) is a special case of property $(M')$ when $S = I$, where $I : K \to X$ is the identity mapping.*

As a consequence of Theorem 3.3, by setting $S = I$, where $I : K \to X$ is the identity mapping, we obtain the following:

**Corollary 3.2.** *Let $X$ be a Banach space, $K$ a nonempty closed subset of $X$ and $G = (V(G), E(G))$ a directed graph such that transitive. Let $T : K \to X$ be a nonself mapping satisfying* (3.12) *for which there exist two constants $\theta \in (0, 1)$ and $L_1 \geq 0$ such that, for any $x, y \in K$,*

(3.14) $$\| Tx - Ty \| \leq \theta \cdot \| x - y \| + L_1 \| x - Tx \|, \text{ for all } (x, y) \in E(G).$$

*Assume that*
*(i) $V(G) = K \cup T(K)$;*
*(ii) $T$ is edge-preserving;*
*(iii) $K$ has property (A);*
*(iv) $T$ has property $(M)$ and satisfies the condition* (3.13)*;*
*Suppose also that $Y_x$ is dominated by $x$ and $Y_x$ dominates $Tx$, for all $x \in K$ with $Tx \notin K$.*
*If there exists $x_0 \in \partial K$ such that $(x_0, Tx_0) \in E(G)$, $(q, p) \in E(G)$, for each $q, p$, which is a fixed point of $T$, then $T$ has a unique fixed point in $K$.*

**Remark 3.2.** *By putting $E(G) = K \cup T(K) \times K \cup T(K)$, from Corollary 3.1, we have Theorem 3.3 of* [24] *and from Corollary 3.2, we get Theorem 3.6 of* [24].

Moreover, from Theorem 3.2 and Theorem 3.3, by putting $E(G) = K \cup T(K) \cup S(K) \times K \cup T(K) \cup S(K)$, we have Theorem 3 and Theorem 4 of [27], respectively.
We denote $d(x, y) = \| x - y \|$.

**Corollary 3.3** ([27], Theorem 3). *Let $X$ be a Banach space, $K$ a nonempty closed subset of $X$ and let $T, S : K \to X$ be two nonself mappings for which there exist two constants $\delta \in (0, 1)$ and $L \geq 0$ such that*

(3.15) $$d(Tx, Ty) \leq \delta \cdot d(Sx, Sy) + Ld(Sy, Tx), \text{ for all } x, y \in K.$$

*If $S(K)$ is closed, the pair $(T, S)$ has property $(M')$ and satisfies the condition: for any $x \in K$,*

(3.16) $$\text{if } Sx \in \partial(S(K)), \text{ then } Tx \in S(K),$$

*then $T$ and $S$ have a point of coincidence in $X$.*

**Corollary 3.4** ([27], Theorem 4)**.** *Let $X$ be a Banach space, $K$ a nonempty closed subset of $X$ and let $T, S : K \to X$ be two nonself mappings satisfying* (3.15) *for which there exist a constant $\theta \in (0, 1)$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta d(Sx, Sy) + L_1 d(Sx, Tx), \text{ for all } x, y \in K. \tag{3.17}$$

*If $S(K)$ is closed and the pair $(T, S)$ has property $(M')$ and satisfies the condition* (3.16)*, then $T$ and $S$ have a unique point of coincidence in $X$.*

*Moreover, if $T$ and $S$ are weakly compatible and there is a unique point of coincidence of $T$ and $S$ in $K$, then $T$ and $S$ have a unique common fixed point in $K$.*

## 4. Conclusions

In this work, we obtain coincidence point theorems and common fixed point theorems for nonself $G$-almost contractions in Banach spaces endowed with a directed graph and give some examples to support the validity of our results.

Our main theorems and corollaries extend the results of [15], [24] and [27].

In addition, our theorems expand and generalize many known results in this area, as for example those in [5], [8], [9], [15], [18], [36].

For other related developments in this area, we refer to [1], [2], [4], [6], [7], [13], [14], [16], [17], [19], [20], [21], [23], [26], [28], [29], [31], [34], [35], [37], [38], [40], [43], [45], [46].

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## References

[1] Abbas, M.; Shatanawi, W.; Nazir, T. Common coupled coincidence and coupled fixed point of $C$-contractive mappings in generalized metric spaces. *Thai J. Math.* **13** (2015), no. 2, 337–351.

[2] Abdulkerim, F.; Koyas, K.; Gebregiorgis, S. Coupled coincidence and coupled common fixed points of $(\psi, \phi)$ contraction type T-coupling in metric spaces. *Iran. J. Math. Sci. Inform.* **19** (2024), no. 2, 61–75.

[3] Alfuraidan, M. R. Fixed points of monotone nonexpansive mappings with a graph. *Fixed Point Theory Appl.* **2015** (2015), no. 49, 6 pp.

[4] Arif, M.; Imdad, M. Coincidence point results on a metric space endowed with a locally $T$-transitive binary relation employing comparison functions. *Miskolc Math. Notes* **25** (2024), no. 1, 63–78.

[5] Assad, N. A.; Kirk, W. A. Fixed point theorems for set-valued mappings of contractive type. *Pacific J. Math.* **43** (1972), 553–562.

[6] Aydi, H.; Karapinar, E.; Mustafa, Z. Some tripled coincidence point theorems for almost generalized contractions in ordered metric spaces. *Tamkang J. Math.* **44** (2013), no. 3, 233–251.

[7] Baghani, H.; Agarwal, Ravi P.; Karapinar, E. On coincidence point and fixed point theorems for a general class of multivalued mappings in incomplete metric spaces with an application. *Filomat* **33** (2019), no. 14, 4493–4508.

[8] Balog, L.; Berinde, V. Fixed point theorems for nonself Kannan type contractions in Banach spaces endowed with a graph. *Carpathian J. Math.* **32** (2016), no. 3, 293–302.

[9] Balog, L.; Berinde, V.; Păcurar, M. Approxmating fixed points of nonself contractive type mappings in Banach spaces endowed with a graph. *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.* **24** (2016), no. 2, 27–43.

[10] Banach, S. Theorie des Operations Lineaires, in: *Monografie Matematyczne*, Warszawa-Lwow, 1932.

[11] Bang-Jensen, J.; Gutin, G. *Digraphs theory, algorithms and applications*. Springer Monographs in Mathematics, Springer-Verlag London Ltd., London, 2007.

[12] Beg, I.; Butt, A. R. Fixed point of set-valued graph contractive mappings. *J. Inequal. Appl.* **2013** (2013), no. 252.

[13] Berinde, V. A common fixed point theorem for quasi contractive type mappings. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **46** (2003), 101–110 (2004).

[14] Berinde, V. A common fixed point theorem for nonself mappings. *Miskolc Math. Notes* **5** (2004), no. 2, 137–144.

[15] Berinde, V. Approximation fixed points of weak contractions using the Picard iteration. *Nonlinear Analysis Forum.* **9** (2004), no. 1, 43–53.

[16] Berinde, V. A common fixed point theorem for compatible quasi contractive self mappings in metric spaces. *Appl. Math. Comput.* **213** (2009), no. 2, 348–354.

[17] Berinde, V. Approximating common fixed points of noncommuting discontinuous weakly contractive mappings in metric spaces. *Carpathian J. Math.* **25** (2009), no. 1, 13–22.

[18] Berinde, V. Approximating common fixed points of noncommuting almost contractions in metric spaces. *Fixed Point Theory.* **11** (2010), no. 2, 179–188.

[19] Berinde, V. Common fixed points of noncommuting almost contractions in cone metric spaces. *Math. Commun.* **15** (2010), no. 1, 229–241.

[20] Berinde, V. Common fixed points of noncommuting discontinuous weakly contractive mappings in cone metric spaces. *Taiwanese J. Math.* **14** (2010), no. 5, 1763–1776.

[21] Berinde, V.; Choban, M. Remarks on some completeness conditions involved in several common fixed point theorems. *Creat. Math. Inform.* **19** (2010), no. 1, 1–10.

[22] Berinde, V.; Măruşter, Şt.; Rus, I. A. An abstract point of view on iterative approximation of fixed points of nonself operators. *J. Nonlinear Convex Anal.* **15** (2014), no. 5, 851–865.

[23] Berinde, V.; Păcurar, M. Coupled fixed point theorems for generalized symmetric Meir-Keeler contractions in ordered metric spaces. *Fixed Point Theory Appl.* **2012**, 2012:115, 11 pp.

[24] Berinde, V.; Păcurar, M. Fixed point theorems for nonself single-valued almost contractions. *Fixed Point Theory.* **14** (2013), no. 2, 301–312.

[25] Berinde, V.; Păcurar, M. The contraction principle for nonself mappings on Banach spaces endowed with a graph. *J. Nonlinear Convex Anal.* **16** (2015), no. 9, 1925–1936.

[26] Berinde, V.; Petruşel, A.; Rus, I. A.; Şerban, M. A. The retraction-displacement condition in the theory of fixed point equation with a convergent iterative algorithm. *Mathematical analysis, approximation theory and their applications*, 75–106, Springer Optim. Appl., 111, Springer, [Cham], 2016.

[27] Berinde, V.; Sridarat, P.; Suantai, S. Coincidence point theorem and common fixed point theorem for nonself single-valued almost contractions. *Bull. Math. Soc. Sci. Math. Roumanie.* **62** (2019), no.1, 51–65.

[28] Berinde, V.; Vetro, F. Common fixed points of mappings satisfying implicit contractive conditions. *Fixed Point Theory Appl.* **2012**, 2012:105, 8 pp.

[29] Bisht, Ravindra K. On coincidence and fixed points of general mappings. *Linear Nonlinear Anal.* **9** (2023), no. 3, 207–212.

[30] Bojor, F. Fixed point theorems for Reich type contractions on metric spaces with a graph. *Nonlinear Anal.* **75** (2012), 3895–3901.

[31] Chifu, C.; Petruşel, G. Fixed point theory for nonself multivalued operators on a set with two metrics. *Creat. Math. Inform.* **17** (2008), no. 3, 339–345 (2009).

[32] Ćirić, Lj. B. Quasi contraction non-self mappings on Banach spaces. *Bull. Cl. Sci. Math. Nat. Sci. Math.* **23** (1998), 25–31.

[33] Ćirić, Lj. B.; Ume, J. S.; Khan, M. S.; Pathak, H. K. On some nonself mappings. *Math. Nachr.* **251** (2003), 28–33.

[34] Filip, A.-D. Coincidence point theorems in some generalized metric spaces. *Stud. Univ. Babeş-Bolyai Math.* **68** (2023), no. 4, 925–930.

[35] Gupta, A.; Kumar, A.; Rohilla, M. A new approach to coincidence points and common fixed points under a homotopy of families of mappings in *b*-metric spaces. *Asian-Eur. J. Math.* **17** (2024), no. 3, Paper No. 2450029, 14 pp.

[36] Horvat-Marc, A.; Balog, L. Fixed point theorems for nonself-Bianchini type contractions in Banach spaces endowed with a graph. *Creat. Math. Inform.* **27** (2018), no. 1, 37–48.

[37] Hussain, N.; Berinde, V. Common fixed point and invariant approximation results in certain metrizable topological vector spaces. *Fixed Point Theory Appl.* **2006**, Art. ID 23582, 13 pp.

[38] Hussain, N.; Berinde, V.; Shafqat, N. Common fixed point and approximation results for generalized $\phi$-contractions. *Fixed Point Theory* **10** (2009), no. 1, 111–124.

[39] Jachymski, J. The contraction principle for mappings on a metric space with a graph. *Proc. American Math. Soc.* **136** (2008), 1359–1373.

[40] Khemphet, A. The existence theorem for a coincidence point of some admissible contraction mappings in a generalized metric space. *Thai J. Math.* **18** (2020), Special issue, 223–235.

[41] Nikolić, R. M.; Ješić, S. N.; Babačev, N. A. Fixed point theorems for non-self mappings with nonlinear contractive condition in strictly convex Menger PM-spaces. *Fixed Point Theory* **18** (2017), no. 1, 315–328.

[42] Pang, C.; Zhang, R.; Zhang, Q.; Wang, J. Dominating sets in directed graphs. *Inform. Sci.* **180** (2010), 3647–3652.

[43] Paunović, M.; Haddadi, M. R.; Parvaneh, V. Generalization of simulation functions for finding best proximity pair, best proximity point and best proximity coincidence point. *Filomat* **38** (2024), no. 8, 2847–2856.

[44] Rus, I. A. Five open problems in fixed point theory in terms of fixed point structures (I): Single valued operators, in: *Fixed Point Theory and Its Applications*, eds. Espinola, R.; Petruşel, A.; Prus, S. 39–60, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2013.

[45] Sager, N.; Sağir, B. Common fixed, coupled coincidence and common coupled fixed point results in hyperbolic valued metric spaces. *Bol. Soc. Parana. Mat. (3)* **41** (2023), 15 pp.

[46] Rathee, S.; Reetu. Common fixed point and best approximation results for subcompatible mappings in hyperbolic ordered metric spaces. *Creat. Math. Inform.* **24** (2015), no. 1, 77–82.

[47] Sridarat, P.; Suantai, S. Caristi fixed point theorem in metric spaces with a graph and its applications. *J. Nonlinear Convex Anal.* **17** (2016), no. 7, 1417–1428.

[48] Sridarat, P.; Suantai, S. Common fixed point theorems for multi-valued weak contractive mappings in metric spaces with graphs. *Filomat* **32** (2018), no. 2, 671–680.

[49] Sridarat, P.; Suparaturatorn, R.; Suantai, S.; Cho, Y. J. Convergence analysis of SP-iteration for G-nonexpansive mappings with directed graphs. *B. Malays. Math. Sci. So.* **42** (2019), no. 5, 2361–2380.

[50] Tiammee, J.; Cho, Y. J.; Suantai, S. Fixed point theorems for nonself G-almost contractive mappings in Banach spaces endowed with graphs. *Carpathian J. Math.* **32** (2016), no.3, 375–382.

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